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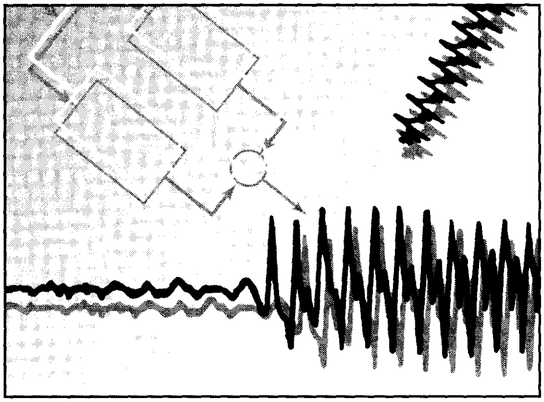
SIGNALS AND SYSTEMS



1.0 INTRODUCTION

As described in the Foreword, the intuitive notions of signals and systems arise in a rich variety of contexts. Moreover, as we will see in this book, there is an analytical framework—that is, a language for describing signals and systems and an extremely powerful set of tools for analyzing them—that applies equally well to problems in many fields. In this chapter, we begin our development of the analytical framework for signals and systems by introducing their mathematical description and representations. In the chapters that follow, we build on this foundation in order to develop and describe additional concepts and methods that add considerably both to our understanding of signals and systems and to our ability to analyze and solve problems involving signals and systems that arise in a broad array of applications.

1.1 CONTINUOUS-TIME AND DISCRETE-TIME SIGNALS

1.1.1 Examples and Mathematical Representation

Signals may describe a wide variety of physical phenomena. Although signals can be represented in many ways, in all cases the information in a signal is contained in a pattern of variations of some form. For example, consider the simple circuit in Figure 1.1. In this case, the patterns of variation over time in the source and capacitor voltages, v_s and v_c, are examples of signals. Similarly, as depicted in Figure 1.2, the variations over time of the applied force f and the resulting automobile velocity v are signals. As another example, consider the human vocal mechanism, which produces speech by creating fluctuations in acoustic pressure. Figure 1.3 is an illustration of a recording of such a speech signal, obtained by

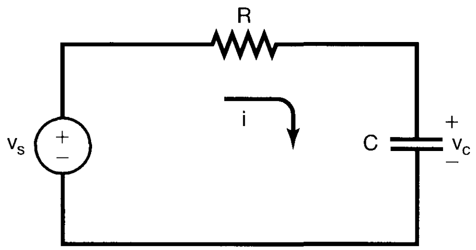


Figure 1.1 A simple *RC* circuit with source voltage v_s and capacitor voltage v_c.

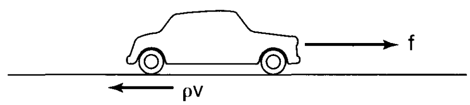


Figure 1.2 An automobile responding to an applied force f from the engine and to a retarding frictional force ρv proportional to the automobile's velocity v.

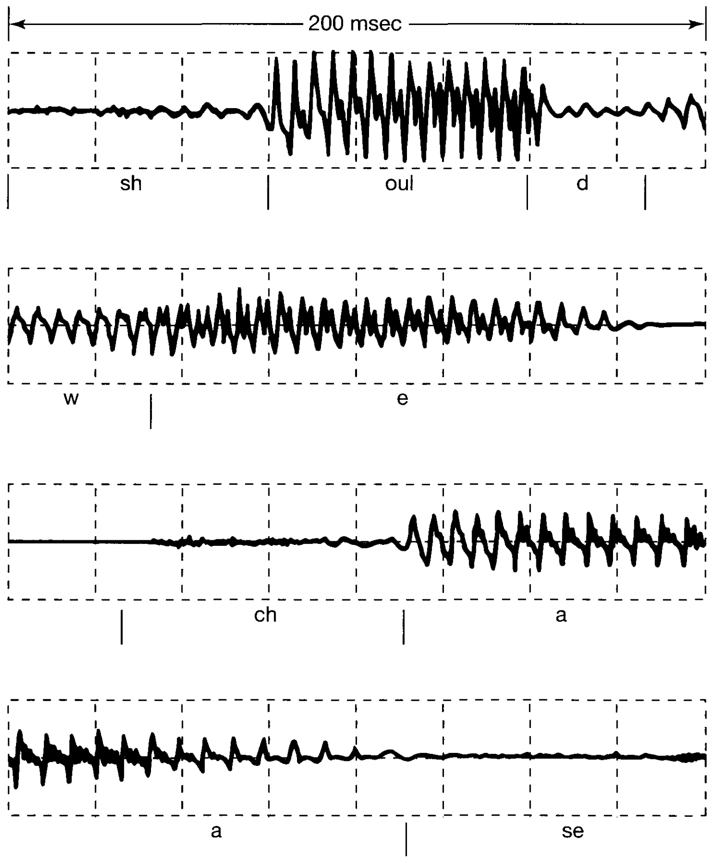


Figure 1.3 Example of a recording of speech. [Adapted from *Applications of Digital Signal Processing*, A.V. Oppenheim, ed. (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1978), p. 121.] The signal represents acoustic pressure variations as a function of time for the spoken words "should we chase." The top line of the figure corresponds to the word "should," the second line to the word "we," and the last two lines to the word "chase." (We have indicated the approximate beginnings and endings of each successive sound in each word.)

using a microphone to sense variations in acoustic pressure, which are then converted into an electrical signal. As can be seen in the figure, different sounds correspond to different patterns in the variations of acoustic pressure, and the human vocal system produces intelligible speech by generating particular sequences of these patterns. Alternatively, for the monochromatic picture, shown in Figure 1.4, it is the pattern of variations in brightness across the image that is important.

Figure 1.4 A monochromatic picture.

Signals are represented mathematically as functions of one or more independent variables. For example, a speech signal can be represented mathematically by acoustic pressure as a function of time, and a picture can be represented by brightness as a function of two spatial variables. In this book, we focus our attention on signals involving a single independent variable. For convenience, we will generally refer to the independent variable as time, although it may not in fact represent time in specific applications. For example, in geophysics, signals representing variations with depth of physical quantities such as density, porosity, and electrical resistivity are used to study the structure of the earth. Also, knowledge of the variations of air pressure, temperature, and wind speed with altitude are extremely important in meteorological investigations. Figure 1.5 depicts a typical example of annual average vertical wind profile as a function of height. The measured variations of wind speed with height are used in examining weather patterns, as well as wind conditions that may affect an aircraft during final approach and landing.

Throughout this book we will be considering two basic types of signals: continuous-time signals and discrete-time signals. In the case of continuous-time signals the independent variable is continuous, and thus these signals are defined for a continuum of values

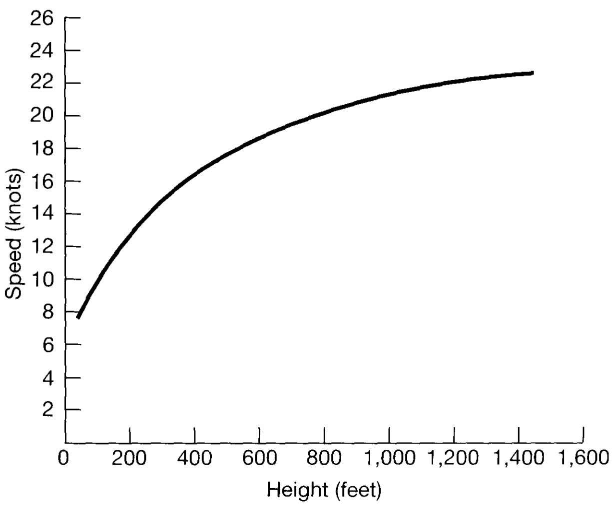


Figure 1.5 Typical annual vertical wind profile. (Adapted from Crawford and Hudson, National Severe Storms Laboratory Report, ESSA ERLTM-NSSL 48, August 1970.)

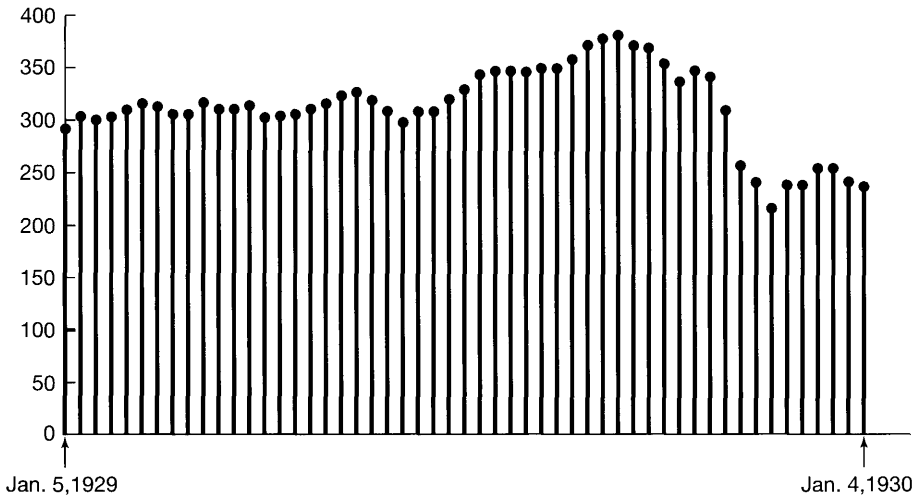


Figure 1.6 An example of a discrete-time signal: The weekly Dow-Jones stock market index from January 5, 1929, to January 4, 1930.

of the independent variable. On the other hand, discrete-time signals are defined only at discrete times, and consequently, for these signals, the independent variable takes on only a discrete set of values. A speech signal as a function of time and atmospheric pressure as a function of altitude are examples of continuous-time signals. The weekly Dow-Jones stock market index, as illustrated in Figure 1.6, is an example of a discrete-time signal. Other examples of discrete-time signals can be found in demographic studies in which various attributes, such as average budget, crime rate, or pounds of fish caught, are tabulated against such discrete variables as family size, total population, or type of fishing vessel, respectively.

To distinguish between continuous-time and discrete-time signals, we will use the symbol t to denote the continuous-time independent variable and n to denote the discrete-time independent variable. In addition, for continuous-time signals we will enclose the independent variable in parentheses $(\cdot)$, whereas for discrete-time signals we will use brackets $[\cdot]$ to enclose the independent variable. We will also have frequent occasions when it will be useful to represent signals graphically. Illustrations of a continuous-time signal $x(t)$ and a discrete-time signal $x[n]$ are shown in Figure 1.7. It is important to note that the discrete-time signal $x[n]$ is defined *only* for integer values of the independent variable. Our choice of graphical representation for $x[n]$ emphasizes this fact, and for further emphasis we will on occasion refer to $x[n]$ as a discrete-time *sequence.*

A discrete-time signal $x[n]$ may represent a phenomenon for which the independent variable is inherently discrete. Signals such as demographic data are examples of this. On the other hand, a very important class of discrete-time signals arises from the *sampling* of continuous-time signals. In this case, the discrete-time signal $x[n]$ represents successive samples of an underlying phenomenon for which the independent variable is continuous. Because of their speed, computational power, and flexibility, modern digital processors are used to implement many practical systems, ranging from digital autopilots to digital audio systems. Such systems require the use of discrete-time sequences representing sampled versions of continuous-time signals—e.g., aircraft position, velocity, and heading for an

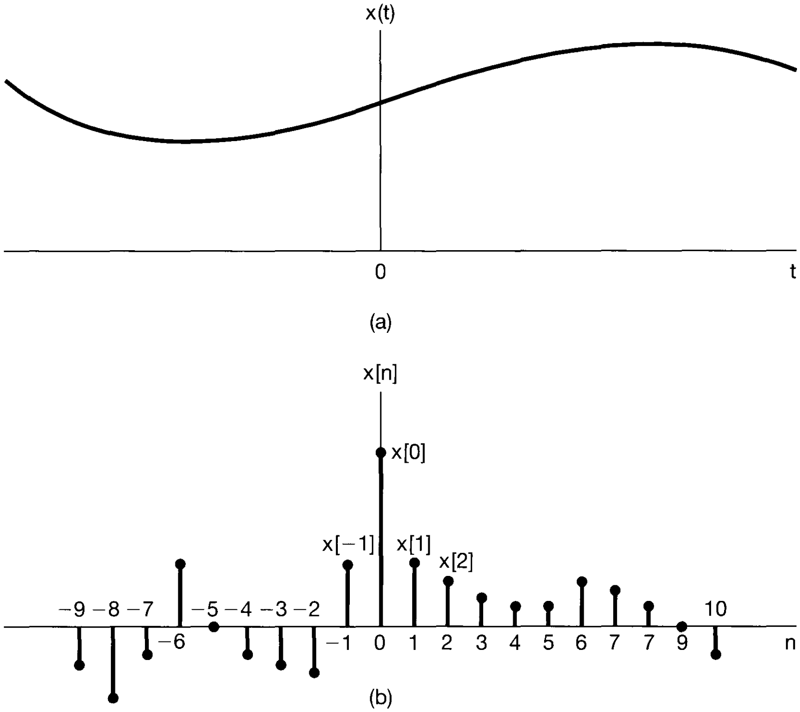


Figure 1.7 Graphical representations of (a) continuous-time and (b) discrete-time signals.

autopilot or speech and music for an audio system. Also, pictures in newspapers—or in this book, for that matter—actually consist of a very fine grid of points, and each of these points represents a sample of the brightness of the corresponding point in the original image. No matter what the source of the data, however, the signal $x[n]$ is defined only for integer values of n. It makes no more sense to refer to the $3\frac{1}{2}$th sample of a digital speech signal than it does to refer to the average budget for a family with $2\frac{1}{2}$ family members.

Throughout most of this book we will treat discrete-time signals and continuous-time signals separately but in parallel, so that we can draw on insights developed in one setting to aid our understanding of another. In Chapter 7 we will return to the question of sampling, and in that context we will bring continuous-time and discrete-time concepts together in order to examine the relationship between a continuous-time signal and a discrete-time signal obtained from it by sampling.

1.1.2 Signal Energy and Power

From the range of examples provided so far, we see that signals may represent a broad variety of phenomena. In many, but not all, applications, the signals we consider are directly related to physical quantities capturing power and energy in a physical system. For example, if $v(t)$ and $i(t)$ are, respectively, the voltage and current across a resistor with resistance R, then the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t). \tag{1.1}$$

The total *energy* expended over the time interval $t_1 \leq t \leq t_2$ is

$$\int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt, \tag{1.2}$$

and the *average power* over this time interval is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt. \tag{1.3}$$

Similarly, for the automobile depicted in Figure 1.2, the instantaneous power dissipated through friction is $p(t) = bv^2(t)$, and we can then define the total energy and average power over a time interval in the same way as in eqs. (1.2) and (1.3).

With simple physical examples such as these as motivation, it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal $x(t)$ or *any* discrete-time signal $x[n]$. Moreover, as we will see shortly, we will frequently find it convenient to consider signals that take on complex values. In this case, the total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt, \tag{1.4}$$

where $|x|$ denotes the magnitude of the (possibly complex) number x. The time-averaged power is obtained by dividing eq. (1.4) by the length, $t_2 - t_1$, of the time interval. Similarly, the total energy in a discrete-time signal $x[n]$ over the time interval $n_1 \leq n \leq n_2$ is defined as

$$\sum_{n=n_1}^{n_2} |x[n]|^2, \tag{1.5}$$

and dividing by the number of points in the interval, $n_2 - n_1 + 1$, yields the average power over the interval. It is important to remember that the terms "power" and "energy" are used here independently of whether the quantities in eqs. (1.4) and (1.5) actually are related to physical energy.[1] Nevertheless, we will find it convenient to use these terms in a general fashion.

Furthermore, in many systems we will be interested in examining power and energy in signals over an infinite time interval, i.e., for $-\infty < t < +\infty$ or for $-\infty < n < +\infty$. In these cases, we define the total energy as limits of eqs. (1.4) and (1.5) as the time interval increases without bound. That is, in continuous time,

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} |x(t)|^2\,dt, \tag{1.6}$$

and in discrete time,

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{1.7}$$

[1] Even if such a relationship does exist, eqs. (1.4) and (1.5) may have the wrong dimensions and scalings. For example, comparing eqs. (1.2) and (1.4), we see that if $x(t)$ represents the voltage across a resistor, then eq. (1.4) must be divided by the resistance (measured, for example, in ohms) to obtain units of physical energy.

Note that for some signals the integral in eq. (1.6) or sum in eq. (1.7) might not converge—e.g., if $x(t)$ or $x[n]$ equals a nonzero constant value for all time. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

In an analogous fashion, we can define the time-averaged power over an infinite interval as

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt \tag{1.8}$$

and

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2 \tag{1.9}$$

in continuous time and discrete time, respectively. With these definitions, we can identify three important classes of signals. The first of these is the class of signals with finite total energy, i.e., those signals for which $E_\infty < \infty$. Such a signal must have zero average power, since in the continuous time case, for example, we see from eq. (1.8) that

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0. \tag{1.10}$$

An example of a finite-energy signal is a signal that takes on the value 1 for $0 \leq t \leq 1$ and 0 otherwise. In this case, $E_\infty = 1$ and $P_\infty = 0$.

A second class of signals are those with finite average power P_∞. From what we have just seen, if $P_\infty > 0$, then, of necessity, $E_\infty = \infty$. This, of course, makes sense, since if there is a nonzero average energy per unit time (i.e., nonzero power), then integrating or summing this over an infinite time interval yields an infinite amount of energy. For example, the constant signal $x[n] = 4$ has infinite energy, but average power $P_\infty = 16$. There are also signals for which neither P_∞ nor E_∞ are finite. A simple example is the signal $x(t) = t$. We will encounter other examples of signals in each of these classes in the remainder of this and the following chapters.

1.2 TRANSFORMATIONS OF THE INDEPENDENT VARIABLE

A central concept in signal and system analysis is that of the transformation of a signal. For example, in an aircraft control system, signals corresponding to the actions of the pilot are transformed by electrical and mechanical systems into changes in aircraft thrust or the positions of aircraft control surfaces such as the rudder or ailerons, which in turn are transformed through the dynamics and kinematics of the vehicle into changes in aircraft velocity and heading. Also, in a high-fidelity audio system, an input signal representing music as recorded on a cassette or compact disc is modified in order to enhance desirable characteristics, to remove recording noise, or to balance the several components of the signal (e.g., treble and bass). In this section, we focus on a very limited but important class of elementary signal transformations that involve simple modification of the independent variable, i.e., the time axis. As we will see in this and subsequent sections of this chapter, these elementary transformations allow us to introduce several basic properties of signals and systems. In later chapters, we will find that they also play an important role in defining and characterizing far richer and important classes of systems.

1.2.1 Examples of Transformations of the Independent Variable

A simple and very important example of transforming the independent variable of a signal is a *time shift.* A time shift in discrete time is illustrated in Figure 1.8, in which we have two signals $x[n]$ and $x[n - n_0]$ that are identical in shape, but that are displaced or shifted relative to each other. We will also encounter time shifts in continuous time, as illustrated in Figure 1.9, in which $x(t - t_0)$ represents a delayed (if t_0 is positive) or advanced (if t_0 is negative) version of $x(t)$. Signals that are related in this fashion arise in applications such as radar, sonar, and seismic signal processing, in which several receivers at different locations observe a signal being transmitted through a medium (water, rock, air, etc.). In this case, the difference in propagation time from the point of origin of the transmitted signal to any two receivers results in a time shift between the signals at the two receivers.

A second basic transformation of the time axis is that of *time reversal.* For example, as illustrated in Figure 1.10, the signal $x[-n]$ is obtained from the signal $x[n]$ by a reflection about $n = 0$ (i.e., by reversing the signal). Similarly, as depicted in Figure 1.11, the signal $x(-t)$ is obtained from the signal $x(t)$ by a reflection about $t = 0$. Thus, if $x(t)$ represents an audio tape recording, then $x(-t)$ is the same tape recording played backward. Another transformation is that of *time scaling.* In Figure 1.12 we have illustrated three signals, $x(t)$, $x(2t)$, and $x(t/2)$, that are related by linear scale changes in the independent variable. If we again think of the example of $x(t)$ as a tape recording, then $x(2t)$ is that recording played at twice the speed, and $x(t/2)$ is the recording played at half-speed.

It is often of interest to determine the effect of transforming the independent variable of a given signal $x(t)$ to obtain a signal of the form $x(\alpha t + \beta)$, where α and β are given numbers. Such a transformation of the independent variable preserves the shape of $x(t)$, except that the resulting signal may be linearly stretched if $|\alpha| < 1$, linearly compressed if $|\alpha| > 1$, reversed in time if $\alpha < 0$, and shifted in time if β is nonzero. This is illustrated in the following set of examples.

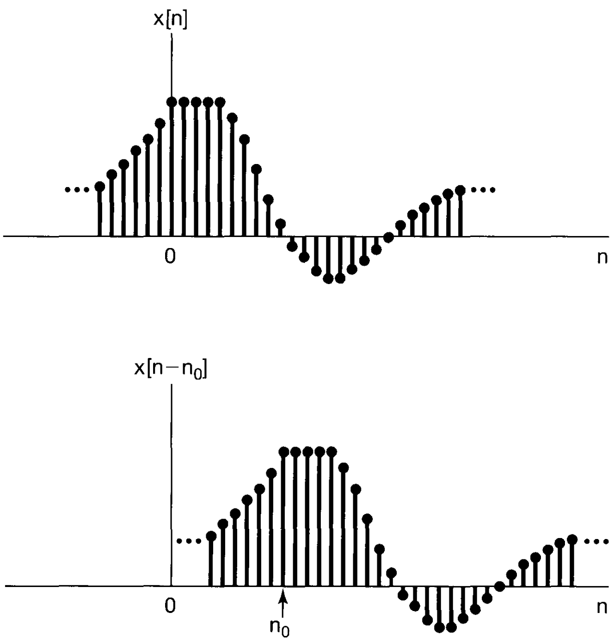


Figure 1.8 Discrete-time signals related by a time shift. In this figure $n_0 > 0$, so that $x[n - n_0]$ is a delayed verson of $x[n]$ (i.e., each point in $x[n]$ occurs later in $x[n - n_0]$).

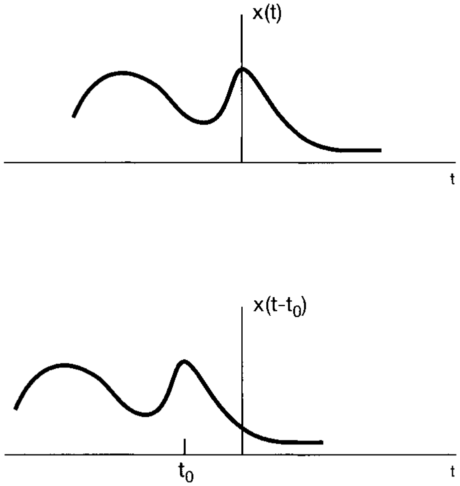


Figure 1.9 Continuous-time signals related by a time shift. In this figure $t_0 < 0$, so that $x(t - t_0)$ is an advanced version of $x(t)$ (i.e., each point in $x(t)$ occurs at an earlier time in $x(t - t_0)$).

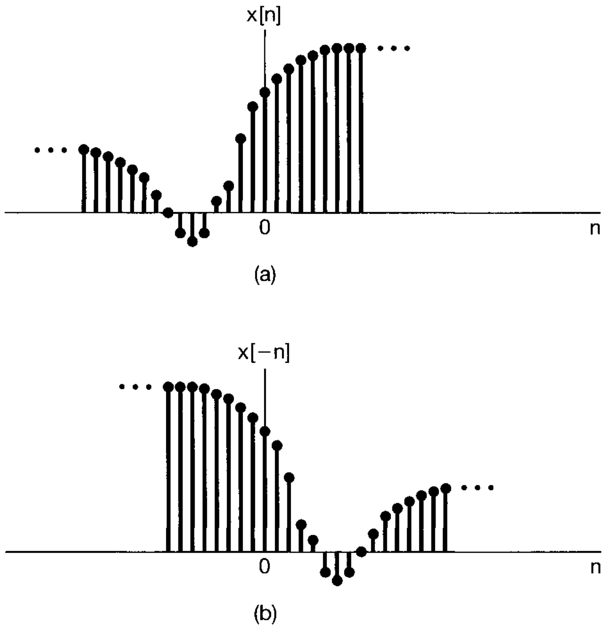


Figure 1.10 (a) A discrete-time signal $x[n]$; (b) its reflection $x[-n]$ about $n = 0$.

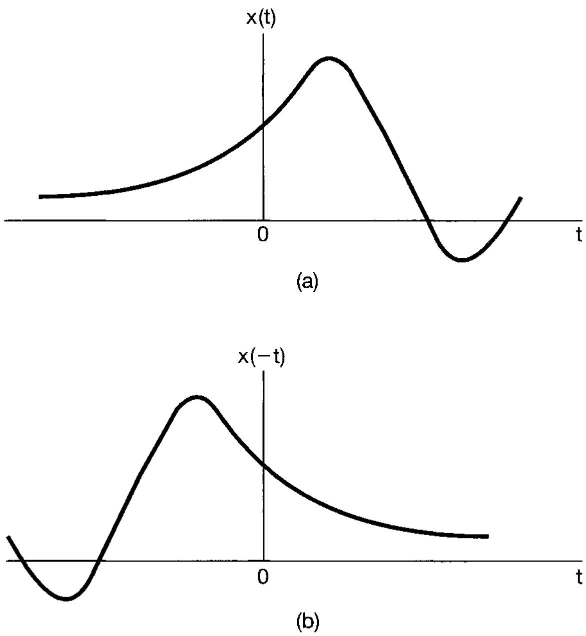


Figure 1.11 (a) A continuous-time signal $x(t)$; (b) its reflection $x(-t)$ about $t = 0$.

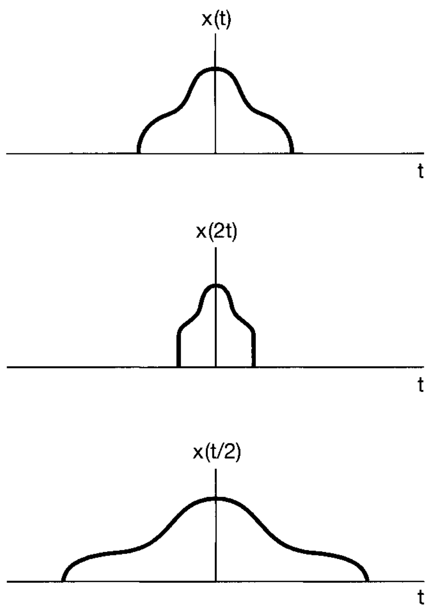


Figure 1.12 Continuous-time signals related by time scaling.

Example 1.1

Given the signal $x(t)$ shown in Figure 1.13(a), the signal $x(t + 1)$ corresponds to an advance (shift to the left) by one unit along the t axis as illustrated in Figure 1.13(b). Specifically, we note that the value of $x(t)$ at $t = t_0$ occurs in $x(t + 1)$ at $t = t_0 - 1$. For

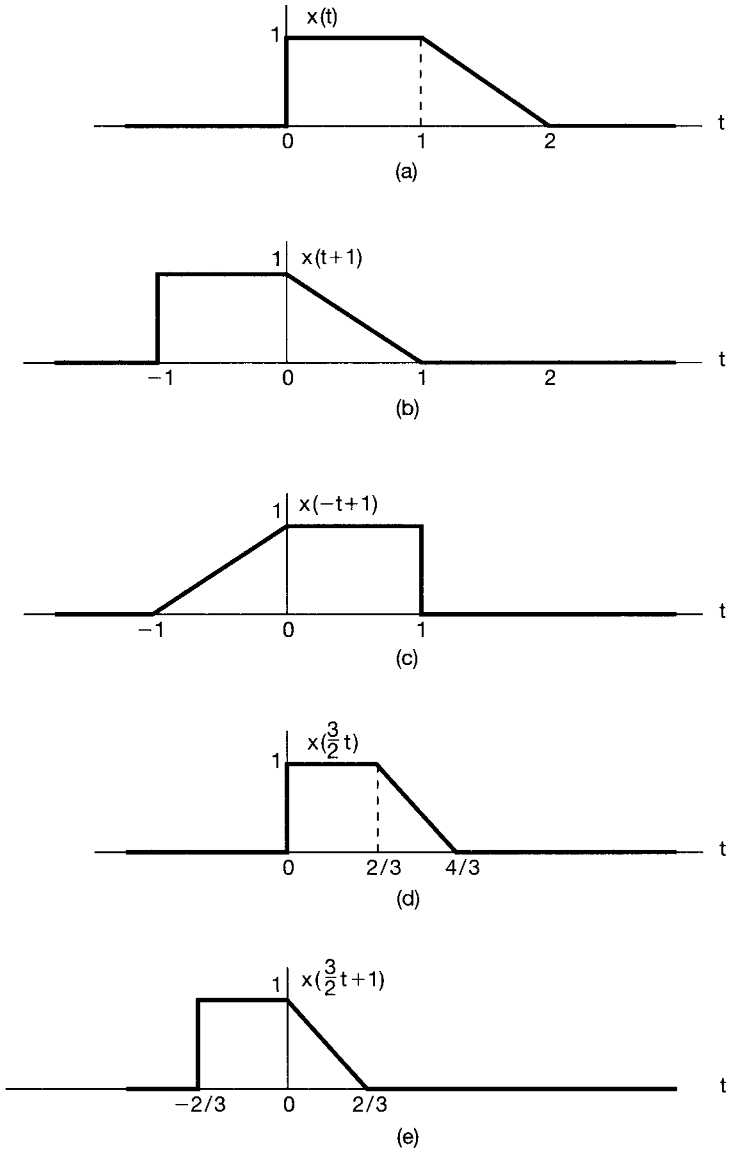


Figure 1.13 (a) The continuous-time signal $x(t)$ used in Examples 1.1–1.3 to illustrate transformations of the independent variable; (b) the time-shifted signal $x(t+1)$; (c) the signal $x(-t+1)$ obtained by a time shift and a time reversal; (d) the time-scaled signal $x(\frac{3}{2}t)$; and (e) the signal $x(\frac{3}{2}t+1)$ obtained by time-shifting and scaling.

example, the value of $x(t)$ at $t = 1$ is found in $x(t+1)$ at $t = 1 - 1 = 0$. Also, since $x(t)$ is zero for $t < 0$, we have $x(t+1)$ zero for $t < -1$. Similarly, since $x(t)$ is zero for $t > 2$, $x(t+1)$ is zero for $t > 1$.

Let us also consider the signal $x(-t+1)$, which may be obtained by replacing t with $-t$ in $x(t+1)$. That is, $x(-t+1)$ is the time reversed version of $x(t+1)$. Thus, $x(-t+1)$ may be obtained graphically by reflecting $x(t+1)$ about the t axis as shown in Figure 1.13(c).

Example 1.2

Given the signal $x(t)$, shown in Figure 1.13(a), the signal $x(\frac{3}{2}t)$ corresponds to a linear compression of $x(t)$ by a factor of $\frac{2}{3}$ as illustrated in Figure 1.13(d). Specifically we note that the value of $x(t)$ at $t = t_0$ occurs in $x(\frac{3}{2}t)$ at $t = \frac{2}{3}t_0$. For example, the value of $x(t)$ at $t = 1$ is found in $x(\frac{3}{2}t)$ at $t = \frac{2}{3}(1) = \frac{2}{3}$. Also, since $x(t)$ is zero for $t < 0$, we have $x(\frac{3}{2}t)$ zero for $t < 0$. Similarly, since $x(t)$ is zero for $t > 2$, $x(\frac{3}{2}t)$ is zero for $t > \frac{4}{3}$.

Example 1.3

Suppose that we would like to determine the effect of transforming the independent variable of a given signal, $x(t)$, to obtain a signal of the form $x(\alpha t + \beta)$, where α and β are given numbers. A systematic approach to doing this is to first delay or advance $x(t)$ in accordance with the value of β, and then to perform time scaling and/or time reversal on the resulting signal in accordance with the value of α. The delayed or advanced signal is linearly stretched if $|\alpha| < 1$, linearly compressed if $|\alpha| > 1$, and reversed in time if $\alpha < 0$.

To illustrate this approach, let us show how $x(\frac{3}{2}t + 1)$ may be determined for the signal $x(t)$ shown in Figure 1.13(a). Since $\beta = 1$, we first advance (shift to the left) $x(t)$ by 1 as shown in Figure 1.13(b). Since $|\alpha| = \frac{3}{2}$, we may linearly compress the shifted signal of Figure 1.13(b) by a factor of $\frac{2}{3}$ to obtain the signal shown in Figure 1.13(e).

In addition to their use in representing physical phenomena such as the time shift in a sonar signal and the speeding up or reversal of an audiotape, transformations of the independent variable are extremely useful in signal and system analysis. In Section 1.6 and in Chapter 2, we will use transformations of the independent variable to introduce and analyze the properties of systems. These transformations are also important in defining and examining some important properties of signals.

1.2.2 Periodic Signals

An important class of signals that we will encounter frequently throughout this book is the class of *periodic* signals. A periodic continuous-time signal $x(t)$ has the property that there is a positive value of T for which

$$x(t) = x(t+T) \tag{1.11}$$

for all values of t. In other words, a periodic signal has the property that it is unchanged by a time shift of T. In this case, we say that $x(t)$ is *periodic with period T*. Periodic continuous-time signals arise in a variety of contexts. For example, as illustrated in Problem 2.61, the natural response of systems in which energy is conserved, such as ideal *LC* circuits without resistive energy dissipation and ideal mechanical systems without frictional losses, are periodic and, in fact, are composed of some of the basic periodic signals that we will introduce in Section 1.3.

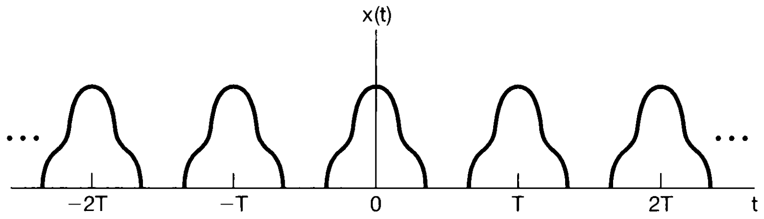


Figure 1.14 A continuous-time periodic signal.

An example of a periodic continuous-time signal is given in Figure 1.14. From the figure or from eq. (1.11), we can readily deduce that if $x(t)$ is periodic with period T, then $x(t) = x(t + mT)$ for all t and for any integer m. Thus, $x(t)$ is also periodic with period $2T, 3T, 4T, \ldots$. The *fundamental period* T_0 of $x(t)$ is the smallest positive value of T for which eq. (1.11) holds. This definition of the fundamental period works, except if $x(t)$ is a constant. In this case the fundamental period is undefined, since $x(t)$ is periodic for *any* choice of T (so there is no smallest positive value). A signal $x(t)$ that is not periodic will be referred to as an *aperiodic* signal.

Periodic signals are defined analogously in discrete time. Specifically, a discrete-time signal $x[n]$ is periodic with period N, where N is a positive integer, if it is unchanged by a time shift of N, i.e., if

$$x[n] = x[n + N] \tag{1.12}$$

for all values of n. If eq. (1.12) holds, then $x[n]$ is also periodic with period $2N, 3N, \ldots$. The *fundamental period* N_0 is the smallest positive value of N for which eq. (1.12) holds. An example of a discrete-time periodic signal with fundamental period $N_0 = 3$ is shown in Figure 1.15.

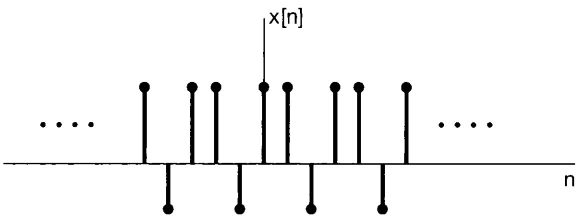


Figure 1.15 A discrete-time periodic signal with fundamental period $N_0 = 3$.

Example 1.4

Let us illustrate the type of problem solving that may be required in determining whether or not a given signal is periodic. The signal whose periodicity we wish to check is given by

$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}. \tag{1.13}$$

From trigonometry, we know that $\cos(t + 2\pi) = \cos(t)$ and $\sin(t + 2\pi) = \sin(t)$. Thus, considering $t > 0$ and $t < 0$ separately, we see that $x(t)$ does repeat itself over every interval of length 2π. However, as illustrated in Figure 1.16, $x(t)$ also has a discontinuity at the time origin that does not recur at any other time. Since every feature in the shape of a periodic signal *must* recur periodically, we conclude that the signal $x(t)$ is not periodic.

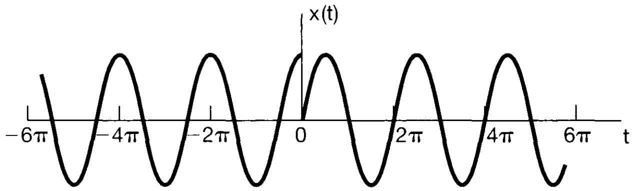


Figure 1.16 The signal $x(t)$ considered in Example 1.4.

1.2.3 Even and Odd Signals

Another set of useful properties of signals relates to their symmetry under time reversal. A signal $x(t)$ or $x[n]$ is referred to as an *even* signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin. In continuous time a signal is even if

$$x(-t) = x(t), \tag{1.14}$$

while a discrete-time signal is even if

$$x[-n] = x[n]. \tag{1.15}$$

A signal is referred to as *odd* if

$$x(-t) = -x(t), \tag{1.16}$$

$$x[-n] = -x[n]. \tag{1.17}$$

An odd signal must necessarily be 0 at $t = 0$ or $n = 0$, since eqs. (1.16) and (1.17) require that $x(0) = -x(0)$ and $x[0] = -x[0]$. Examples of even and odd continuous-time signals are shown in Figure 1.17.

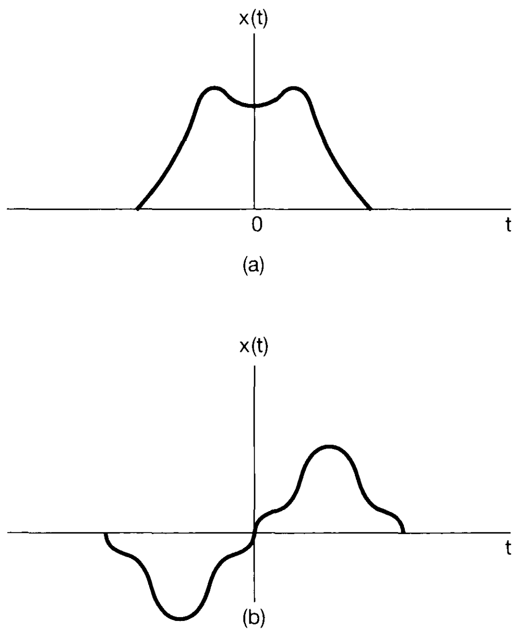


Figure 1.17 (a) An even continuous-time signal; (b) an odd continuous-time signal.

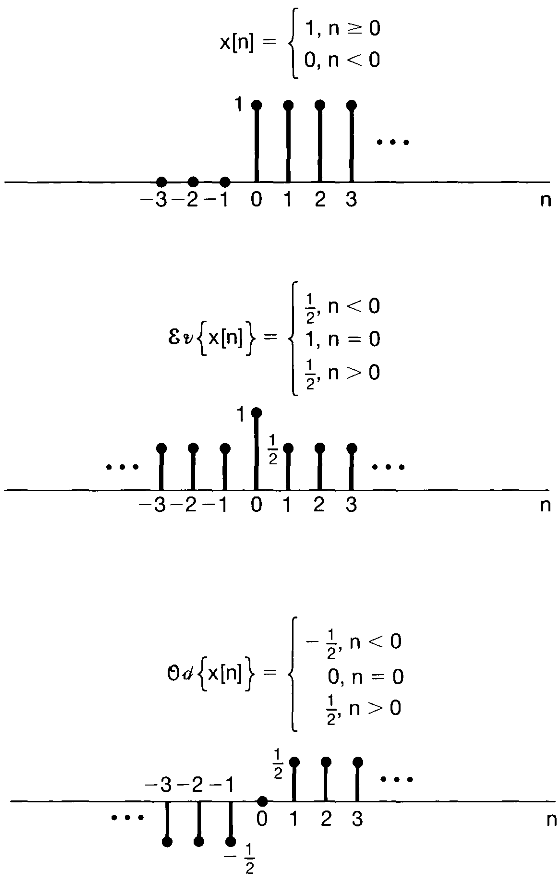


Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

An important fact is that any signal can be broken into a sum of two signals, one of which is even and one of which is odd. To see this, consider the signal

$$\mathcal{E}v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)], \tag{1.18}$$

which is referred to as the *even part* of $x(t)$. Similarly, the *odd part* of $x(t)$ is given by

$$\mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]. \tag{1.19}$$

It is a simple exercise to check that the even part is in fact even, that the odd part is odd, and that $x(t)$ is the sum of the two. Exactly analogous definitions hold in the discrete-time case. An example of the even-odd decomposition of a discrete-time signal is given in Figure 1.18.

1.3 EXPONENTIAL AND SINUSOIDAL SIGNALS

In this section and the next, we introduce several basic continuous-time and discrete-time signals. Not only do these signals occur frequently, but they also serve as basic building blocks from which we can construct many other signals.

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}, \tag{1.20}$$

where C and a are, in general, complex numbers. Depending upon the values of these parameters, the complex exponential can exhibit several different characteristics.

Real Exponential Signals

As illustrated in Figure 1.19, if C and a are real [in which case $x(t)$ is called a *real exponential*], there are basically two types of behavior. If a is positive, then as t increases $x(t)$ is a growing exponential, a form that is used in describing many different physical processes, including chain reactions in atomic explosions and complex chemical reactions. If a is negative, then $x(t)$ is a decaying exponential, a signal that is also used to describe a wide variety of phenomena, including the process of radioactive decay and the responses of RC circuits and damped mechanical systems. In particular, as shown in Problems 2.61 and 2.62, the natural responses of the circuit in Figure 1.1 and the automobile in Figure 1.2 are decaying exponentials. Also, we note that for $a = 0$, $x(t)$ is constant.

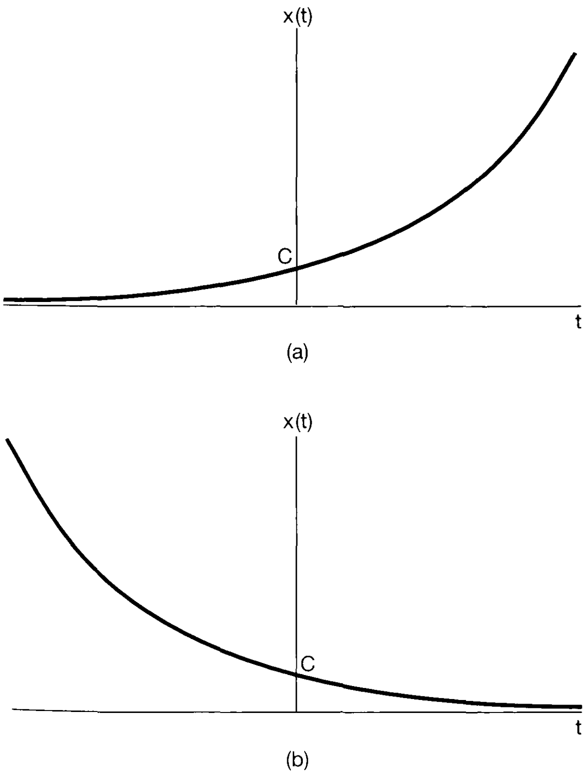


Figure 1.19 Continuous-time real exponential $x(t) = Ce^{at}$: (a) $a > 0$; (b) $a < 0$.

Periodic Complex Exponential and Sinusoidal Signals

A second important class of complex exponentials is obtained by constraining a to be purely imaginary. Specifically, consider

$$x(t) = e^{j\omega_0 t}. \tag{1.21}$$

An important property of this signal is that it is periodic. To verify this, we recall from eq. (1.11) that $x(t)$ will be periodic with period T if

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}. \tag{1.22}$$

Or, since

$$e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T},$$

it follows that for periodicity, we must have

$$e^{j\omega_0 T} = 1. \tag{1.23}$$

If $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T. If $\omega_0 \neq 0$, then the fundamental period T_0 of $x(t)$—that is, the smallest positive value of T for which eq. (1.23) holds—is

$$T_0 = \frac{2\pi}{|\omega_0|}. \tag{1.24}$$

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

A signal closely related to the periodic complex exponential is the *sinusoidal signal*

$$x(t) = A\cos(\omega_0 t + \phi), \tag{1.25}$$

as illustrated in Figure 1.20. With seconds as the units of t, the units of ϕ and ω_0 are radians and radians per second, respectively. It is also common to write $\omega_0 = 2\pi f_0$, where f_0 has the units of cycles per second, or hertz (Hz). Like the complex exponential signal, the sinusoidal signal is periodic with fundamental period T_0 given by eq. (1.24). Sinusoidal and

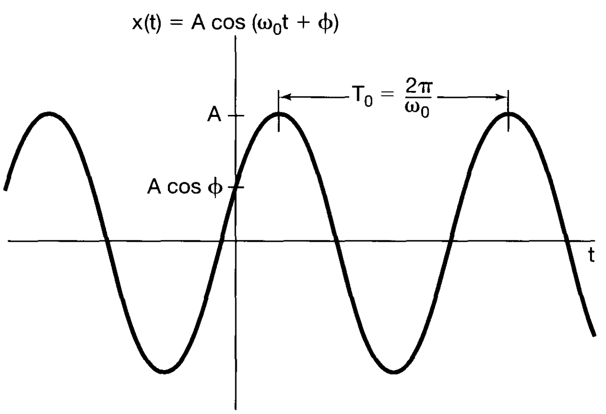


Figure 1.20 Continuous-time sinusoidal signal.

complex exponential signals are also used to describe the characteristics of many physical processes—in particular, physical systems in which energy is conserved. For example, as shown in Problem 2.61, the natural response of an *LC* circuit is sinusoidal, as is the simple harmonic motion of a mechanical system consisting of a mass connected by a spring to a stationary support. The acoustic pressure variations corresponding to a single musical tone are also sinusoidal.

By using Euler's relation,[2] the complex exponential in eq. (1.21) can be written in terms of sinusoidal signals with the same fundamental period:

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t. \tag{1.26}$$

Similarly, the sinusoidal signal of eq. (1.25) can be written in terms of periodic complex exponentials, again with the same fundamental period:

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}. \tag{1.27}$$

Note that the two exponentials in eq. (1.27) have complex amplitudes. Alternatively, we can express a sinusoid in terms of a complex exponential signal as

$$A\cos(\omega_0 t + \phi) = A\,\mathcal{R}e\{e^{j(\omega_0 t+\phi)}\}, \tag{1.28}$$

where, if c is a complex number, $\mathcal{R}e\{c\}$ denotes its real part. We will also use the notation $\mathcal{I}m\{c\}$ for the imaginary part of c, so that, for example,

$$A\sin(\omega_0 t + \phi) = A\,\mathcal{I}m\{e^{j(\omega_0 t+\phi)}\}. \tag{1.29}$$

From eq. (1.24), we see that the fundamental period T_0 of a continuous-time sinusoidal signal or a periodic complex exponential is inversely proportional to $|\omega_0|$, which we will refer to as the *fundamental frequency*. From Figure 1.21, we see graphically what this means. If we decrease the magnitude of ω_0, we slow down the rate of oscillation and therefore increase the period. Exactly the opposite effects occur if we increase the magnitude of ω_0. Consider now the case $\omega_0 = 0$. In this case, as we mentioned earlier, $x(t)$ is constant and therefore is periodic with period T for any positive value of T. Thus, the fundamental period of a constant signal is undefined. On the other hand, there is no ambiguity in defining the fundamental frequency of a constant signal to be zero. That is, a constant signal has a zero rate of oscillation.

Periodic signals—and in particular, the complex periodic exponential signal in eq. (1.21) and the sinusoidal signal in eq. (1.25)—provide important examples of signals with infinite total energy but finite average power. For example, consider the periodic exponential signal of eq. (1.21), and suppose that we calculate the total energy and average power in this signal over one period:

$$\begin{aligned} E_{\text{period}} &= \int_0^{T_0} |e^{j\omega_0 t}|^2\,dt \\ &= \int_0^{T_0} 1\cdot dt = T_0, \end{aligned} \tag{1.30}$$

[2]Euler's relation and other basic ideas related to the manipulation of complex numbers and exponentials are considered in the mathematical review section of the problems at the end of the chapter.

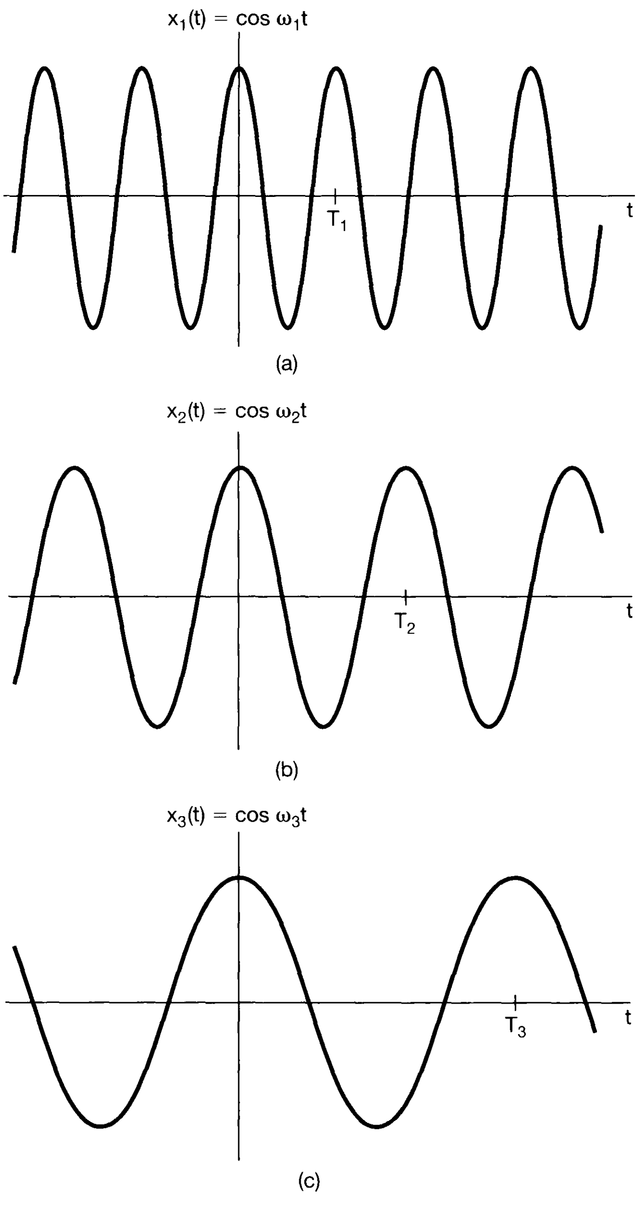


Figure 1.21 Relationship between the fundamental frequency and period for continuous-time sinusoidal signals; here, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1. \tag{1.31}$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. However, each period of the signal looks exactly the same. Since the average power of the signal equals 1 over each period, averaging over multiple periods always yields an average power of 1. That is, the complex periodic exponential signal has finite average power equal to

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_0 t}|^2 dt = 1. \tag{1.32}$$

Problem 1.3 provides additional examples of energy and power calculations for periodic and aperiodic signals.

Periodic complex exponentials will play a central role in much of our treatment of signals and systems, in part because they serve as extremely useful building blocks for many other signals. We will often find it useful to consider sets of *harmonically related* complex exponentials—that is, sets of periodic exponentials, all of which are periodic with a common period T_0. Specifically, a necessary condition for a complex exponential $e^{j\omega t}$ to be periodic with period T_0 is that

$$e^{j\omega T_0} = 1, \tag{1.33}$$

which implies that ωT_0 is a multiple of 2π, i.e.,

$$\omega T_0 = 2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.34}$$

Thus, if we define

$$\omega_0 = \frac{2\pi}{T_0}, \tag{1.35}$$

we see that, to satisfy eq. (1.34), ω must be an integer multiple of ω_0. That is, a harmonically related set of complex exponentials is a set of periodic exponentials with fundamental frequencies that are all multiples of a single positive frequency ω_0:

$$\phi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.36}$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k, $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}. \tag{1.37}$$

The kth harmonic $\phi_k(t)$ is still periodic with period T_0 as well, as it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0.

Our use of the term "harmonic" is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency. For example, the pattern of vibrations of a string on an instrument such as a violin can be described as a superposition—i.e., a weighted sum—of harmonically related periodic exponentials. In Chapter 3, we will see that we can build a very rich class of periodic signals using the harmonically related signals of eq. (1.36) as the building blocks.

Example 1.5

It is sometimes desirable to express the sum of two complex exponentials as the product of a single complex exponential and a single sinusoid. For example, suppose we wish to plot the magnitude of the signal

$$x(t) = e^{j2t} + e^{j3t}. \tag{1.38}$$

To do this, we first factor out a complex exponential from the right side of eq. (1.38), where the frequency of this exponential factor is taken as the average of the frequencies of the two exponentials in the sum. Doing this, we obtain

$$x(t) = e^{j2.5t}(e^{-j0.5t} + e^{j0.5t}), \tag{1.39}$$

which, because of Euler's relation, can be rewritten as

$$x(t) = 2e^{j2.5t}\cos(0.5t). \tag{1.40}$$

From this, we can directly obtain an expression for the magnitude of $x(t)$:

$$|x(t)| = 2|\cos(0.5t)|. \tag{1.41}$$

Here, we have used the fact that the magnitude of the complex exponential $e^{j2.5t}$ is always unity. Thus, $|x(t)|$ is what is commonly referred to as a full-wave rectified sinusoid, as shown in Figure 1.22.

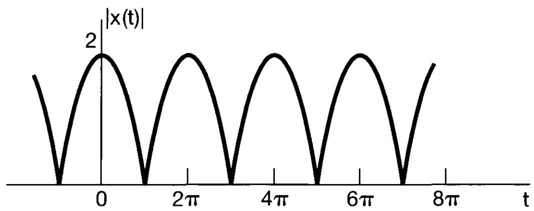


Figure 1.22 The full-wave rectified sinusoid of Example 1.5.

General Complex Exponential Signals

The most general case of a complex exponential can be expressed and interpreted in terms of the two cases we have examined so far: the real exponential and the periodic complex exponential. Specifically, consider a complex exponential Ce^{at}, where C is expressed in polar form and a in rectangular form. That is,

$$C = |C|e^{j\theta}$$

and

$$a = r + j\omega_0.$$

Then

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}. \tag{1.42}$$

Using Euler's relation, we can expand this further as

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta). \tag{1.43}$$

Thus, for $r = 0$, the real and imaginary parts of a complex exponential are sinusoidal. For $r > 0$ they correspond to sinusoidal signals multiplied by a growing exponential, and for $r < 0$ they correspond to sinusoidal signals multiplied by a decaying exponential. These two cases are shown in Figure 1.23. The dashed lines in the figure correspond to the functions $\pm|C|e^{rt}$. From eq. (1.42), we see that $|C|e^{rt}$ is the magnitude of the complex exponential. Thus, the dashed curves act as an envelope for the oscillatory curve in the figure in that the peaks of the oscillations just reach these curves, and in this way the envelope provides us with a convenient way to visualize the general trend in the amplitude of the oscillations.

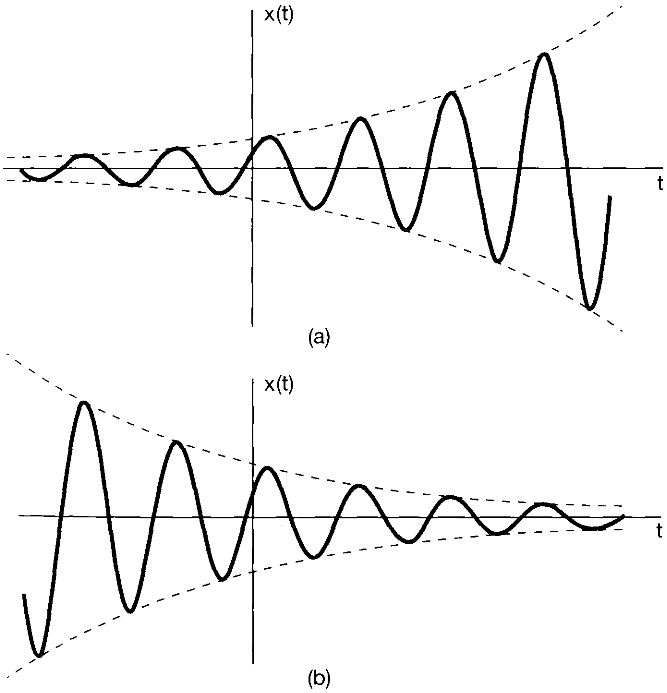


Figure 1.23 (a) Growing sinusoidal signal $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r > 0$; (b) decaying sinusoid $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r < 0$.

Sinusoidal signals multiplied by decaying exponentials are commonly referred to as *damped sinusoids.* Examples of damped sinusoids arise in the response of *RLC* circuits and in mechanical systems containing both damping and restoring forces, such as automotive suspension systems. These kinds of systems have mechanisms that dissipate energy (resistors, damping forces such as friction) with oscillations that decay in time. Examples illustrating such systems and their damped sinusoidal natural responses can be found in Problems 2.61 and 2.62.

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals

As in continuous time, an important signal in discrete time is the *complex exponential signal* or *sequence,* defined by

$$x[n] = C\alpha^n, \tag{1.44}$$

where C and α are, in general, complex numbers. This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n}, \tag{1.45}$$

where

$$\alpha = e^{\beta}.$$

Although the form of the discrete-time complex exponential sequence given in eq. (1.45) is more analogous to the form of the continuous-time exponential, it is often more convenient to express the discrete-time complex exponential sequence in the form of eq. (1.44).

Real Exponential Signals

If C and α are real, we can have one of several types of behavior, as illustrated in Figure 1.24. If $|\alpha| > 1$ the magnitude of the signal grows exponentially with n, while if $|\alpha| < 1$ we have a decaying exponential. Furthermore, if α is positive, all the values of $C\alpha^n$ are of the same sign, but if α is negative then the sign of $x[n]$ alternates. Note also that if $\alpha = 1$ then $x[n]$ is a constant, whereas if $\alpha = -1$, $x[n]$ alternates in value between $+C$ and $-C$. Real-valued discrete-time exponentials are often used to describe population growth as a function of generation and total return on investment as a function of day, month, or quarter.

Sinusoidal Signals

Another important complex exponential is obtained by using the form given in eq. (1.45) and by constraining β to be purely imaginary (so that $|\alpha| = 1$). Specifically, consider

$$x[n] = e^{j\omega_0 n}. \tag{1.46}$$

As in the continuous-time case, this signal is closely related to the sinusoidal signal

$$x[n] = A\cos(\omega_0 n + \phi). \tag{1.47}$$

If we take n to be dimensionless, then both ω_0 and ϕ have units of radians. Three examples of sinusoidal sequences are shown in Figure 1.25.

As before, Euler's relation allows us to relate complex exponentials and sinusoids:

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n \tag{1.48}$$

and

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{1.49}$$

The signals in eqs. (1.46) and (1.47) are examples of discrete-time signals with infinite total energy but finite average power. For example, since $|e^{j\omega_0 n}|^2 = 1$, every sample of the signal in eq. (1.46) contributes 1 to the signal's energy. Thus, the total energy for $-\infty < n < \infty$ is infinite, while the average power per time point is obviously equal to 1. Other examples of energy and power calculations for discrete-time signals are given in Problem 1.3.

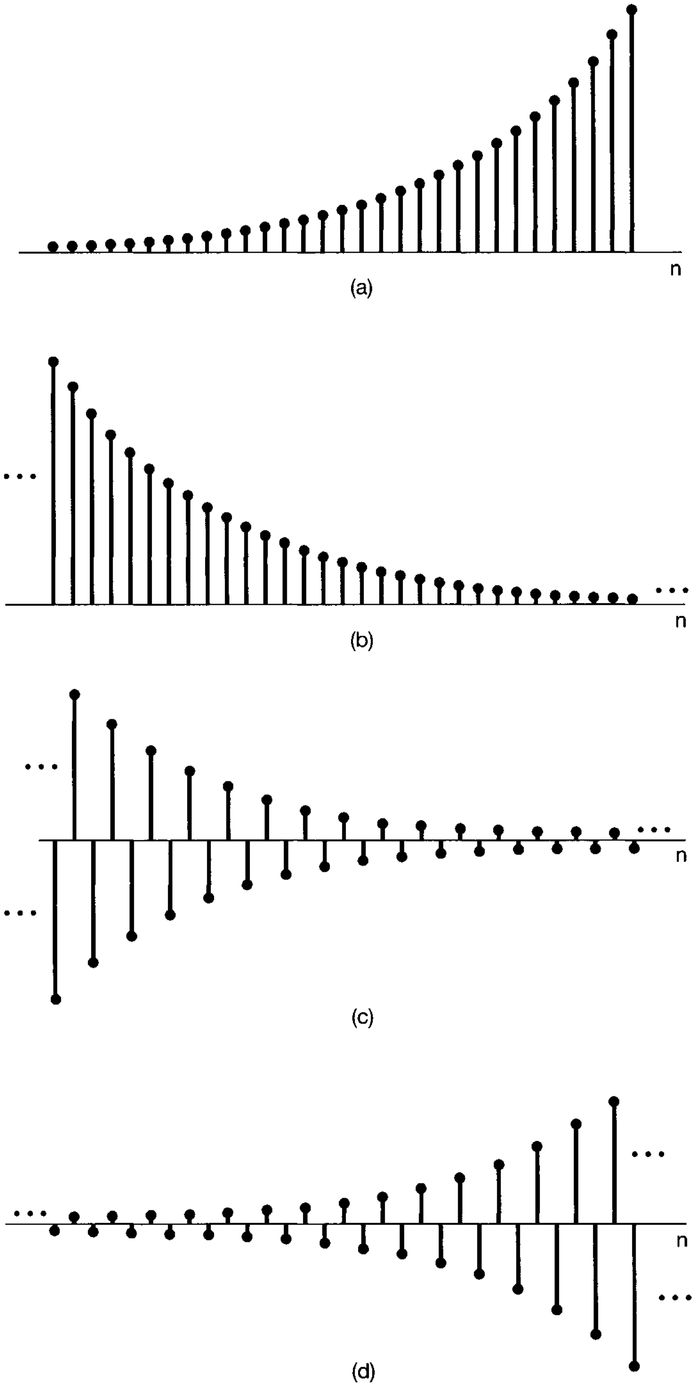


Figure 1.24 The real exponential signal $x[n] = C\alpha^n$: (a) $\alpha > 1$; (b) $0 < \alpha < 1$; (c) $-1 < \alpha < 0$; (d) $\alpha < -1$.

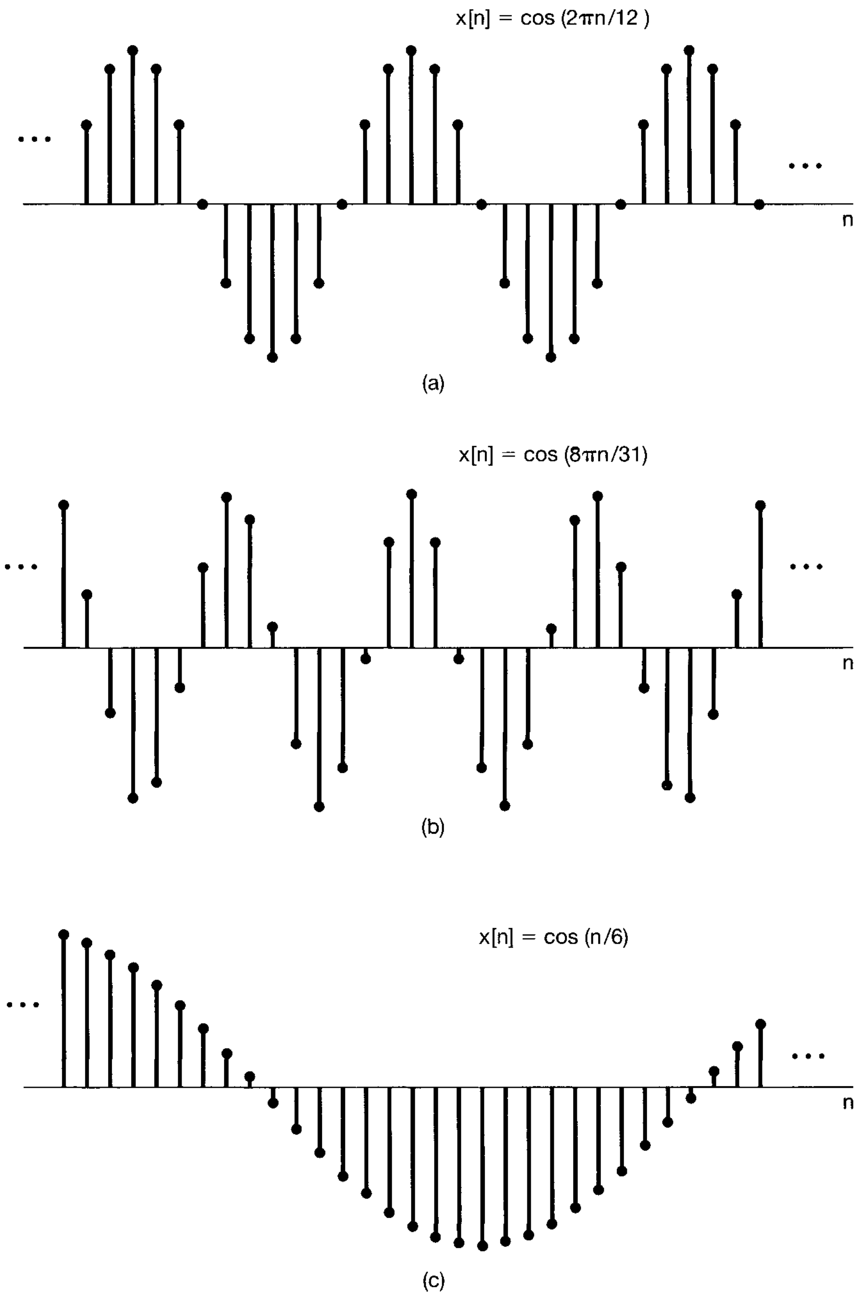


Figure 1.25 Discrete-time sinusoidal signals.

General Complex Exponential Signals

The general discrete-time complex exponential can be written and interpreted in terms of real exponentials and sinusoidal signals. Specifically, if we write C and α in polar form, viz.,

$$C = |C|e^{j\theta}$$

and

$$\alpha = |\alpha|e^{j\omega_0},$$

then

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta). \tag{1.50}$$

Thus, for $|\alpha| = 1$, the real and imaginary parts of a complex exponential sequence are sinusoidal. For $|\alpha| < 1$ they correspond to sinusoidal sequences multiplied by a decaying exponential, while for $|\alpha| > 1$ they correspond to sinusoidal sequences multiplied by a growing exponential. Examples of these signals are depicted in Figure 1.26.

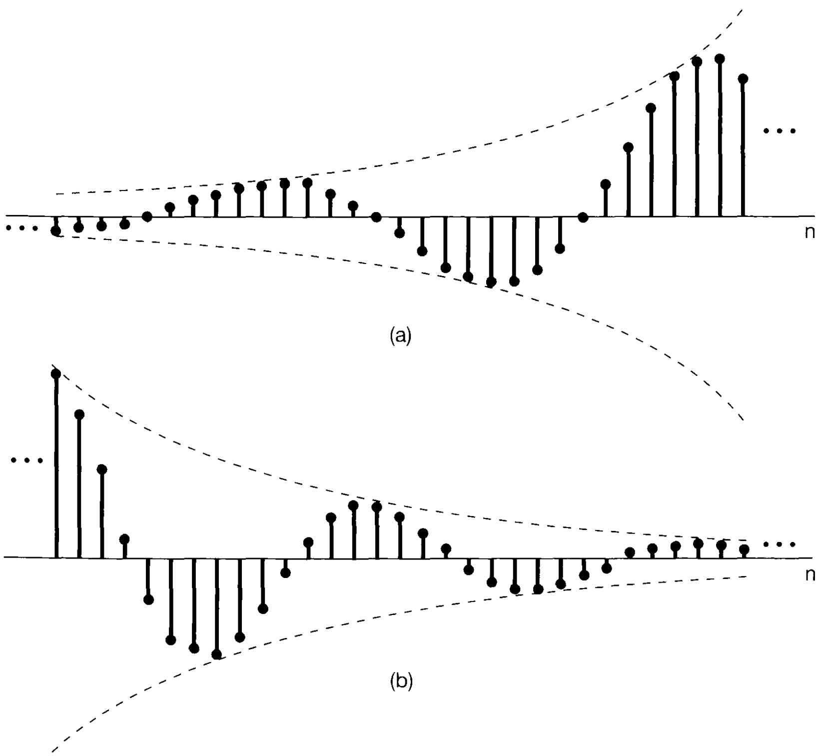


Figure 1.26 (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

1.3.3 Periodicity Properties of Discrete-Time Complex Exponentials

While there are many similarities between continuous-time and discrete-time signals, there are also a number of important differences. One of these concerns the discrete-time exponential signal $e^{j\omega_0 n}$. In Section 1.3.1, we identified the following two properties of its continuous-time counterpart $e^{j\omega_0 t}$: (1) the larger the magnitude of ω_0, the higher is the rate of oscillation in the signal; and (2) $e^{j\omega_0 t}$ is periodic for any value of ω_0. In this section we describe the discrete-time versions of both of these properties, and as we will see, there are definite differences between each of these and its continuous-time counterpart.

The fact that the first of these properties is different in discrete time is a direct consequence of another extremely important distinction between discrete-time and continuous-time complex exponentials. Specifically, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}. \tag{1.51}$$

From eq. (1.51), we see that the exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0. Thus, we have a very different situation from the continuous-time case, in which the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0. In discrete time, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. Therefore, in considering discrete-time complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0. Although, according to eq. (1.51), any interval of length 2π will do, on most occasions we will use the interval $0 \le \omega_0 < 2\pi$ or the interval $-\pi \le \omega_0 < \pi$.

Because of the periodicity implied by eq. (1.51), the signal $e^{j\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 is increased in magnitude. Rather, as illustrated in Figure 1.27, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we *decrease* the rate of oscillation until we reach $\omega_0 = 2\pi$, which produces the same constant sequence as $\omega_0 = 0$. Therefore, the low-frequency (that is, slowly varying) discrete-time exponentials have values of ω_0 near 0, 2π, and any other even multiple of π, while the high frequencies (corresponding to rapid variations) are located near $\omega_0 = \pm\pi$ and other odd multiples of π. Note in particular that for $\omega_0 = \pi$ or any other odd multiple of π,

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n, \tag{1.52}$$

so that this signal oscillates rapidly, changing sign at each point in time [as illustrated in Figure 1.27(e)].

The second property we wish to consider concerns the periodicity of the discrete-time complex exponential. In order for the signal $e^{j\omega_0 n}$ to be periodic with period $N > 0$, we must have

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}, \tag{1.53}$$

or equivalently,

$$e^{j\omega_0 N} = 1. \tag{1.54}$$

For eq. (1.54) to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer m such that

$$\omega_0 N = 2\pi m, \tag{1.55}$$

or equivalently,

$$\frac{\omega_0}{2\pi} = \frac{m}{N}. \tag{1.56}$$

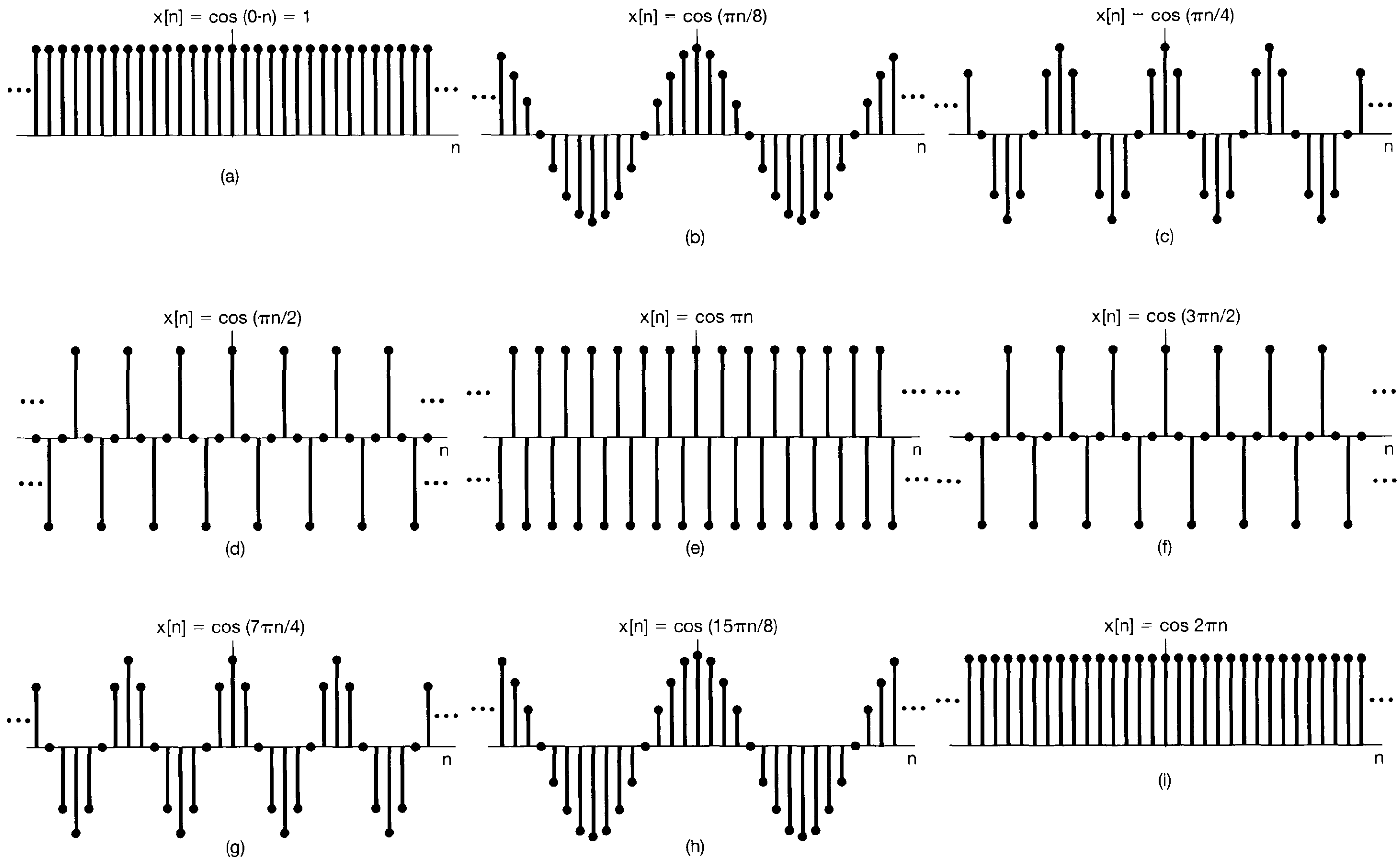


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

According to eq. (1.56), the signal $e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise. These same observations also hold for discrete-time sinusoids. For example, the signals depicted in Figure 1.25(a) and (b) are periodic, while the signal in Figure 1.25(c) is not.

Using the calculations that we have just made, we can also determine the fundamental period and frequency of discrete-time complex exponentials, where we define the fundamental frequency of a discrete-time periodic signal as we did in continuous time. That is, if $x[n]$ is periodic with fundamental period N, its fundamental frequency is $2\pi/N$. Consider, then, a periodic complex exponential $x[n] = e^{j\omega_0 n}$ with $\omega_0 \neq 0$. As we have just seen, ω_0 must satisfy eq. (1.56) for some pair of integers m and N, with $N > 0$. In Problem 1.35, it is shown that if $\omega_0 \neq 0$ and if N and m have no factors in common, then the fundamental period of $x[n]$ is N. Using this fact together with eq. (1.56), we find that the fundamental frequency of the periodic signal $e^{j\omega_0 n}$ is

$$\frac{2\pi}{N} = \frac{\omega_0}{m}. \tag{1.57}$$

Note that the fundamental period can also be written as

$$N = m\left(\frac{2\pi}{\omega_0}\right). \tag{1.58}$$

These last two expressions again differ from their continuous-time counterparts. In Table 1.1, we have summarized some of the differences between the continuous-time signal $e^{j\omega_0 t}$ and the discrete-time signal $e^{j\omega_0 n}$. Note that, as in the continuous-time case, the constant discrete-time signal resulting from setting $\omega_0 = 0$ has a fundamental frequency of zero, and its fundamental period is undefined.

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

To gain some additional insight into these properties, let us examine again the signals depicted in Figure 1.25. First, consider the sequence $x[n] = \cos(2\pi n/12)$, depicted in Figure 1.25(a), which we can think of as the set of samples of the continuous-time sinusoid $x(t) = \cos(2\pi t/12)$ at integer time points. In this case, $x(t)$ is periodic with fundamental period 12 and $x[n]$ is also periodic with fundamental period 12. That is, the values of $x[n]$ repeat every 12 points, exactly in step with the fundamental period of $x(t)$.

In contrast, consider the signal $x[n] = \cos(8\pi n/31)$, depicted in Figure 1.25(b), which we can view as the set of samples of $x(t) = \cos(8\pi t/31)$ at integer points in time. In this case, $x(t)$ is periodic with fundamental period 31/4. On the other hand, $x[n]$ is periodic with fundamental period 31. The reason for this difference is that the discrete-time signal is defined only for integer values of the independent variable. Thus, there is no sample at time $t = 31/4$, when $x(t)$ completes one period (starting from $t = 0$). Similarly, there is no sample at $t = 2 \cdot 31/4$ or $t = 3 \cdot 31/4$, when $x(t)$ has completed two or three periods, but there is a sample at $t = 4 \cdot 31/4 = 31$, when $x(t)$ has completed *four* periods. This can be seen in Figure 1.25(b), where the pattern of $x[n]$ values does *not* repeat with each single cycle of positive and negative values. Rather, the pattern repeats after four such cycles, namely, every 31 points.

Similarly, the signal $x[n] = \cos(n/6)$ can be viewed as the set of samples of the signal $x(t) = \cos(t/6)$ at integer time points. In this case, the values of $x(t)$ at integer sample points never repeat, as these sample points never span an interval that is an exact multiple of the period, 12π, of $x(t)$. Thus, $x[n]$ is not periodic, although the eye visually interpolates between the sample points, suggesting the envelope $x(t)$, which *is* periodic. The use of the concept of sampling to gain insight into the periodicity of discrete-time sinusoidal sequences is explored further in Problem 1.36.

Example 1.6

Suppose that we wish to determine the fundamental period of the discrete-time signal

$$x[n] = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}. \tag{1.59}$$

The first exponential on the right-hand side of eq. (1.59) has a fundamental period of 3. While this can be verified from eq. (1.58), there is a simpler way to obtain that answer. In particular, note that the angle $(2\pi/3)n$ of the first term must be incremented by a multiple of 2π for the values of this exponential to begin repeating. We then immediately see that if n is incremented by 3, the angle will be incremented by a single multiple of 2π. With regard to the second term, we see that incrementing the angle $(3\pi/4)n$ by 2π would require n to be incremented by 8/3, which is impossible, since n is restricted to being an integer. Similarly, incrementing the angle by 4π would require a noninteger increment of 16/3 to n. However, incrementing the angle by 6π requires an increment of 8 to n, and thus the fundamental period of the second term is 8.

Now, for the entire signal $x[n]$ to repeat, each of the terms in eq. (1.59) must go through an integer number of its own fundamental period. The smallest increment of n that accomplishes this is 24. That is, over an interval of 24 points, the first term on the right-hand side of eq. (1.59) will have gone through eight of its fundamental periods, the second term through three of its fundamental periods, and the overall signal $x[n]$ through exactly one of its fundamental periods.

As in continuous time, it is also of considerable value in discrete-time signal and system analysis to consider sets of harmonically related periodic exponentials—that is, periodic exponentials with a common period N. From eq. (1.56), we know that these are precisely the signals which are at frequencies which are multiples of $2\pi/N$. That is,

$$\phi_k[n] = e^{jk(2\pi/N)n}, \qquad k = 0, \pm 1, \ldots. \tag{1.60}$$

In the continuous-time case, all of the harmonically related complex exponentials $e^{jk(2\pi/T)t}$, $k = 0, \pm 1, \pm 2, \ldots$, are distinct. However, because of eq. (1.51), this is *not* the case in discrete time. Specifically,

$$\begin{aligned}\phi_{k+N}[n] &= e^{j(k+N)(2\pi/N)n} \\ &= e^{jk(2\pi/N)n}e^{j2\pi n} = \phi_k[n].\end{aligned} \tag{1.61}$$

This implies that there are only N distinct periodic exponentials in the set given in eq. (1.60). For example,

$$\phi_0[n] = 1, \phi_1[n] = e^{j2\pi n/N}, \phi_2[n] = e^{j4\pi n/N}, \ldots, \phi_{N-1}[n] = e^{j2\pi(N-1)n/N} \tag{1.62}$$

are all distinct, and any other $\phi_k[n]$ is identical to one of these (e.g., $\phi_N[n] = \phi_0[n]$ and $\phi_{-1}[n] = \phi_{N-1}[n]$).

1.4 THE UNIT IMPULSE AND UNIT STEP FUNCTIONS

In this section, we introduce several other basic signals—specifically, the unit impulse and step functions in continuous and discrete time—that are also of considerable importance in signal and system analysis. In Chapter 2, we will see how we can use unit impulse signals as basic building blocks for the construction and representation of other signals. We begin with the discrete-time case.

1.4.1 The Discrete-Time Unit Impulse and Unit Step Sequences

One of the simplest discrete-time signals is the *unit impulse* (or *unit sample*), which is defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{1.63}$$

and which is shown in Figure 1.28. Throughout the book, we will refer to $\delta[n]$ interchangeably as the unit impulse or unit sample.

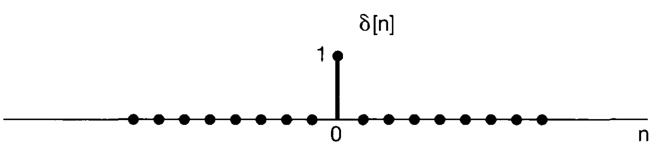


Figure 1.28 Discrete-time unit impulse (sample).

A second basic discrete-time signal is the discrete-time *unit step,* denoted by $u[n]$ and defined by

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}. \tag{1.64}$$

The unit step sequence is shown in Figure 1.29.

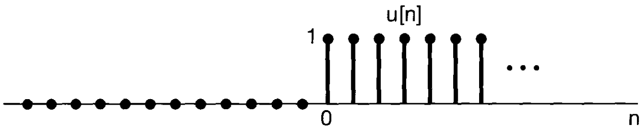


Figure 1.29 Discrete-time unit step sequence.

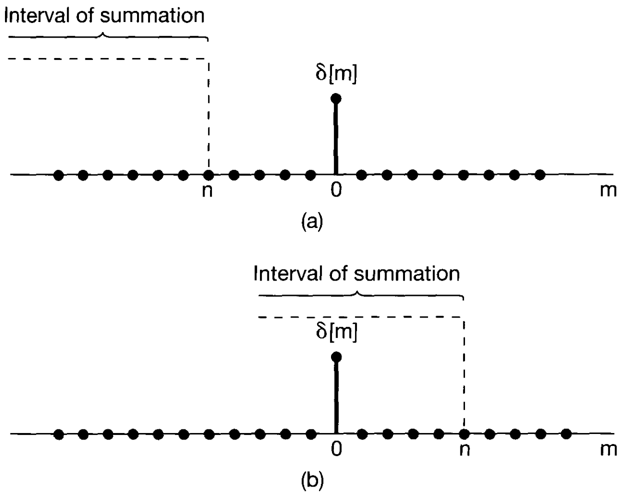


Figure 1.30 Running sum of eq. (1.66): (a) $n < 0$; (b) $n > 0$.

There is a close relationship between the discrete-time unit impulse and unit step. In particular, the discrete-time unit impulse is the *first difference* of the discrete-time step

$$\delta[n] = u[n] - u[n-1]. \tag{1.65}$$

Conversely, the discrete-time unit step is the *running sum* of the unit sample. That is,

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]. \tag{1.66}$$

Equation (1.66) is illustrated graphically in Figure 1.30. Since the only nonzero value of the unit sample is at the point at which its argument is zero, we see from the figure that the running sum in eq. (1.66) is 0 for $n < 0$ and 1 for $n \geq 0$. Furthermore, by changing the variable of summation from m to $k = n - m$ in eq. (1.66), we find that the discrete-time unit step can also be written in terms of the unit sample as

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k],$$

or equivalently,

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{1.67}$$

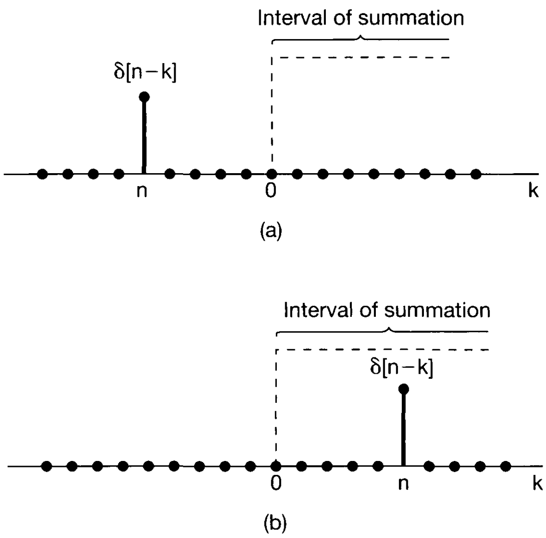


Figure 1.31 Relationship given in eq. (1.67): (a) $n < 0$; (b) $n > 0$.

Equation (1.67) is illustrated in Figure 1.31. In this case the nonzero value of $\delta[n-k]$ is at the value of k equal to n, so that again we see that the summation in eq. (1.67) is 0 for $n < 0$ and 1 for $n \geq 0$.

An interpretation of eq. (1.67) is as a superposition of delayed impulses; i.e., we can view the equation as the sum of a unit impulse $\delta[n]$ at $n = 0$, a unit impulse $\delta[n-1]$ at $n = 1$, another, $\delta[n-2]$, at $n = 2$, etc. We will make explicit use of this interpretation in Chapter 2.

The unit impulse sequence can be used to sample the value of a signal at $n = 0$. In particular, since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n]. \tag{1.68}$$

More generally, if we consider a unit impulse $\delta[n-n_0]$ at $n = n_0$, then

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]. \tag{1.69}$$

This sampling property of the unit impulse will play an important role in Chapters 2 and 7.

1.4.2 The Continuous-Time Unit Step and Unit Impulse Functions

The continuous-time *unit step function* $u(t)$ is defined in a manner similar to its discrete-time counterpart. Specifically,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}, \tag{1.70}$$

as is shown in Figure 1.32. Note that the unit step is discontinuous at $t = 0$. The continuous-time *unit impulse function* $\delta(t)$ is related to the unit step in a manner analogous

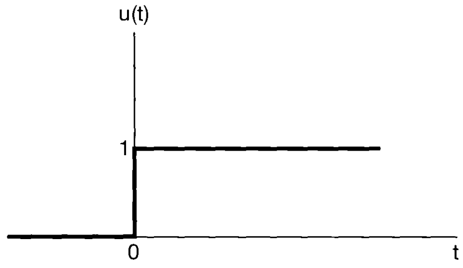


Figure 1.32 Continuous-time unit step function.

to the relationship between the discrete-time unit impulse and step functions. In particular, the continuous-time unit step is the *running integral* of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau. \tag{1.71}$$

This also suggests a relationship between $\delta(t)$ and $u(t)$ analogous to the expression for $\delta[n]$ in eq. (1.65). In particular, it follows from eq. (1.71) that the continuous-time unit impulse can be thought of as the *first derivative* of the continuous-time unit step:

$$\delta(t) = \frac{du(t)}{dt}. \tag{1.72}$$

In contrast to the discrete-time case, there is some formal difficulty with this equation as a representation of the unit impulse function, since $u(t)$ is discontinuous at $t = 0$ and consequently is formally not differentiable. We can, however, interpret eq. (1.72) by considering an approximation to the unit step $u_\Delta(t)$, as illustrated in Figure 1.33, which rises from the value 0 to the value 1 in a short time interval of length Δ. The unit step, of course, changes values instantaneously and thus can be thought of as an idealization of $u_\Delta(t)$ for Δ so short that its duration doesn't matter for any practical purpose. Formally, $u(t)$ is the limit of $u_\Delta(t)$ as $\Delta \to 0$. Let us now consider the derivative

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}, \tag{1.73}$$

as shown in Figure 1.34.

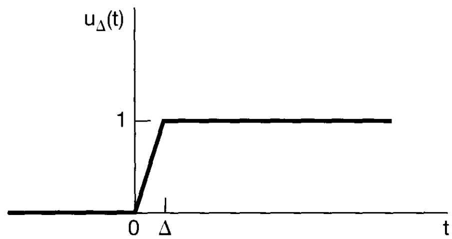


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

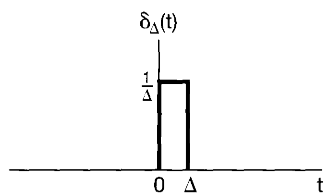


Figure 1.34 Derivative of $u_\Delta(t)$.

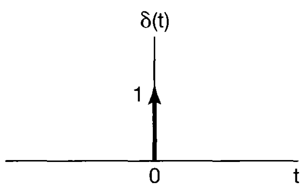


Figure 1.35 Continuous-time unit impulse.

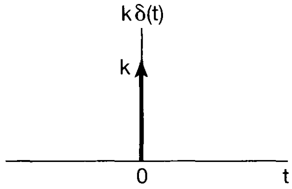


Figure 1.36 Scaled impulse.

Note that $\delta_\Delta(t)$ is a short pulse, of duration Δ and with unit area for any value of Δ. As $\Delta \rightarrow 0$, $\delta_\Delta(t)$ becomes narrower and higher, maintaining its unit area. Its limiting form,

$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_\Delta(t), \tag{1.74}$$

can then be thought of as an idealization of the short pulse $\delta_\Delta(t)$ as the duration Δ becomes insignificant. Since $\delta(t)$ has, in effect, no duration but unit area, we adopt the graphical notation for it shown in Figure 1.35, where the arrow at $t = 0$ indicates that the area of the pulse is concentrated at $t = 0$ and the height of the arrow and the "1" next to the arrow are used to represent the *area* of the impulse. More generally, a scaled impulse $k\delta(t)$ will have an area k, and thus,

$$\int_{-\infty}^{t} k\delta(\tau)\,d\tau = ku(t).$$

A scaled impulse with area k is shown in Figure 1.36, where the height of the arrow used to depict the scaled impulse is chosen to be proportional to the area of the impulse.

As with discrete time, we can provide a simple graphical interpretation of the running integral of eq. (1.71); this is shown in Figure 1.37. Since the area of the continuous-time unit impulse $\delta(\tau)$ is concentrated at $\tau = 0$, we see that the running integral is 0 for $t < 0$ and 1 for $t > 0$. Also, we note that the relationship in eq. (1.71) between the continuous-time unit step and impulse can be rewritten in a different form, analogous to the discrete-time form in eq. (1.67), by changing the variable of integration from τ to $\sigma = t - \tau$:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{\infty}^{0} \delta(t - \sigma)(-d\sigma),$$

or equivalently,

$$u(t) = \int_{0}^{\infty} \delta(t - \sigma)\,d\sigma. \tag{1.75}$$

The graphical interpretation of this form of the relationship between $u(t)$ and $\delta(t)$ is given in Figure 1.38. Since in this case the area of $\delta(t - \sigma)$ is concentrated at the point $\sigma = t$, we again see that the integral in eq. (1.75) is 0 for $t < 0$ and 1 for $t > 0$. This type of graphical interpretation of the behavior of the unit impulse under integration will be extremely useful in Chapter 2.

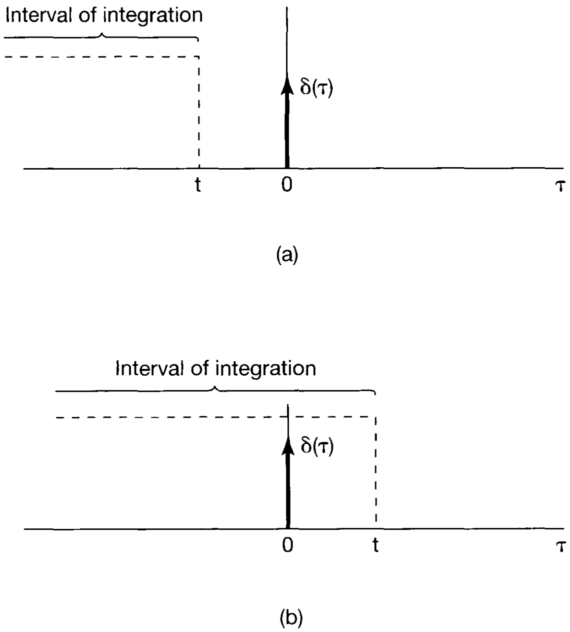


Figure 1.37 Running integral given in eq. (1.71): (a) $t < 0$; (b) $t > 0$.

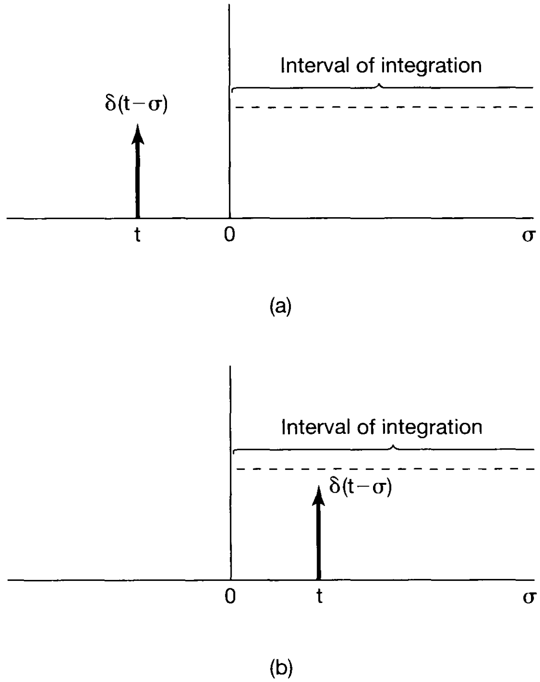


Figure 1.38 Relationship given in eq. (1.75): (a) $t < 0$; (b) $t > 0$.

As with the discrete-time impulse, the continuous-time impulse has a very important sampling property. In particular, for a number of reasons it will be important to consider the product of an impulse and more well-behaved continuous-time functions $x(t)$. The interpretation of this quantity is most readily developed using the definition of $\delta(t)$ according to eq. (1.74). Specifically, consider

$$x_1(t) = x(t)\delta_\Delta(t).$$

In Figure 1.39(a) we have depicted the two time functions $x(t)$ and $\delta_\Delta(t)$, and in Figure 1.39(b) we see an enlarged view of the nonzero portion of their product. By construction, $x_1(t)$ is zero outside the interval $0 \leq t \leq \Delta$. For Δ sufficiently small so that $x(t)$ is approximately constant over this interval,

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t).$$

Since $\delta(t)$ is the limit as $\Delta \rightarrow 0$ of $\delta_\Delta(t)$, it follows that

$$x(t)\delta(t) = x(0)\delta(t). \tag{1.76}$$

By the same argument, we have an analogous expression for an impulse concentrated at an arbitrary point, say, t_0. That is,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$

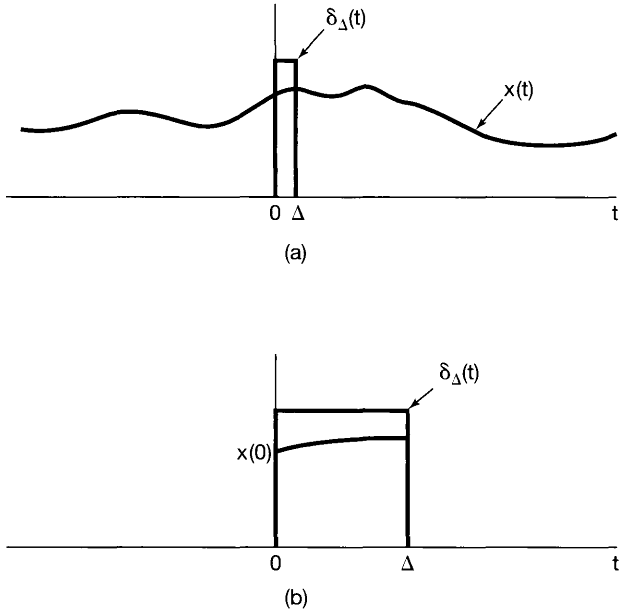


Figure 1.39 The product $x(t)\delta_\Delta(t)$: (a) graphs of both functions; (b) enlarged view of the nonzero portion of their product.

Although our discussion of the unit impulse in this section has been somewhat informal, it does provide us with some important intuition about this signal that will be useful throughout the book. As we have stated, the unit impulse should be viewed as an idealization. As we illustrate and discuss in more detail in Section 2.5, any real physical system has some inertia associated with it and thus does not respond instantaneously to inputs. Consequently, if a pulse of sufficiently short duration is applied to such a system, the system response will not be noticeably influenced by the pulse's duration or by the details of the shape of the pulse, for that matter. Instead, the primary characteristic of the pulse that will matter is the net, integrated effect of the pulse—i.e., its area. For systems that respond much more quickly than others, the pulse will have to be of much shorter duration before the details of the pulse shape or its duration no longer matter. Nevertheless, for any physical system, we can always find a pulse that is "short enough." The unit impulse then is an idealization of this concept—the pulse that is short enough for *any* system. As we will see in Chapter 2, the response of a system to this idealized pulse plays a crucial role in signal and system analysis, and in the process of developing and understanding this role, we will develop additional insight into the idealized signal.[3]

[3]The unit impulse and other related functions (which are often collectively referred to as *singularity functions*) have been thoroughly studied in the field of mathematics under the alternative names of *generalized functions* and the *theory of distributions.* For more detailed discussions of this subject, see *Distribution Theory and Transform Analysis,* by A. H. Zemanian (New York: McGraw-Hill Book Company, 1965), *Generalised Functions,* by R.F. Hoskins (New York: Halsted Press, 1979), or the more advanced text, *Fourier Analysis and Generalized Functions,* by M. J. Lighthill (New York: Cambridge University Press, 1958). Our discussion of singularity functions in Section 2.5 is closely related in spirit to the mathematical theory described in these texts and thus provides an informal introduction to concepts that underlie this topic in mathematics.

Example 1.7

Consider the discontinuous signal $x(t)$ depicted in Figure 1.40(a). Because of the relationship between the continuous-time unit impulse and unit step, we can readily calculate and graph the derivative of this signal. Specifically, the derivative of $x(t)$ is clearly 0, except at the discontinuities. In the case of the unit step, we have seen [eq. (1.72)] that differentiation gives rise to a unit impulse located at the point of discontinuity. Furthermore, by multiplying both sides of eq. (1.72) by any number k, we see that the derivative of a unit step with a discontinuity of size k gives rise to an impulse of area k at the point of discontinuity. This rule also holds for any other signal with a jump discontinuity, such as $x(t)$ in Figure 1.40(a). Consequently, we can sketch its derivative $\dot{x}(t)$, as in Figure 1.40(b), where an impulse is placed at each discontinuity of $x(t)$, with area equal to the size of the discontinuity. Note, for example, that the discontinuity in $x(t)$ at $t = 2$ has a value of -3, so that an impulse scaled by -3 is located at $t = 2$ in the signal $\dot{x}(t)$.

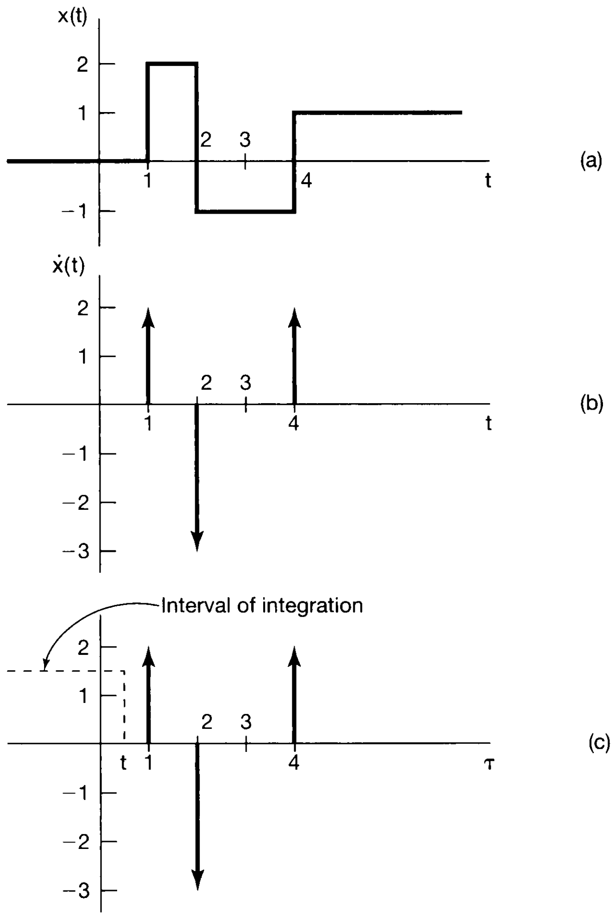


Figure 1.40 (a) The discontinuous signal $x(t)$ analyzed in Example 1.7; (b) its derivative $\dot{x}(t)$; (c) depiction of the recovery of $x(t)$ as the running integral of $\dot{x}(t)$, illustrated for a value of t between 0 and 1.

As a check of our result, we can verify that we can recover $x(t)$ from $\dot{x}(t)$. Specifically, since $x(t)$ and $\dot{x}(t)$ are both zero for $t \leq 0$, we need only check that for $t > 0$,

$$x(t) = \int_0^t \dot{x}(\tau)\, d\tau. \tag{1.77}$$

As illustrated in Figure 1.40(c), for $t < 1$, the integral on the right-hand side of eq. (1.77) is zero, since none of the impulses that constitute $\dot{x}(t)$ are within the interval of integration. For $1 < t < 2$, the first impulse (located at $t = 1$) is the only one within the integration interval, and thus the integral in eq. (1.77) equals 2, the area of this impulse. For $2 < t < 4$, the first two impulses are within the interval of integration, and the integral accumulates the sum of both of their areas, namely, $2 - 3 = -1$. Finally, for $t > 4$, all three impulses are within the integration interval, so that the integral equals the sum of all three areas—that is, $2 - 3 + 2 = +1$. The result is exactly the signal $x(t)$ depicted in Figure 1.40(a).

1.5 CONTINUOUS-TIME AND DISCRETE-TIME SYSTEMS

Physical systems in the broadest sense are an interconnection of components, devices, or subsystems. In contexts ranging from signal processing and communications to electromechanical motors, automotive vehicles, and chemical-processing plants, a *system* can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs. For example, a high-fidelity system takes a recorded audio signal and generates a reproduction of that signal. If the hi-fi system has tone controls, we can change the tonal quality of the reproduced signal. Similarly, the circuit in Figure 1.1 can be viewed as a system with input voltage $v_s(t)$ and output voltage $v_c(t)$, while the automobile in Figure 1.2 can be thought of as a system with input equal to the force $f(t)$ and output equal to the velocity $v(t)$ of the vehicle. An image-enhancement system transforms an input image into an output image that has some desired properties, such as improved contrast.

A *continuous-time system* is a system in which continuous-time input signals are applied and result in continuous-time output signals. Such a system will be represented pictorially as in Figure 1.41(a), where $x(t)$ is the input and $y(t)$ is the output. Alternatively, we will often represent the input-output relation of a continuous-time system by the notation

$$x(t) \rightarrow y(t). \tag{1.78}$$

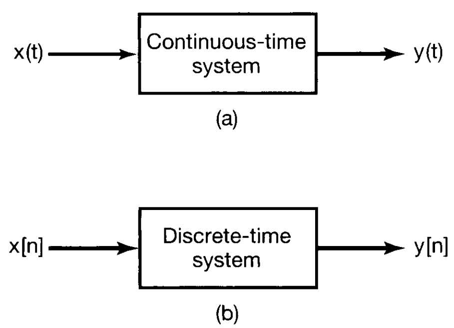


Figure 1.41 (a) Continuous-time system; (b) discrete-time system.

Similarly, a *discrete-time system*—that is, a system that transforms discrete-time inputs into discrete-time outputs—will be depicted as in Figure 1.41(b) and will sometimes be represented symbolically as

$$x[n] \rightarrow y[n]. \tag{1.79}$$

In most of this book, we will treat discrete-time systems and continuous-time systems separately but in parallel. In Chapter 7, we will bring continuous-time and discrete-time systems together through the concept of sampling, and we will develop some insights into the use of discrete-time systems to process continuous-time signals that have been sampled.

1.5.1 Simple Examples of Systems

One of the most important motivations for the development of general tools for analyzing and designing systems is that systems from many different applications have very similar mathematical descriptions. To illustrate this, we begin with a few simple examples.

Example 1.8

Consider the RC circuit depicted in Figure 1.1. If we regard $v_s(t)$ as the input signal and $v_c(t)$ as the output signal, then we can use simple circuit analysis to derive an equation describing the relationship between the input and output. Specifically, from Ohm's law, the current $i(t)$ through the resistor is proportional (with proportionality constant $1/R$) to the voltage drop across the resistor; i.e.,

$$i(t) = \frac{v_s(t) - v_c(t)}{R}. \tag{1.80}$$

Similarly, using the defining constitutive relation for a capacitor, we can relate $i(t)$ to the rate of change with time of the voltage across the capacitor:

$$i(t) = C\frac{dv_c(t)}{dt}. \tag{1.81}$$

Equating the right-hand sides of eqs. (1.80) and (1.81), we obtain a differential equation describing the relationship between the input $v_s(t)$ and the output $v_c(t)$:

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t). \tag{1.82}$$

Example 1.9

Consider Figure 1.2, in which we regard the force $f(t)$ as the input and the velocity $v(t)$ as the output. If we let m denote the mass of the automobile and $m\rho v$ the resistance due to friction, then equating acceleration—i.e., the time derivative of velocity—with net force divided by mass, we obtain

$$\frac{dv(t)}{dt} = \frac{1}{m}\left[f(t) - \rho v(t)\right], \tag{1.83}$$

i.e.,

$$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t). \tag{1.84}$$

Examining and comparing eqs. (1.82) and (1.84) in the above examples, we see that the input-output relationships captured in these two equations for these two very different physical systems are basically the same. In particular, they are both examples of first-order linear differential equations of the form

$$\frac{dy(t)}{dt} + ay(t) = bx(t), \tag{1.85}$$

where $x(t)$ is the input, $y(t)$ is the output, and a and b are constants. This is one very simple example of the fact that, by developing methods for analyzing general classes of systems such as that represented by eq. (1.85), we will be able to use them in a wide variety of applications.

Example 1.10

As a simple example of a discrete-time system, consider a simple model for the balance in a bank account from month to month. Specifically, let $y[n]$ denote the balance at the end of the nth month, and suppose that $y[n]$ evolves from month to month according to the equation

$$y[n] = 1.01y[n-1] + x[n], \tag{1.86}$$

or equivalently,

$$y[n] - 1.01y[n-1] = x[n], \tag{1.87}$$

where $x[n]$ represents the net deposit (i.e., deposits minus withdrawals) during the nth month and the term $1.01y[n-1]$ models the fact that we accrue 1% interest each month.

Example 1.11

As a second example, consider a simple digital simulation of the differential equation in eq. (1.84) in which we resolve time into discrete intervals of length Δ and approximate $dv(t)/dt$ at $t = n\Delta$ by the first backward difference, i.e.,

$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta}.$$

In this case, if we let $v[n] = v(n\Delta)$ and $f[n] = f(n\Delta)$, we obtain the following discrete-time model relating the sampled signals $f[n]$ and $v[n]$:

$$v[n] - \frac{m}{(m+\rho\Delta)}v[n-1] = \frac{\Delta}{(m+\rho\Delta)}f[n]. \tag{1.88}$$

Comparing eqs. (1.87) and (1.88), we see that they are both examples of the same general first-order linear difference equation, namely,

$$y[n] + ay[n-1] = bx[n]. \tag{1.89}$$

As the preceding examples suggest, the mathematical descriptions of systems from a wide variety of applications frequently have a great deal in common, and it is this fact that provides considerable motivation for the development of broadly applicable tools for signal and system analysis. The key to doing this successfully is identifying classes of systems that have two important characteristics: (1) The systems in this class have properties and structures that we can exploit to gain insight into their behavior and to develop effective tools for their analysis; and (2) many systems of practical importance can be accurately modeled using systems in this class. It is on the first of these characteristics that most of this book focuses, as we develop tools for a particular class of systems referred to as linear, time-invariant systems. In the next section, we will introduce the properties that characterize this class, as well as a number of other very important basic system properties.

The second characteristic mentioned in the preceding paragraph is of obvious importance for any system analysis technique to be of value in practice. It is a well-established fact that a wide range of physical systems (including those in Examples 1.8–1.10) can be well modeled within the class of systems on which we focus in this book. However, a critical point is that *any* model used in describing or analyzing a physical system represents an idealization of that system, and thus, any resulting analysis is only as good as the model itself. For example, the simple linear model of a resistor in eq. (1.80) and that of a capacitor in eq. (1.81) are idealizations. However, these idealizations are quite accurate for real resistors and capacitors in many applications, and thus, analyses employing such idealizations provide useful results and conclusions, as long as the voltages and currents remain within the operating conditions under which these simple linear models are valid. Similarly, the use of a linear retarding force to represent frictional effects in eq. (1.83) is an approximation with a range of validity. Consequently, although we will not address this issue in the book, it is important to remember that an essential component of engineering practice in using the methods we develop here consists of identifying the range of validity of the assumptions that have gone into a model and ensuring that any analysis or design based on that model does not violate those assumptions.

1.5.2 Interconnections of Systems

An important idea that we will use throughout this book is the concept of the interconnection of systems. Many real systems are built as interconnections of several subsystems. One example is an audio system, which involves the interconnection of a radio receiver, compact disc player, or tape deck with an amplifier and one or more speakers. Another is a digitally controlled aircraft, which is an interconnection of the aircraft, described by its equations of motion and the aerodynamic forces affecting it; the sensors, which measure various aircraft variables such as accelerations, rotation rates, and heading; a digital autopilot, which responds to the measured variables and to command inputs from the pilot (e.g., the desired course, altitude, and speed); and the aircraft's actuators, which respond to inputs provided by the autopilot in order to use the aircraft control surfaces (rudder, tail, ailerons) to change the aerodynamic forces on the aircraft. By viewing such a system as an interconnection of its components, we can use our understanding of the component

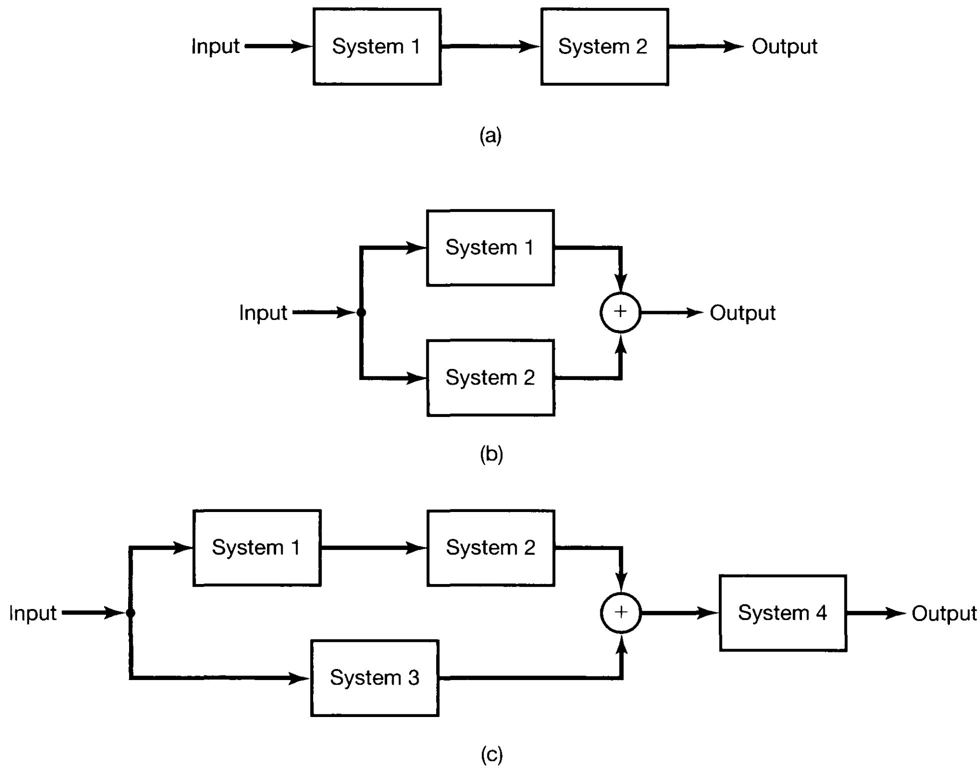


Figure 1.42 Interconnection of two systems: (a) series (cascade) interconnection; (b) parallel interconnection; (c) series-parallel interconnection.

systems and of how they are interconnected in order to analyze the operation and behavior of the overall system. In addition, by describing a system in terms of an interconnection of simpler subsystems, we may in fact be able to define useful ways in which to synthesize complex systems out of simpler, basic building blocks.

While one can construct a variety of system interconnections, there are several basic ones that are frequently encountered. A *series* or *cascade interconnection* of two systems is illustrated in Figure 1.42(a). Diagrams such as this are referred to as *block diagrams.* Here, the output of System 1 is the input to System 2, and the overall system transforms an input by processing it first by System 1 and then by System 2. An example of a series interconnection is a radio receiver followed by an amplifier. Similarly, one can define a series interconnection of three or more systems.

A *parallel interconnection* of two systems is illustrated in Figure 1.42(b). Here, the same input signal is applied to Systems 1 and 2. The symbol "$\oplus$" in the figure denotes addition, so that the output of the parallel interconnection is the sum of the outputs of Systems 1 and 2. An example of a parallel interconnection is a simple audio system with several microphones feeding into a single amplifier and speaker system. In addition to the simple parallel interconnection in Figure 1.42(b), we can define parallel interconnections of more than two systems, and we can combine both cascade and parallel interconnections

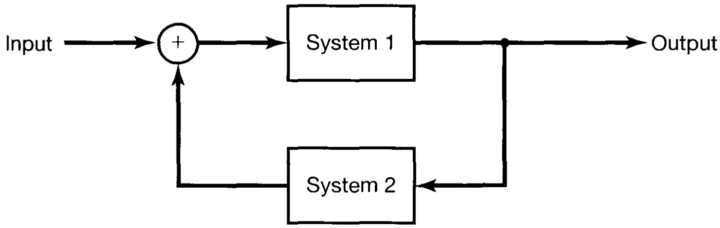
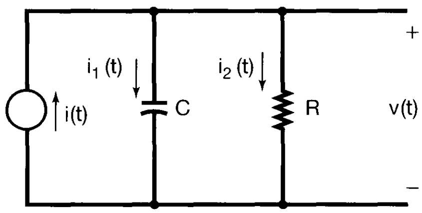


Figure 1.43 Feedback interconnection.

to obtain more complicated interconnections. An example of such an interconnection is given in Figure 1.42(c).[4]

Another important type of system interconnection is a *feedback interconnection,* an example of which is illustrated in Figure 1.43. Here, the output of System 1 is the input to System 2, while the output of System 2 is fed back and added to the external input to produce the actual input to System 1. Feedback systems arise in a wide variety of applications. For example, a cruise control system on an automobile senses the vehicle's velocity and adjusts the fuel flow in order to keep the speed at the desired level. Similarly, a digitally controlled aircraft is most naturally thought of as a feedback system in which differences between actual and desired speed, heading, or altitude are fed back through the autopilot in order to correct these discrepancies. Also, electrical circuits are often usefully viewed as containing feedback interconnections. As an example, consider the circuit depicted in Figure 1.44(a). As indicated in Figure 1.44(b), this system can be viewed as the feedback interconnection of the two circuit elements.



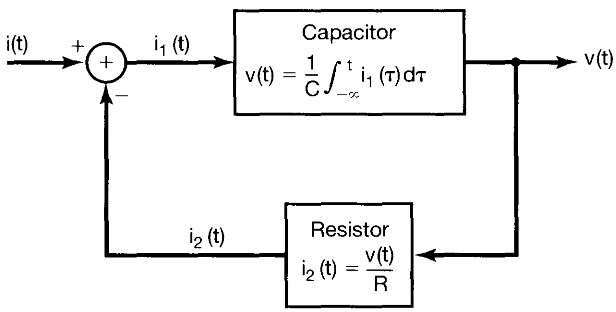


Figure 1.44 (a) Simple electrical circuit; (b) block diagram in which the circuit is depicted as the feedback interconnection of two circuit elements.

[4]On occasion, we will also use the symbol ⊗ in our pictorial representation of systems to denote the operation of multiplying two signals (see, for example, Figure 4.26).

1.6 BASIC SYSTEM PROPERTIES

In this section we introduce and discuss a number of basic properties of continuous-time and discrete-time systems. These properties have important physical interpretations and relatively simple mathematical descriptions using the signals and systems language that we have begun to develop.

1.6.1 Systems with and without Memory

A system is said to be *memoryless* if its output for each value of the independent variable at a given time is dependent only on the input at that same time. For example, the system specified by the relationship

$$y[n] = (2x[n] - x^2[n])^2 \tag{1.90}$$

is memoryless, as the value of $y[n]$ at any particular time n_0 depends only on the value of $x[n]$ at that time. Similarly, a resistor is a memoryless system; with the input $x(t)$ taken as the current and with the voltage taken as the output $y(t)$, the input-output relationship of a resistor is

$$y(t) = Rx(t), \tag{1.91}$$

where R is the resistance. One particularly simple memoryless system is the *identity system,* whose output is identical to its input. That is, the input-output relationship for the continuous-time identity system is

$$y(t) = x(t),$$

and the corresponding relationship in discrete time is

$$y[n] = x[n].$$

An example of a discrete-time system with memory is an *accumulator* or *summer*

$$y[n] = \sum_{k=-\infty}^{n} x[k], \tag{1.92}$$

and a second example is a *delay*

$$y[n] = x[n-1]. \tag{1.93}$$

A capacitor is an example of a continuous-time system with memory, since if the input is taken to be the current and the output is the voltage, then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau, \tag{1.94}$$

where C is the capacitance.

Roughly speaking, the concept of memory in a system corresponds to the presence of a mechanism in the system that retains or stores information about input values at times other than the current time. For example, the delay in eq. (1.93) must retain or store the preceding value of the input. Similarly, the accumulator in eq. (1.92) must "remember" or store information about past inputs. In particular, the accumulator computes the running sum of all inputs up to the current time, and thus, at each instant of time, the accumulator must add the current input value to the preceding value of the running sum. In other words, the relationship between the input and output of an accumulator can be described as

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n], \tag{1.95}$$

or equivalently,

$$y[n] = y[n-1] + x[n]. \tag{1.96}$$

Represented in the latter way, to obtain the output at the current time n, the accumulator must remember the running sum of previous input values, which is exactly the preceding value of the accumulator output.

In many physical systems, memory is directly associated with the storage of energy. For example, the capacitor in eq. (1.94) stores energy by accumulating electrical charge, represented as the integral of the current. Thus, the simple *RC* circuit in Example 1.8 and Figure 1.1 has memory physically stored in the capacitor. Similarly, the automobile in Figure 1.2 has memory stored in its kinetic energy. In discrete-time systems implemented with computers or digital microprocessors, memory is typically directly associated with storage registers that retain values between clock pulses.

While the concept of memory in a system would typically suggest storing *past* input and output values, our formal definition also leads to our referring to a system as having memory if the current output is dependent on *future* values of the input and output. While systems having this dependence on future values might at first seem unnatural, they in fact form an important class of systems, as we discuss further in Section 1.6.3.

1.6.2 Invertibility and Inverse Systems

A system is said to be *invertible* if distinct inputs lead to distinct outputs. As illustrated in Figure 1.45(a) for the discrete-time case, if a system is invertible, then an *inverse system* exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ to the first system. Thus, the series interconnection in Figure 1.45(a) has an overall input-output relationship which is the same as that for the identity system.

An example of an invertible continuous-time system is

$$y(t) = 2x(t), \tag{1.97}$$

for which the inverse system is

$$w(t) = \frac{1}{2}y(t). \tag{1.98}$$

This example is illustrated in Figure 1.45(b). Another example of an invertible system is the accumulator of eq. (1.92). For this system, the difference between two successive

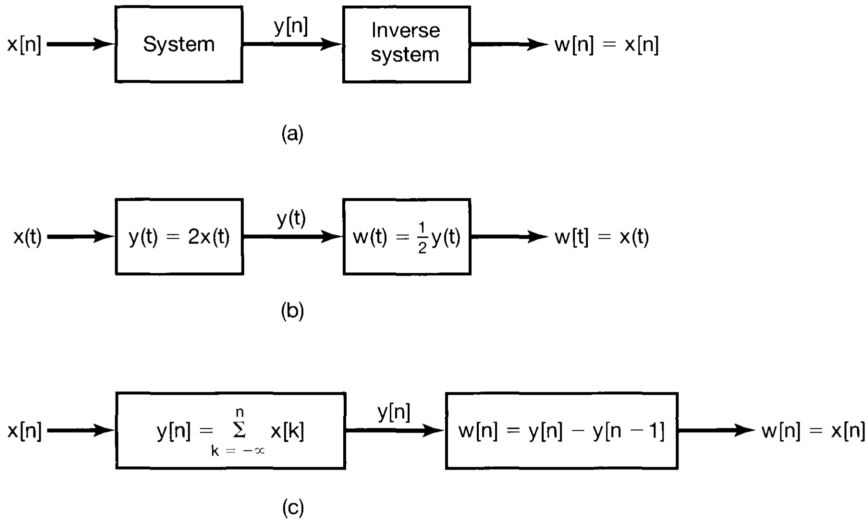


Figure 1.45 Concept of an inverse system for: (a) a general invertible system; (b) the invertible system described by eq. (1.97); (c) the invertible system defined in eq. (1.92).

values of the output is precisely the last input value. Therefore, in this case, the inverse system is

$$w[n] = y[n] - y[n-1], \tag{1.99}$$

as illustrated in Figure 1.45(c). Examples of noninvertible systems are

$$y[n] = 0, \tag{1.100}$$

that is, the system that produces the zero output sequence for any input sequence, and

$$y(t) = x^2(t), \tag{1.101}$$

in which case we cannot determine the sign of the input from knowledge of the output.

The concept of invertibility is important in many contexts. One example arises in systems for encoding used in a wide variety of communications applications. In such a system, a signal that we wish to transmit is first applied as the input to a system known as an encoder. There are many reasons for doing this, ranging from the desire to encrypt the original message for secure or private communication to the objective of providing some redundancy in the signal (for example, by adding what are known as parity bits) so that any errors that occur in transmission can be detected and, possibly, corrected. For *lossless* coding, the input to the encoder must be exactly recoverable from the output; i.e., the encoder must be invertible.

1.6.3 Causality

A system is *causal* if the output at any time depends only on values of the input at the present time and in the past. Such a system is often referred to as being *nonanticipative,* as the system output does not anticipate future values of the input. Consequently, if two inputs to a causal system are identical up to some point in time t_0 or n_0, the corresponding outputs must also be equal up to this same time. The RC circuit of Figure 1.1 is causal, since the capacitor voltage responds only to the present and past values of the source voltage. Similarly, the motion of an automobile is causal, since it does not anticipate future actions of the driver. The systems described in eqs. (1.92) – (1.94) are also causal, but the systems defined by

$$y[n] = x[n] - x[n+1] \tag{1.102}$$

and

$$y(t) = x(t+1) \tag{1.103}$$

are not. All memoryless systems are causal, since the output responds only to the current value of the input.

Although causal systems are of great importance, they do not by any means constitute the only systems that are of practical significance. For example, causality is not often an essential constraint in applications in which the independent variable is not time, such as in image processing. Furthermore, in processing data that have been recorded previously, as often happens with speech, geophysical, or meteorological signals, to name a few, we are by no means constrained to causal processing. As another example, in many applications, including historical stock market analysis and demographic studies, we may be interested in determining a slowly varying trend in data that also contain high-frequency fluctuations about that trend. In this case, a commonly used approach is to average data over an interval in order to smooth out the fluctuations and keep only the trend. An example of a noncausal averaging system is

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k]. \tag{1.104}$$

Example 1.12

When checking the causality of a system, it is important to look carefully at the input-output relation. To illustrate some of the issues involved in doing this, we will check the causality of two particular systems.

The first system is defined by

$$y[n] = x[-n]. \tag{1.105}$$

Note that the output $y[n_0]$ at a positive time n_0 depends only on the value of the input signal $x[-n_0]$ at time $(-n_0)$, which is negative and therefore in the past of n_0. We may be tempted to conclude at this point that the given system is causal. However, we should always be careful to check the input-output relation for *all* times. In particular, for $n < 0$, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of the input. Hence, the system is not causal.

It is also important to distinguish carefully the effects of the input from those of any other functions used in the definition of the system. For example, consider the system

$$y(t) = x(t)\cos(t+1). \tag{1.106}$$

In this system, the output at any time t equals the input at that same time multiplied by a number that varies with time. Specifically, we can rewrite eq. (1.106) as

$$y(t) = x(t)g(t),$$

where g(t) is a time-varying function, namely $g(t) = \cos(t + 1)$. Thus, only the current value of the input $x(t)$ influences the current value of the output $y(t)$, and we conclude that this system is causal (and, in fact, memoryless).

1.6.4 Stability

Stability is another important system property. Informally, a stable system is one in which small inputs lead to responses that do not diverge. For example, consider the pendulum in Figure 1.46(a), in which the input is the applied force $x(t)$ and the output is the angular deviation $y(t)$ from the vertical. In this case, gravity applies a restoring force that tends to return the pendulum to the vertical position, and frictional losses due to drag tend to slow it down. Consequently, if a small force $x(t)$ is applied, the resulting deflection from vertical will also be small. In contrast, for the inverted pendulum in Figure 1.46(b), the effect of gravity is to apply a force that tends to *increase* the deviation from vertical. Thus, a small applied force leads to a large vertical deflection causing the pendulum to topple over, despite any retarding forces due to friction.

The system in Figure 1.46(a) is an example of a stable system, while that in Figure 1.46(b) is unstable. Models for chain reactions or for population growth with unlimited food supplies and no predators are examples of unstable systems, since the system response grows without bound in response to small inputs. Another example of an unstable system is the model for a bank account balance in eq. (1.86), since if an initial deposit is made (i.e., $x[0]$ = a positive amount) and there are no subsequent withdrawals, then that deposit will grow each month without bound, because of the compounding effect of interest payments.

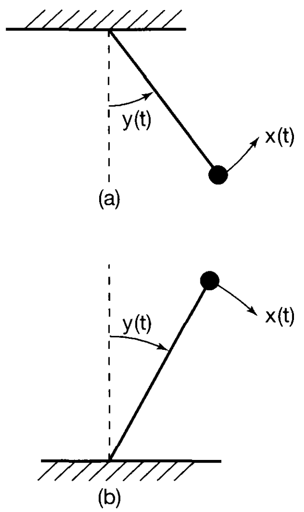


Figure 1.46 (a) A stable pendulum; (b) an unstable inverted pendulum.

There are also numerous examples of stable systems. Stability of physical systems generally results from the presence of mechanisms that dissipate energy. For example, assuming positive component values in the simple *RC* circuit of Example 1.8, the resistor dissipates energy and this circuit is a stable system. The system in Example 1.9 is also stable because of the dissipation of energy through friction.

The preceding examples provide us with an intuitive understanding of the concept of stability. More formally, if the input to a stable system is bounded (i.e., if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge. This is the definition of stability that we will use throughout this book. For example, consider applying a constant force $f(t) = F$ to the automobile in Figure 1.2, with the vehicle initially at rest. In this case the velocity of the car will increase, but not without bound, since the retarding frictional force also increases with velocity. In fact, the velocity will continue to increase until the frictional force exactly balances the applied force; so, from eq. (1.84), we see that this terminal velocity value V must satisfy

$$\frac{\rho}{m}V = \frac{1}{m}F, \tag{1.107}$$

i.e.,

$$V = \frac{F}{\rho}. \tag{1.108}$$

As another example, consider the discrete-time system defined by eq. (1.104), and suppose that the input $x[n]$ is bounded in magnitude by some number, say, B, for all values of n. Then the largest possible magnitude for $y[n]$ is also B, because $y[n]$ is the average of a finite set of values of the input. Therefore, $y[n]$ is bounded and the system is stable. On the other hand, consider the accumulator described by eq. (1.92). Unlike the system in eq. (1.104), this system sums *all* of the past values of the input rather than just a finite set of values, and the system is unstable, since the sum can grow continually even if $x[n]$ is bounded. For example, if the input to the accumulator is a unit step $u[n]$, the output will be

$$y[n] = \sum_{k=-\infty}^{n} u[k] = (n+1)u[n].$$

That is, $y[0] = 1$, $y[1] = 2$, $y[2] = 3$, and so on, and $y[n]$ grows without bound.

Example 1.13

If we suspect that a system is unstable, then a useful strategy to verify this is to look for a *specific* bounded input that leads to an unbounded output. Finding one such example enables us to conclude that the given system is unstable. If such an example does not exist or is difficult to find, we must check for stability by using a method that does not utilize specific examples of input signals. To illustrate this approach, let us check the stability of two systems,

$$S_1: y(t) = tx(t) \tag{1.109}$$

and

$$S_2: y(t) = e^{x(t)}. \tag{1.110}$$

In seeking a specific counterexample in order to disprove stability, we might try simple bounded inputs such as a constant or a unit step. For system S_1 in eq. (1.109), a constant input $x(t) = 1$ yields $y(t) = t$, which is unbounded, since no matter what finite constant we pick, $|y(t)|$ will exceed that constant for some t. We conclude that system S_1 is unstable.

For system S_2, which happens to be stable, we would be unable to find a bounded input that results in an unbounded output. So we proceed to verify that all bounded inputs result in bounded outputs. Specifically, let B be an arbitrary positive number, and let $x(t)$ be an arbitrary signal bounded by B; that is, we are making no assumption about $x(t)$, except that

$$|x(t)| < B, \tag{1.111}$$

or

$$-B < x(t) < B, \tag{1.112}$$

for all t. Using the definition of S_2 in eq. (1.110), we then see that if $x(t)$ satisfies eq. (1.111), then $y(t)$ must satisfy

$$e^{-B} < |y(t)| < e^{B}. \tag{1.113}$$

We conclude that if any input to S_2 is bounded by an arbitrary positive number B, the corresponding output is guaranteed to be bounded by e^B. Thus, S_2 is stable.

The system properties and concepts that we have introduced so far in this section are of great importance, and we will examine some of these in more detail later in the book. There remain, however, two additional properties—time invariance and linearity—that play a particularly central role in the subsequent chapters of the book, and in the remainder of this section we introduce and provide initial discussions of these two very important concepts.

1.6.5 Time Invariance

Conceptually, a system is time invariant if the behavior and characteristics of the system are fixed over time. For example, the RC circuit of Figure 1.1 is time invariant if the resistance and capacitance values R and C are constant over time: We would expect to get the same results from an experiment with this circuit today as we would if we ran the identical experiment tomorrow. On the other hand, if the values of R and C are changed or fluctuate over time, then we would expect the results of our experiment to depend on the time at which we run it. Similarly, if the frictional coefficient b and mass m of the automobile in Figure 1.2 are constant, we would expect the vehicle to respond identically independently of when we drive it. On the other hand, if we load the auto's trunk with heavy suitcases one day, thus increasing m, we would expect the car to behave differently than at other times when it is not so heavily loaded.

The property of time invariance can be described very simply in terms of the signals and systems language that we have introduced. Specifically, a system is time invariant if a time shift in the input signal results in an identical time shift in the output signal. That is, if $y[n]$ is the output of a discrete-time, time-invariant system when $x[n]$ is the input, then $y[n - n_0]$ is the output when $x[n - n_0]$ is applied. In continuous time with $y(t)$ the output corresponding to the input $x(t)$, a time-invariant system will have $y(t - t_0)$ as the output when $x(t - t_0)$ is the input.

To see how to determine whether a system is time invariant or not, and to gain some insight into this property, consider the following examples:

Example 1.14

Consider the continuous-time system defined by

$$y(t) = \sin[x(t)]. \tag{1.114}$$

To check that this system is time invariant, we must determine whether the time-invariance property holds for *any* input and *any* time shift t_0. Thus, let $x_1(t)$ be an arbitrary input to this system, and let

$$y_1(t) = \sin[x_1(t)] \tag{1.115}$$

be the corresponding output. Then consider a second input obtained by shifting $x_1(t)$ in time:

$$x_2(t) = x_1(t - t_0). \tag{1.116}$$

The output corresponding to this input is

$$y_2(t) = \sin[x_2(t)] = \sin[x_1(t - t_0)]. \tag{1.117}$$

Similarly, from eq. (1.115),

$$y_1(t - t_0) = \sin[x_1(t - t_0)]. \tag{1.118}$$

Comparing eqs. (1.117) and (1.118), we see that $y_2(t) = y_1(t - t_0)$, and therefore, this system is time invariant.

Example 1.15

As a second example, consider the discrete-time system

$$y[n] = nx[n]. \tag{1.119}$$

This is a time-varying system, a fact that can be verified using the same formal procedure as that used in the preceding example (see Problem 1.28). However, when a system is suspected of being time varying, an approach to showing this that is often very useful is to seek a counterexample—i.e., to use our intuition to find an input signal for which the condition of time invariance is violated. In particular, the system in this example represents a system with a time-varying gain. For example, if we know that the current input value is 1, we cannot determine the current output value without knowing the current time.

Consequently, consider the input signal $x_1[n] = \delta[n]$, which yields an output $y_1[n]$ that is identically 0 (since $n\delta[n] = 0$). However, the input $x_2[n] = \delta[n-1]$ yields the output $y_2[n] = n\delta[n-1] = \delta[n-1]$. Thus, while $x_2[n]$ is a shifted version of $x_1[n]$, $y_2[n]$ is *not* a shifted version of $y_1[n]$.

While the system in the preceding example has a time-varying gain and as a result is a time-varying system, the system in eq. (1.97) has a constant gain and, in fact, is time invariant. Other examples of time-invariant systems are given by eqs. (1.91)–(1.104). The following example illustrates a time-varying system.

Example 1.16

Consider the system

$$y(t) = x(2t). \tag{1.120}$$

This system represents a time scaling. That is, $y(t)$ is a time-compressed (by a factor of 2) version of $x(t)$. Intuitively, then, any time shift in the input will also be compressed by a factor of 2, and it is for this reason that the system is not time invariant. To demonstrate this by counterexample, consider the input $x_1(t)$ shown in Figure 1.47(a) and the resulting output $y_1(t)$ depicted in Figure 1.47(b). If we then shift the input by 2—i.e., consider $x_2(t) = x_1(t-2)$, as shown in Figure 1.47(c)—we obtain the resulting output

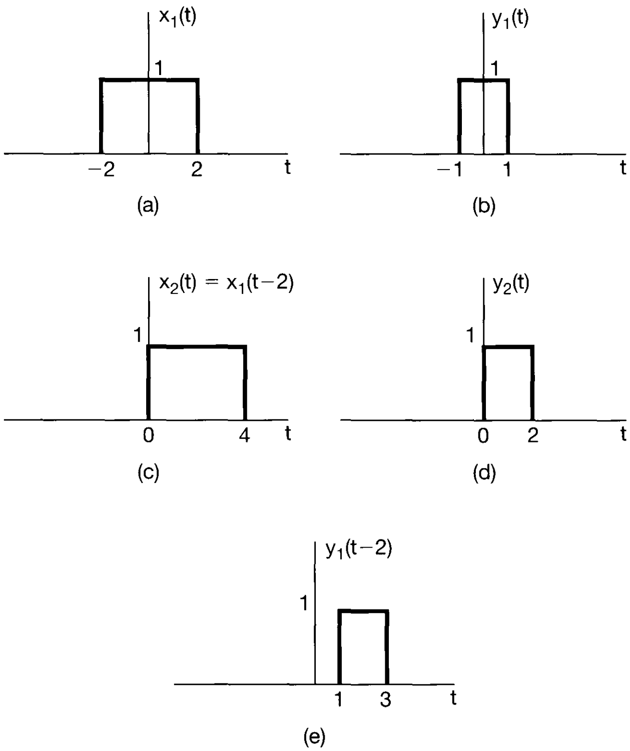


Figure 1.47 (a) The input $x_1(t)$ to the system in Example 1.16; (b) the output $y_1(t)$ corresponding to $x_1(t)$; (c) the shifted input $x_2(t) = x_1(t-2)$; (d) the output $y_2(t)$ corresponding to $x_2(t)$; (e) the shifted signal $y_1(t-2)$. Note that $y_2(t) \neq y_1(t-2)$, showing that the system is not time invariant.

$y_2(t) = x_2(2t)$ shown in Figure 1.47(d). Comparing Figures 1.47(d) and (e), we see that $y_2(t) \neq y_1(t-2)$, so that the system is not time invariant. (In fact, $y_2(t) = y_1(t-1)$, so that the output time shift is only half as big as it should be for time invariance, due to the time compression imparted by the system.)

1.6.6 Linearity

A *linear system,* in continuous time or discrete time, is a system that possesses the important property of superposition: If an input consists of the weighted sum of several signals, then the output is the superposition—that is, the weighted sum—of the responses of the system to each of those signals. More precisely, let $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and let $y_2(t)$ be the output corresponding to the input $x_2(t)$. Then the system is linear if:

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. The response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant.

The first of these two properties is known as the *additivity* property; the second is known as the *scaling* or *homogeneity* property. Although we have written this description using continuous-time signals, the same definition holds in discrete time. The systems specified by eqs. (1.91)–(1.100), (1.102)–(1.104), and (1.119) are linear, while those defined by eqs. (1.101) and (1.114) are nonlinear. Note that a system can be linear without being time invariant, as in eq. (1.119), and it can be time invariant without being linear, as in eqs. (1.101) and (1.114).

The two properties defining a linear system can be combined into a single statement:

$$\text{continuous time: } ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t), \tag{1.121}$$

$$\text{discrete time: } ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]. \tag{1.122}$$

Here, a and b are any complex constants. Furthermore, it is straightforward to show from the definition of linearity that if $x_k[n]$, $k = 1, 2, 3, \ldots$, are a set of inputs to a discrete-time linear system with corresponding outputs $y_k[n]$, $k = 1, 2, 3, \ldots$, then the response to a linear combination of these inputs given by

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \ldots \tag{1.123}$$

is

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \ldots. \tag{1.124}$$

This very important fact is known as the *superposition property*, which holds for linear systems in both continuous and discrete time.

A direct consequence of the superposition property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time. For example, if $x[n] \rightarrow y[n]$, then the homogeneity property tells us that

$$0 = 0 \cdot x[n] \rightarrow 0 \cdot y[n] = 0. \tag{1.125}$$

In the following examples we illustrate how the linearity of a given system can be checked by directly applying the definition of linearity.

Example 1.17

Consider a system S whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = tx(t)$$

To determine whether or not S is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$
$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to S, then the corresponding output may be expressed as

$$\begin{aligned} y_3(t) &= tx_3(t) \\ &= t(ax_1(t) + bx_2(t)) \\ &= atx_1(t) + btx_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

We conclude that the system S is linear.

Example 1.18

Let us apply the linearity-checking procedure of the previous example to another system S whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = x^2(t)$$

Defining $x_1(t)$, $x_2(t)$, and $x_3(t)$ as in the previous example, we have

$$x_1(t) \rightarrow y_1(t) = x_1^2(t)$$
$$x_2(t) \rightarrow y_2(t) = x_2^2(t)$$

and

$$\begin{aligned} x_3(t) \rightarrow y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

Clearly, we can specify $x_1(t)$, $x_2(t)$, a, and b such that $y_3(t)$ is not the same as $ay_1(t) + by_2(t)$. For example, if $x_1(t) = 1$, $x_2(t) = 0$, $a = 2$, and $b = 0$, then $y_3(t) = (2x_1(t))^2 = 4$, but $2y_1(t) = 2(x_1(t))^2 = 2$. We conclude that the system S is not linear.

Example 1.19

In checking the linearity of a system, it is important to remember that the system must satisfy both the additivity and homogeneity properties and that the signals, as well as any scaling constants, are allowed to be complex. To emphasize the importance of these points, consider the system specified by

$$y[n] = \mathcal{R}e\{x[n]\}. \tag{1.126}$$

As shown in Problem 1.29, this system is additive; however, it does not satisfy the homogeneity property, as we now demonstrate. Let

$$x_1[n] = r[n] + js[n] \tag{1.127}$$

be an arbitrary complex input with real and imaginary parts $r[n]$ and $s[n]$, respectively, so that the corresponding output is

$$y_1[n] = r[n]. \tag{1.128}$$

Now, consider scaling $x_1[n]$ by a complex number, for example, $a = j$; i.e., consider the input

$$\begin{aligned} x_2[n] &= jx_1[n] = j(r[n] + js[n]) \\ &= -s[n] + jr[n]. \end{aligned} \tag{1.129}$$

The output corresponding to $x_2[n]$ is

$$y_2[n] = \mathcal{R}e\{x_2[n]\} = -s[n], \tag{1.130}$$

which is not equal to the scaled version of $y_1[n]$,

$$ay_1[n] = jr[n]. \tag{1.131}$$

We conclude that the system violates the homogeneity property and hence is not linear.

Example 1.20

Consider the system

$$y[n] = 2x[n] + 3. \tag{1.132}$$

This system is not linear, as can be verified in several ways. For example, the system violates the additivity property: If $x_1[n] = 2$ and $x_2[n] = 3$, then

$$x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3 = 7, \tag{1.133}$$

$$x_2[n] \rightarrow y_2[n] = 2x_2[n] + 3 = 9. \tag{1.134}$$

However, the response to $x_3[n] = x_1[n] + x_2[n]$ is

$$y_3[n] = 2[x_1[n] + x_2[n]] + 3 = 13, \tag{1.135}$$

which does not equal $y_1[n] + y_2[n] = 16$. Alternatively, since $y[n] = 3$ if $x[n] = 0$, we see that the system violates the "zero-in/zero-out" property of linear systems given in eq. (1.125).

It may seem surprising that the system in the above example is nonlinear, since eq. (1.132) is a linear equation. On the other hand, as depicted in Figure 1.48, the output of this system can be represented as the sum of the output of a linear system and another signal equal to the *zero-input response* of the system. For the system in eq. (1.132), the linear system is

$$x[n] \rightarrow 2x[n],$$

and the zero-input response is

$$y_0[n] = 3.$$

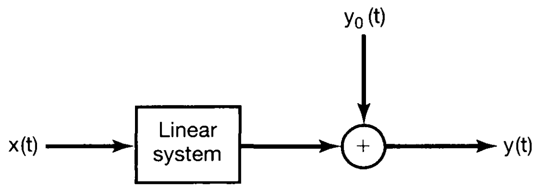


Figure 1.48 Structure of an incrementally linear system. Here, $y_0[n]$ is the zero-input response of the system.

There are, in fact, large classes of systems in both continuous and discrete time that can be represented as in Figure 1.48—i.e., for which the overall system output consists of the superposition of the response of a linear system with a zero-input response. As shown in Problem 1.47, such systems correspond to the class of *incrementally linear systems*—i.e., systems in continuous or discrete time that respond linearly to *changes* in the input. In other words, the *difference* between the responses to any two inputs to an incrementally linear system is a linear (i.e., additive and homogeneous) function of the *difference* between the two inputs. For example, if $x_1[n]$ and $x_2[n]$ are two inputs to the system specified by eq. (1.132), and if $y_1[n]$ and $y_2[n]$ are the corresponding outputs, then

$$y_1[n] - y_2[n] = 2x_1[n] + 3 - \{2x_2[n] + 3\} = 2\{x_1[n] - x_2[n]\}. \qquad (1.136)$$

1.7 SUMMARY

In this chapter, we have developed a number of basic concepts related to continuous-time and discrete-time signals and systems. We have presented both an intuitive picture of what signals and systems are through several examples and a mathematical representation for signals and systems that we will use throughout the book. Specifically, we introduced graphical and mathematical representations of signals and used these representations in performing transformations of the independent variable. We also defined and examined several basic signals, both in continuous time and in discrete time. These included complex exponential signals, sinusoidal signals, and unit impulse and step functions. In addition, we investigated the concept of periodicity for continuous-time and discrete-time signals.

In developing some of the elementary ideas related to systems, we introduced block diagrams to facilitate our discussions concerning the interconnection of systems, and we defined a number of important properties of systems, including causality, stability, time invariance, and linearity.

The primary focus in most of this book will be on the class of linear, time-invariant (LTI) systems, both in continuous time and in discrete time. These systems play a particularly important role in system analysis and design, in part due to the fact that many systems encountered in nature can be successfully modeled as linear and time invariant. Furthermore, as we shall see in the following chapters, the properties of linearity and time invariance allow us to analyze in detail the behavior of LTI systems.

Chapter 1 Problems

Basic problems emphasize the mechanics of using concepts and methods in a manner similar to that illustrated in the examples that are solved in the text.

Advanced problems explore and elaborate upon the foundations and practical implications of the textual material.

The first section of problems belongs to the basic category, and the answers are provided in the back of the book. The next two sections contain problems belonging to the basic and advanced categories, respectively. A final section, **Mathematical Review**, provides practice problems on the fundamental ideas of complex arithmetic and algebra.

BASIC PROBLEMS WITH ANSWERS

1.1. Express each of the following complex numbers in Cartesian form $(x + jy)$: $\frac{1}{2}e^{j\pi}$, $\frac{1}{2}e^{-j\pi}$, $e^{j\pi/2}$, $e^{-j\pi/2}$, $e^{j5\pi/2}$, $\sqrt{2}e^{j\pi/4}$, $\sqrt{2}e^{j9\pi/4}$, $\sqrt{2}e^{-j9\pi/4}$, $\sqrt{2}e^{-j\pi/4}$.

1.2. Express each of the following complex numbers in polar form ($re^{j\theta}$, with $-\pi < \theta \leq \pi$): 5, -2, $-3j$, $\frac{1}{2} - j\frac{\sqrt{3}}{2}$, $1 + j$, $(1-j)^2$, $j(1-j)$, $(1+j)/(1-j)$, $(\sqrt{2} + j\sqrt{2})/(1 + j\sqrt{3})$.

1.3. Determine the values of P_∞ and E_∞ for each of the following signals:

(a) $x_1(t) = e^{-2t}u(t)$ **(b)** $x_2(t) = e^{j(2t+\pi/4)}$ **(c)** $x_3(t) = \cos(t)$

(d) $x_1[n] = (\frac{1}{2})^n u[n]$ **(e)** $x_2[n] = e^{j(\pi/2n+\pi/8)}$ **(f)** $x_3[n] = \cos(\frac{\pi}{4}n)$

1.4. Let $x[n]$ be a signal with $x[n] = 0$ for $n < -2$ and $n > 4$. For each signal given below, determine the values of n for which it is guaranteed to be zero.

(a) $x[n-3]$ **(b)** $x[n+4]$ **(c)** $x[-n]$

(d) $x[-n+2]$ **(e)** $x[-n-2]$

1.5. Let $x(t)$ be a signal with $x(t) = 0$ for $t < 3$. For each signal given below, determine the values of t for which it is guaranteed to be zero.

(a) $x(1-t)$ **(b)** $x(1-t) + x(2-t)$ **(c)** $x(1-t)x(2-t)$

(d) $x(3t)$ **(e)** $x(t/3)$

1.6. Determine whether or not each of the following signals is periodic:

(a) $x_1(t) = 2e^{j(t+\pi/4)}u(t)$ **(b)** $x_2[n] = u[n] + u[-n]$

(c) $x_3[n] = \sum_{k=-\infty}^{\infty}\{\delta[n-4k] - \delta[n-1-4k]\}$

1.7. For each signal given below, determine all the values of the independent variable at which the even part of the signal is guaranteed to be zero.

(a) $x_1[n] = u[n] - u[n-4]$ **(b)** $x_2(t) = \sin(\frac{1}{2}t)$

(c) $x_3[n] = (\frac{1}{2})^n u[n-3]$ **(d)** $x_4(t) = e^{-5t}u(t+2)$

1.8. Express the real part of each of the following signals in the form $Ae^{-at}\cos(\omega t + \phi)$, where A, a, ω, and ϕ are real numbers with $A > 0$ and $-\pi < \phi \leq \pi$:

(a) $x_1(t) = -2$ **(b)** $x_2(t) = \sqrt{2}e^{j\pi/4}\cos(3t + 2\pi)$

(c) $x_3(t) = e^{-t}\sin(3t + \pi)$ **(d)** $x_4(t) = je^{(-2+j100)t}$

1.9. Determine whether or not each of the following signals is periodic. If a signal is periodic, specify its fundamental period.

(a) $x_1(t) = je^{j10t}$ **(b)** $x_2(t) = e^{(-1+j)t}$ **(c)** $x_3[n] = e^{j7\pi n}$

(d) $x_4[n] = 3e^{j3\pi(n+1/2)/5}$ **(e)** $x_5[n] = 3e^{j3/5(n+1/2)}$

1.10. Determine the fundamental period of the signal $x(t) = 2\cos(10t + 1) - \sin(4t - 1)$.

1.11. Determine the fundamental period of the signal $x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$.

1.12. Consider the discrete-time signal

$$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n - 1 - k].$$

Determine the values of the integers M and n_0 so that $x[n]$ may be expressed as

$$x[n] = u[Mn - n_0].$$

1.13. Consider the continuous-time signal

$$x(t) = \delta(t + 2) - \delta(t - 2).$$

Calculate the value of E_∞ for the signal

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

1.14. Consider a periodic signal

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -2, & 1 < t < 2 \end{cases}$$

with period $T = 2$. The derivative of this signal is related to the "impulse train"

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

with period $T = 2$. It can be shown that

$$\frac{dx(t)}{dt} = A_1 g(t - t_1) + A_2 g(t - t_2).$$

Determine the values of A_1, t_1, A_2, and t_2.

1.15. Consider a system S with input $x[n]$ and output $y[n]$. This system is obtained through a series interconnection of a system S_1 followed by a system S_2. The input-output relationships for S_1 and S_2 are

$$S_1: \quad y_1[n] = 2x_1[n] + 4x_1[n - 1],$$

$$S_2: \quad y_2[n] = x_2[n - 2] + \frac{1}{2}x_2[n - 3],$$

where $x_1[n]$ and $x_2[n]$ denote input signals.

(a) Determine the input-output relationship for system S.

(b) Does the input-output relationship of system S change if the order in which S_1 and S_2 are connected in series is reversed (i.e., if S_2 follows S_1)?

1.16. Consider a discrete-time system with input $x[n]$ and output $y[n]$. The input-output relationship for this system is

$$y[n] = x[n]x[n - 2].$$

(a) Is the system memoryless?
(b) Determine the output of the system when the input is $A\delta[n]$, where A is any real or complex number.
(c) Is the system invertible?

1.17. Consider a continuous-time system with input $x(t)$ and output $y(t)$ related by

$$y(t) = x(\sin(t)).$$

(a) Is this system causal?
(b) Is this system linear?

1.18. Consider a discrete-time system with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k],$$

where n_0 is a finite positive integer.
(a) Is this system linear?
(a) Is this system time-invariant?
(c) If $x[n]$ is known to be bounded by a finite integer B (i.e., $|x[n]| < B$ for all n), it can be shown that $y[n]$ is bounded by a finite number C. We conclude that the given system is stable. Express C in terms of B and n_0.

1.19. For each of the following input-output relationships, determine whether the corresponding system is linear, time invariant or both.
(a) $y(t) = t^2 x(t-1)$ **(b)** $y[n] = x^2[n-2]$
(c) $y[n] = x[n+1] - x[n-1]$ **(d)** $y[n] = \mathcal{O}d\{x(t)\}$

1.20. A continuous-time linear system S with input $x(t)$ and output $y(t)$ yields the following input-output pairs:

$$x(t) = e^{j2t} \xrightarrow{S} y(t) = e^{j3t},$$

$$x(t) = e^{-j2t} \xrightarrow{S} y(t) = e^{-j3t}.$$

(a) If $x_1(t) = \cos(2t)$, determine the corresponding output $y_1(t)$ for system S.
(b) If $x_2(t) = \cos(2(t - \frac{1}{2}))$, determine the corresponding output $y_2(t)$ for system S.

BASIC PROBLEMS

1.21. A continuous-time signal $x(t)$ is shown in Figure P1.21. Sketch and label carefully each of the following signals:
(a) $x(t-1)$ **(b)** $x(2-t)$ **(c)** $x(2t+1)$
(d) $x(4 - \frac{t}{2})$ **(e)** $[x(t) + x(-t)]u(t)$ **(f)** $x(t)[\delta(t + \frac{3}{2}) - \delta(t - \frac{3}{2})]$

1.22. A discrete-time signal is shown in Figure P1.22. Sketch and label carefully each of the following signals:
(a) $x[n-4]$ **(b)** $x[3-n]$ **(c)** $x[3n]$
(d) $x[3n+1]$ **(e)** $x[n]u[3-n]$ **(f)** $x[n-2]\delta[n-2]$
(g) $\frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n]$ **(h)** $x[(n-1)^2]$

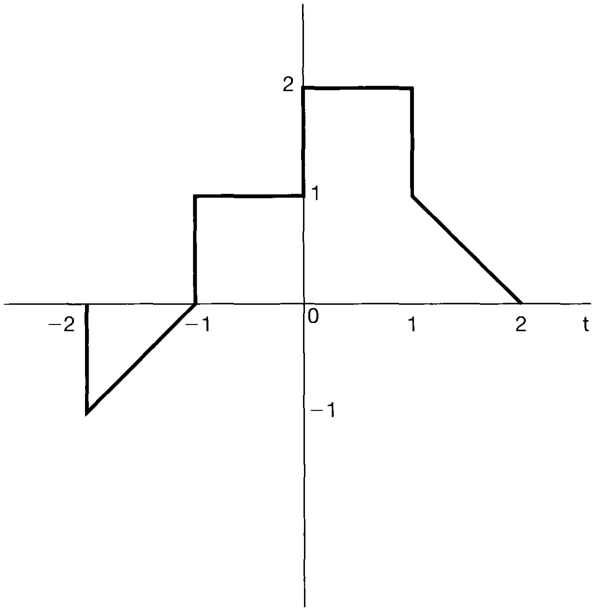


Figure P1.21

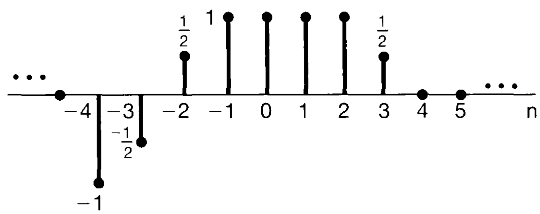


Figure P1.22

1.23. Determine and sketch the even and odd parts of the signals depicted in Figure P1.23. Label your sketches carefully.

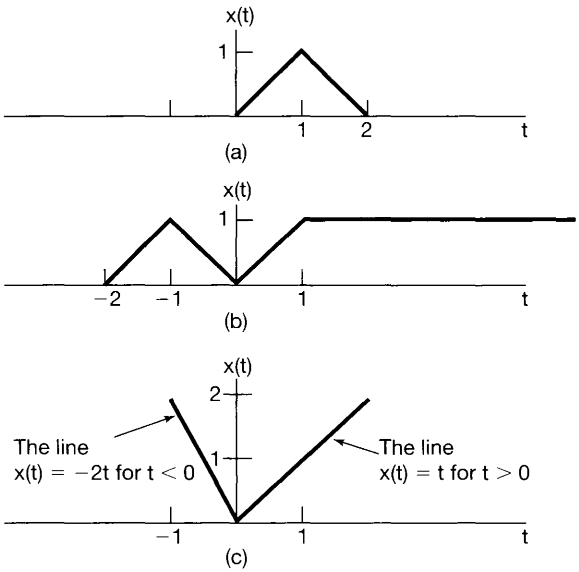


Figure P1.23

1.24. Determine and sketch the even and odd parts of the signals depicted in Figure P1.24. Label your sketches carefully.

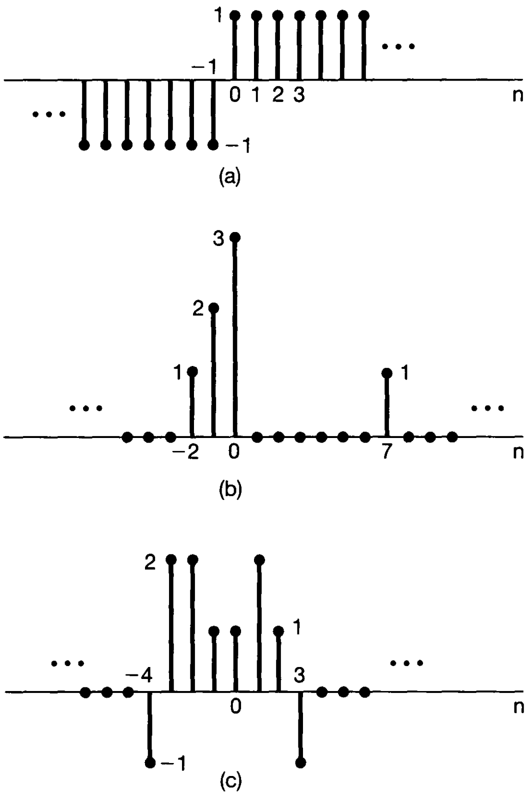


Figure P1.24

1.25. Determine whether or not each of the following continuous-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x(t) = 3\cos(4t + \frac{\pi}{3})$ **(b)** $x(t) = e^{j(\pi t - 1)}$

(c) $x(t) = [\cos(2t - \frac{\pi}{3})]^2$ **(d)** $x(t) = \mathcal{E}v\{\cos(4\pi t)u(t)\}$

(e) $x(t) = \mathcal{E}v\{\sin(4\pi t)u(t)\}$ **(f)** $x(t) = \sum_{n=-\infty}^{\infty} e^{-(2t-n)}u(2t - n)$

1.26. Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x[n] = \sin(\frac{6\pi}{7}n + 1)$ **(b)** $x[n] = \cos(\frac{n}{8} - \pi)$ **(c)** $x[n] = \cos(\frac{\pi}{8}n^2)$

(d) $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$ **(e)** $x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$

1.27. In this chapter, we introduced a number of general properties of systems. In particular, a system may or may not be

(1) Memoryless
(2) Time invariant
(3) Linear
(4) Causal
(5) Stable

Determine which of these properties hold and which do not hold for each of the following continuous-time systems. Justify your answers. In each example, $y(t)$ denotes the system output and $x(t)$ is the system input.

(a) $y(t) = x(t-2) + x(2-t)$ (b) $y(t) = [\cos(3t)]x(t)$

(c) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$ (d) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-2), & t \geq 0 \end{cases}$

(e) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-2), & x(t) \geq 0 \end{cases}$ (f) $y(t) = x(t/3)$

(g) $y(t) = \frac{dx(t)}{dt}$

1.28. Determine which of the properties listed in Problem 1.27 hold and which do not hold for each of the following discrete-time systems. Justify your answers. In each example, $y[n]$ denotes the system output and $x[n]$ is the system input.

(a) $y[n] = x[-n]$ (b) $y[n] = x[n-2] - 2x[n-8]$

(c) $y[n] = nx[n]$ (d) $y[n] = \mathcal{E}v\{x[n-1]\}$

(e) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$ (f) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$

(g) $y[n] = x[4n+1]$

1.29. (a) Show that the discrete-time system whose input $x[n]$ and output $y[n]$ are related by $y[n] = \mathcal{R}e\{x[n]\}$ is additive. Does this system remain additive if its input-output relationship is changed to $y[n] = \mathcal{R}e\{e^{j\pi n/4}x[n]\}$? (Do not assume that $x[n]$ is real in this problem.)

(b) In the text, we discussed the fact that the property of linearity for a system is equivalent to the system possessing both the additivity property and homogeneity property. Determine whether each of the systems defined below is additive and/or homogeneous. Justify your answers by providing a proof for each property if it holds or a counterexample if it does not.

(i) $y(t) = \frac{1}{x(t)}\left[\frac{dx(t)}{dt}\right]^2$ (ii) $y[n] = \begin{cases} \frac{x[n]x[n-2]}{x[n-1]}, & x[n-1] \neq 0 \\ 0, & x[n-1] = 0 \end{cases}$

1.30. Determine if each of the following systems is invertible. If it is, construct the inverse system. If it is not, find two input signals to the system that have the same output.

(a) $y(t) = x(t-4)$ (b) $y(t) = \cos[x(t)]$

(c) $y[n] = nx[n]$ (d) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

(e) $y[n] = \begin{cases} x[n-1], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$ (f) $y[n] = x[n]x[n-1]$

(g) $y[n] = x[1-n]$ (h) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau)d\tau$

(i) $y[n] = \sum_{k=-\infty}^{n} (\frac{1}{2})^{n-k}x[k]$ (j) $y(t) = \frac{dx(t)}{dt}$

(k) $y[n] = \begin{cases} x[n+1], & n \geq 0 \\ x[n], & n \leq -1 \end{cases}$ (l) $y(t) = x(2t)$

(m) $y[n] = x[2n]$ (n) $y[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

1.31. In this problem, we illustrate one of the most important consequences of the properties of linearity and time invariance. Specifically, once we know the response of a linear system or a linear time-invariant (LTI) system to a single input or the responses to several inputs, we can directly compute the responses to many other input signals. Much of the remainder of this book deals with a thorough exploitation of this fact in order to develop results and techniques for analyzing and synthesizing LTI systems.

(a) Consider an LTI system whose response to the signal $x_1(t)$ in Figure P1.31(a) is the signal $y_1(t)$ illustrated in Figure P1.31(b). Determine and sketch carefully the response of the system to the input $x_2(t)$ depicted in Figure P1.31(c).

(b) Determine and sketch the response of the system considered in part (a) to the input $x_3(t)$ shown in Figure P1.31(d).

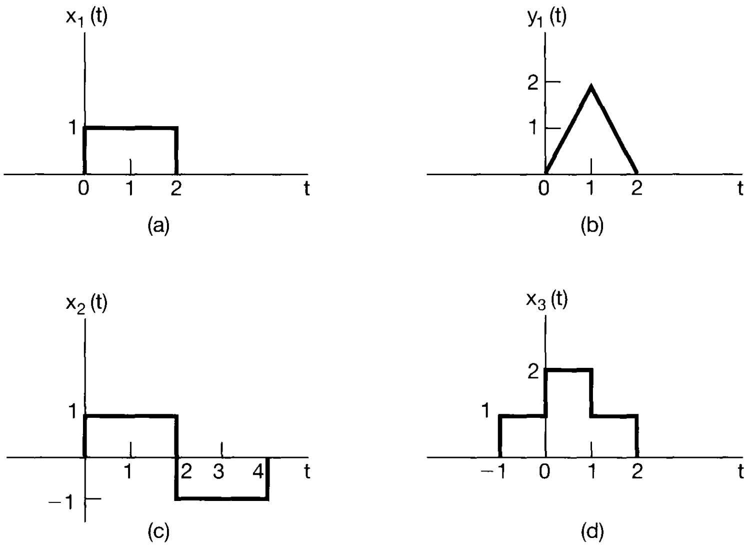


Figure P1.31

ADVANCED PROBLEMS

1.32. Let $x(t)$ be a continuous-time signal, and let

$$y_1(t) = x(2t) \text{ and } y_2(t) = x(t/2).$$

The signal $y_1(t)$ represents a speeded up version of $x(t)$ in the sense that the duration of the signal is cut in half. Similarly, $y_2(t)$ represents a slowed down version of $x(t)$ in the sense that the duration of the signal is doubled. Consider the following statements:

(1) If $x(t)$ is periodic, then $y_1(t)$ is periodic.

(2) If $y_1(t)$ is periodic, then $x(t)$ is periodic.

(3) If $x(t)$ is periodic, then $y_2(t)$ is periodic.

(4) If $y_2(t)$ is periodic, then $x(t)$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

1.33. Let $x[n]$ be a discrete-time signal, and let

$$y_1[n] = x[2n] \text{ and } y_2[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

The signals $y_1[n]$ and $y_2[n]$ respectively represent in some sense the speeded up and slowed down versions of $x[n]$. However, it should be noted that the discrete-time notions of speeded up and slowed down have subtle differences with respect to their continuous-time counterparts. Consider the following statements:

(1) If $x[n]$ is periodic, then $y_1[n]$ is periodic.
(2) If $y_1[n]$ is periodic, then $x[n]$ is periodic.
(3) If $x[n]$ is periodic, then $y_2[n]$ is periodic.
(4) If $y_2[n]$ is periodic, then $x[n]$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

1.34. In this problem, we explore several of the properties of even and odd signals.

(a) Show that if $x[n]$ is an odd signal, then

$$\sum_{n=-\infty}^{+\infty} x[n] = 0.$$

(b) Show that if $x_1[n]$ is an odd signal and $x_2[n]$ is an even signal, then $x_1[n]x_2[n]$ is an odd signal.

(c) Let $x[n]$ be an arbitrary signal with even and odd parts denoted by

$$x_e[n] = \mathcal{E}v\{x[n]\}$$

and

$$x_o[n] = \mathcal{O}d\{x[n]\}.$$

Show that

$$\sum_{n=-\infty}^{+\infty} x^2[n] = \sum_{n=-\infty}^{+\infty} x_e^2[n] + \sum_{n=-\infty}^{+\infty} x_o^2[n].$$

(d) Although parts (a)–(c) have been stated in terms of discrete-time signals, the analogous properties are also valid in continuous time. To demonstrate this, show that

$$\int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} x_e^2(t)dt + \int_{-\infty}^{+\infty} x_o^2(t)dt,$$

where $x_e(t)$ and $x_o(t)$ are, respectively, the even and odd parts of $x(t)$.

1.35. Consider the periodic discrete-time exponential time signal

$$x[n] = e^{jm(2\pi/N)n}.$$

Show that the fundamental period of this signal is

$$N_0 = N/\gcd(m, N),$$

where $\gcd(m, N)$ is the *greatest common divisor* of m and N—that is, the largest integer that divides both m and N an integral number of times. For example,

$$\gcd(2, 3) = 1, \gcd(2, 4) = 2, \gcd(8, 12) = 4.$$

Note that $N_0 = N$ if m and N have no factors in common.

1.36. Let $x(t)$ be the continuous-time complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

with fundamental frequency ω_0 and fundamental period $T_0 = 2\pi/\omega_0$. Consider the discrete-time signal obtained by taking equally spaced samples of $x(t)$—that is,

$$x[n] = x(nT) = e^{j\omega_0 nT}.$$

(a) Show that $x[n]$ is periodic if and only if T/T_0 is a rational number—that is, if and only if some multiple of the sampling interval *exactly equals* a multiple of the period of $x(t)$.

(b) Suppose that $x[n]$ is periodic—that is, that

$$\frac{T}{T_0} = \frac{p}{q}, \qquad \text{(P1.36–1)}$$

where p and q are integers. What are the fundamental period and fundamental frequency of $x[n]$? Express the fundamental frequency as a fraction of $\omega_0 T$.

(c) Again assuming that T/T_0 satisfies eq. (P1.36–1), determine precisely how many periods of $x(t)$ are needed to obtain the samples that form a single period of $x[n]$.

1.37. An important concept in many communications applications is the *correlation* between two signals. In the problems at the end of Chapter 2, we will have more to say about this topic and will provide some indication of how it is used in practice. For now, we content ourselves with a brief introduction to correlation functions and some of their properties.

Let $x(t)$ and $y(t)$ be two signals; then the *correlation function* is defined as

$$\phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau.$$

The function $\phi_{xx}(t)$ is usually referred to as the *autocorrelation function* of the signal $x(t)$, while $\phi_{xy}(t)$ is often called a *cross-correlation function*.

(a) What is the relationship between $\phi_{xy}(t)$ and $\phi_{yx}(t)$?

(b) Compute the odd part of $\phi_{xx}(t)$.

(c) Suppose that $y(t) = x(t+T)$. Express $\phi_{xy}(t)$ and $\phi_{yy}(t)$ in terms of $\phi_{xx}(t)$.

1.38. In this problem, we examine a few of the properties of the unit impulse function.

(a) Show that

$$\delta(2t) = \frac{1}{2}\delta(t).$$

Hint: Examine $\delta_\Delta(t)$. (See Figure 1.34.)

(b) In Section 1.4, we defined the continuous-time unit impulse as the limit of the signal $\delta_\Delta(t)$. More precisely, we defined several of the *properties* of $\delta(t)$ by examining the corresponding properties of $\delta_\Delta(t)$. For example, since the signal

$$u_\Delta(t) = \int_{-\infty}^{t} \delta_\Delta(\tau)d\tau$$

converges to the unit step

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t), \tag{P1.38–1}$$

we could interpret $\delta(t)$ through the equation

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

or by viewing $\delta(t)$ as the formal derivative of $u(t)$.

This type of discussion is important, as we are in effect trying to define $\delta(t)$ through its properties rather than by specifying its value for each t, which is not possible. In Chapter 2, we provide a very simple characterization of the behavior of the unit impulse that is extremely useful in the study of linear time-invariant systems. For the present, however, we concentrate on demonstrating that the important concept in using the unit impulse is to understand *how* it behaves. To do this, consider the six signals depicted in Figure P1.38. Show

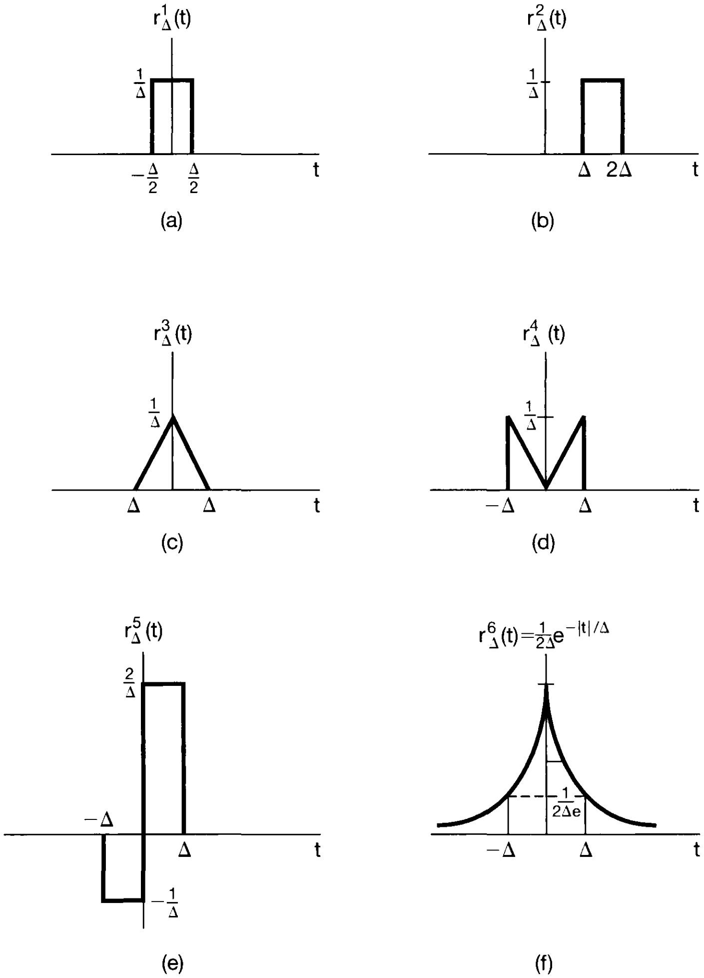


Figure P1.38

that each "behaves like an impulse" as $\Delta \to 0$ in that, if we let

$$u_\Delta^i(t) = \int_{-\infty}^{t} r_\Delta^i(\tau)d\tau,$$

then

$$\lim_{\Delta \to 0} u_\Delta^i(t) = u(t).$$

In each case, sketch and label carefully the signal $u_\Delta^i(t)$. Note that

$$r_\Delta^2(0) = r_\Delta^4(0) = 0 \text{ for all } \Delta.$$

Therefore, it is not enough to define or to think of $\delta(t)$ as being zero for $t \neq 0$ and infinite for $t = 0$. Rather, it is properties such as eq. (P1.38–1) that define the impulse. In Section 2.5 we will define a whole class of signals known as *singularity functions*, which are related to the unit impulse and which are also defined in terms of their properties rather than their values.

1.39. The role played by $u(t)$, $\delta(t)$, and other singularity functions in the study of linear time-invariant systems is that of an *idealization* of a physical phenomenon, and, as we will see, the use of these idealizations allow us to obtain an exceedingly important and very simple representation of such systems. In using singularity functions, we need, however, to be careful. In particular, we must remember that they are idealizations, and thus, whenever we perform a calculation using them, we are implicitly assuming that this calculation represents an accurate description of the behavior of the signals that they are intended to idealize. To illustrate, consider the equation

$$x(t)\delta(t) = x(0)\delta(t). \tag{P1.39–1}$$

This equation is based on the observation that

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t). \tag{P1.39–2}$$

Taking the limit of this relationship then yields the idealized one given by eq. (P1.39–1). However, a more careful examination of our derivation of eq. (P1.39–2) shows that that equation really makes sense only if $x(t)$ is continuous at $t = 0$. If it is not, then we will not have $x(t) \approx x(0)$ for t small.

To make this point clearer, consider the unit step signal $u(t)$. Recall from eq. (1.70) that $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t > 0$, but that its value at $t = 0$ is not defined. [Note, for example, that $u_\Delta(0) = 0$ for all Δ, while $u_\Delta^1(0) = \frac{1}{2}$ (from Problem 1.38(b)).] The fact that $u(0)$ is not defined is not particularly bothersome, as long as the calculations we perform using $u(t)$ do not rely on a specific choice for $u(0)$. For example, if $f(t)$ is a signal that is continuous at $t = 0$, then the value of

$$\int_{-\infty}^{+\infty} f(\sigma)u(\sigma)d\sigma$$

does not depend upon a choice for $u(0)$. On the other hand, the fact that $u(0)$ is undefined is significant in that it means that certain calculations involving singularity functions are undefined. Consider trying to define a value for the product $u(t)\delta(t)$.

To see that this *cannot* be defined, show that

$$\lim_{\Delta \to 0} [u_\Delta(t)\delta(t)] = 0,$$

but

$$\lim_{\Delta \to 0} [u_\Delta(t)\delta_\Delta(t)] = \frac{1}{2}\delta(t).$$

In general, we can define the product of two signals without any difficulty, as long as the signals do not contain singularities (discontinuities, impulses, or the other singularities introduced in Section 2.5) whose locations coincide. When the locations do coincide, the product is undefined. As an example, show that the signal

$$g(t) = \int_{-\infty}^{+\infty} u(\tau)\delta(t-\tau)d\tau$$

is identical to $u(t)$; that is, it is 0 for $t < 0$, it equals 1 for $t > 0$, and it is undefined for $t = 0$.

1.40. **(a)** Show that if a system is *either* additive or homogeneous, it has the property that if the input is identically zero, then the output is also identically zero.

(b) Determine a system (either in continuous or discrete time) that is *neither* additive *nor* homogeneous but which has a zero output if the input is identically zero.

(c) From part (a), can you conclude that if the input to a linear system is zero between times t_1 and t_2 in continuous time or between times n_1 and n_2 in discrete time, then its output must also be zero between these same times? Explain your answer.

1.41. Consider a system S with input $x[n]$ and output $y[n]$ related by

$$y[n] = x[n]\{g[n] + g[n-1]\}.$$

(a) If $g[n] = 1$ for all n, show that S is time invariant.
(b) If $g[n] = n$, show that S is not time invariant.
(c) If $g[n] = 1 + (-1)^n$, show that S is time invariant.

1.42. **(a)** Is the following statement true or false?

The series interconnection of two linear time-invariant systems is itself a linear, time-invariant system.

Justify your answer.

(b) Is the following statement true or false?

The series interconnection of two nonlinear systems is itself nonlinear.

Justify your answer.

(c) Consider three systems with the following input-output relationships:

$$\text{System 1:} \qquad y[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases},$$

System 2: $y[n] = x[n] + \frac{1}{2}x[n-1] + \frac{1}{4}x[n-2]$,

System 3: $y[n] = x[2n]$.

Suppose that these systems are connected in series as depicted in Figure P1.42. Find the input-output relationship for the overall interconnected system. Is this system linear? Is it time invariant?

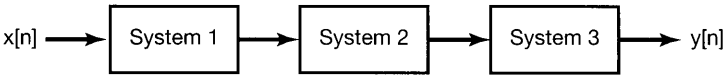


Figure P1.42

1.43. **(a)** Consider a time-invariant system with input $x(t)$ and output $y(t)$. Show that if $x(t)$ is periodic with period T, then so is $y(t)$. Show that the analogous result also holds in discrete time.

(b) Give an example of a time-invariant system and a nonperiodic input signal $x(t)$ such that the corresponding output $y(t)$ is periodic.

1.44. **(a)** Show that causality for a continuous-time linear system is equivalent to the following statement:

For any time t_0 and any input $x(t)$ such that $x(t) = 0$ for $t < t_0$, the corresponding output $y(t)$ must also be zero for $t < t_0$.

The analogous statement can be made for a discrete-time linear system.

(b) Find a nonlinear system that satisfies the foregoing condition but is not causal.

(c) Find a nonlinear system that is causal but does not satisfy the condition.

(d) Show that invertibility for a discrete-time linear system is equivalent to the following statement:

The only input that produces $y[n] = 0$ for all n is $x[n] = 0$ for all n.

The analogous statement is also true for a continuous-time linear system.

(e) Find a nonlinear system that satisfies the condition of part (d) but is not invertible.

1.45. In Problem 1.37, we introduced the concept of correlation functions. It is often important in practice to compute the correlation function $\phi_{hx}(t)$, where $h(t)$ is a fixed given signal, but where $x(t)$ may be any of a wide variety of signals. In this case, what is done is to design a system S with input $x(t)$ and output $\phi_{hx}(t)$.

(a) Is S linear? Is S time invariant? Is S causal? Explain your answers.

(b) Do any of your answers to part (a) change if we take as the output $\phi_{xh}(t)$ rather than $\phi_{hx}(t)$?

1.46. Consider the feedback system of Figure P1.46. Assume that $y[n] = 0$ for $n < 0$.

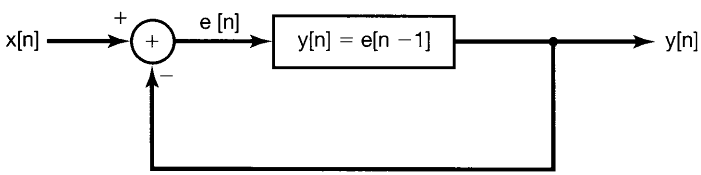


Figure P1.46

(a) Sketch the output when $x[n] = \delta[n]$.
(b) Sketch the output when $x[n] = u[n]$.

1.47. **(a)** Let S denote an incrementally linear system, and let $x_1[n]$ be an arbitrary input signal to S with corresponding output $y_1[n]$. Consider the system illustrated in Figure P1.47(a). Show that this system is linear and that, in fact, the overall input-output relationship between $x[n]$ and $y[n]$ does not depend on the particular choice of $x_1[n]$.

(b) Use the result of part (a) to show that S can be represented in the form shown in Figure 1.48.

(c) Which of the following systems are incrementally linear? Justify your answers, and if a system is incrementally linear, identify the linear system L and the zero-input response $y_0[n]$ or $y_0(t)$ for the representation of the system as shown in Figure 1.48.

(i) $y[n] = n + x[n] + 2x[n+4]$

(ii) $y[n] = \begin{cases} n/2, & n \text{ even} \\ (n-1)/2 + \sum_{k=-\infty}^{(n-1)/2} x[k], & n \text{ odd} \end{cases}$

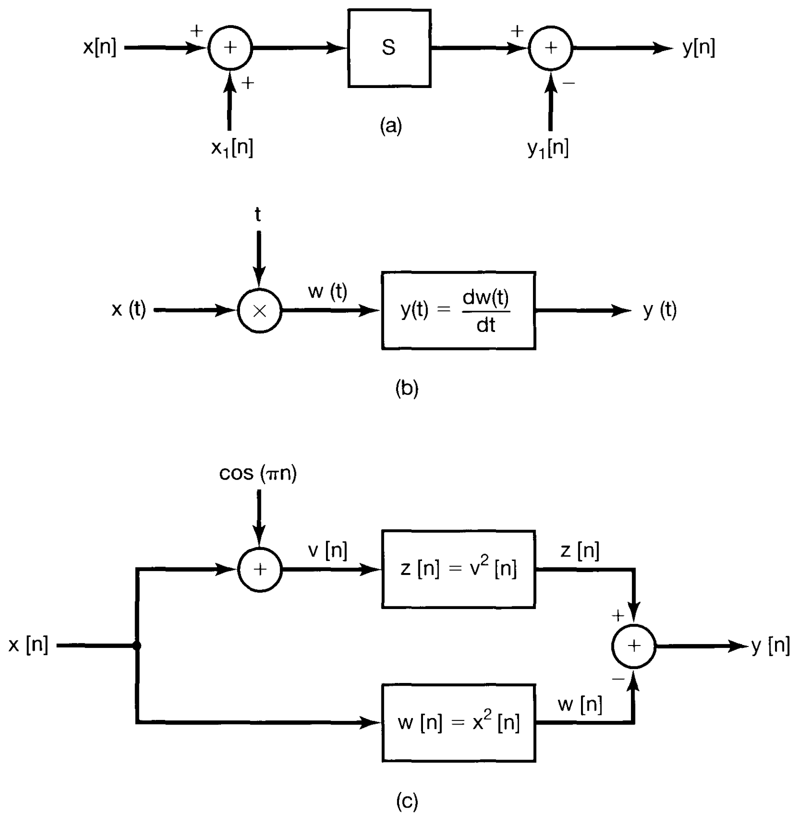


Figure P1.47

(iii) $y[n] = \begin{cases} x[n] - x[n-1] + 3, & \text{if } x[0] \geq 0 \\ x[n] - x[n-1] - 3, & \text{if } x[0] < 0 \end{cases}$

(iv) The system depicted in Figure P1.47(b).

(v) The system depicted in Figure P1.47(c).

(d) Suppose that a particular incrementally linear system has a representation as in Figure 1.48, with L denoting the linear system and $y_0[n]$ the zero-input response. Show that S is time invariant if and only if L is a time-invariant system *and* $y_0[n]$ is constant.

MATHEMATICAL REVIEW

The complex number z can be expressed in several ways. The *Cartesian* or *rectangular* form for z is

$$z = x + jy,$$

where $j = \sqrt{-1}$ and x and y are real numbers referred to respectively as the *real part* and the *imaginary part* of z. As we indicated earlier, we will often use the notation

$$x = \mathcal{Re}\{z\}, \; y = \mathcal{Im}\{z\}.$$

The complex number z can also be represented in *polar form* as

$$z = re^{j\theta},$$

where $r > 0$ is the *magnitude* of z and θ is the *angle* or *phase* of z. These quantities will often be written as

$$r = |z|, \theta = \sphericalangle z.$$

The relationship between these two representations of complex numbers can be determined either from *Euler's relation*,

$$e^{j\theta} = \cos\theta + j\sin\theta,$$

or by plotting z in the complex plane, as shown in Figure P1.48, in which the coordinate axes are $\mathcal{Re}\{z\}$ along the horizontal axis and $\mathcal{Im}\{z\}$ along the vertical axis. With respect to this graphical representation, x and y are the Cartesian coordinates of z, and r and θ are its polar coordinates.

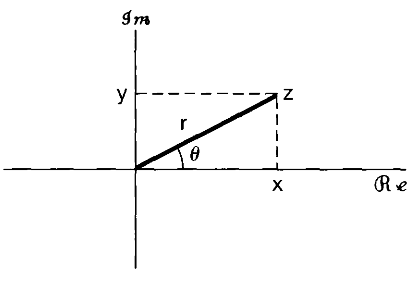


Figure P1.48

1.48. Let z_0 be a complex number with polar coordinates (r_0, θ_0) and Cartesian coordinates (x_0, y_0). Determine expressions for the Cartesian coordinates of the following complex numbers in terms of x_0 and y_0. Plot the points z_0, z_1, z_2, z_3, z_4, and z_5 in the complex plane when $r_0 = 2$ and $\theta_0 = \pi/4$ and when $r_0 = 2$ and $\theta_0 = \pi/2$. Indicate on your plots the real and imaginary parts of each point.

(a) $z_1 = r_0 e^{-j\theta_0}$ **(b)** $z_2 = r_0$ **(c)** $z_3 = r_0 e^{j(\theta_0+\pi)}$
(d) $z_4 = r_0 e^{j(-\theta_0+\pi)}$ **(e)** $z_5 = r_0 e^{j(\theta_0+2\pi)}$

1.49. Express each of the following complex numbers in polar form, and plot them in the complex plane, indicating the magnitude and angle of each number:

(a) $1 + j\sqrt{3}$ **(b)** -5 **(c)** $-5 - 5j$
(d) $3 + 4j$ **(e)** $(1 - j\sqrt{3})^3$ **(f)** $(1 + j)^5$
(g) $(\sqrt{3} + j^3)(1 - j)$ **(h)** $\frac{2-j(6/\sqrt{3})}{2+j(6/\sqrt{3})}$ **(i)** $\frac{1+j\sqrt{3}}{\sqrt{3}+j}$
(j) $j(1 + j)e^{j\pi/6}$ **(k)** $(\sqrt{3} + j)2\sqrt{2}e^{-j\pi/4}$ **(l)** $\frac{e^{j\pi/3}-1}{1+j\sqrt{3}}$

1.50. **(a)** Using Euler's relationship or Figure P1.48, determine expressions for x and y in terms of r and θ.
(b) Determine expressions for r and θ in terms of x and y.
(c) If we are given only r and $\tan\theta$, can we uniquely determine x and y? Explain your answer.

1.51. Using Euler's relation, derive the following relationships:
(a) $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
(b) $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$
(c) $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
(d) $(\sin\theta)(\sin\phi) = \frac{1}{2}\cos(\theta - \phi) - \frac{1}{2}\cos(\theta + \phi)$
(e) $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

1.52. Let z denote a complex variable; that is,

$$z = x + jy = re^{j\theta}.$$

The *complex conjugate* of z is

$$z^* = x - jy = re^{-j\theta}.$$

Derive each of the following relations, where z, z_1, and z_2 are arbitrary complex numbers:
(a) $zz^* = r^2$
(b) $\frac{z}{z^*} = e^{j2\theta}$
(c) $z + z^* = 2\mathcal{R}e\{z\}$
(d) $z - z^* = 2j\mathcal{I}m\{z\}$
(e) $(z_1 + z_2)^* = z_1^* + z_2^*$
(f) $(az_1z_2)^* = az_1^*z_2^*$, where a is any real number
(g) $(\frac{z_1}{z_2})^* = \frac{z_1^*}{z_2^*}$
(h) $\mathcal{R}e\{\frac{z_1}{z_2}\} = \frac{1}{2}[\frac{z_1z_2^*+z_1^*z_2}{z_2z_2^*}]$

1.53. Derive the following relations, where z, z_1, and z_2 are arbitrary complex numbers:
(a) $(e^z)^* = e^{z^*}$
(b) $z_1z_2^* + z_1^*z_2 = 2\mathcal{R}e\{z_1z_2^*\} = 2\mathcal{R}e\{z_1^*z_2\}$

(c) $|z| = |z^*|$
(d) $|z_1 z_2| = |z_1||z_2|$
(e) $\mathcal{R}e\{z\} \le |z|, \mathcal{I}m\{z\} \le |z|$
(f) $|z_1 z_2^* + z_1^* z_2| \le 2|z_1 z_2|$
(g) $(|z_1| - |z_2|)^2 \le |z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$

1.54. The relations considered in this problem are used on many occasions throughout the book.

(a) Prove the validity of the following expression:

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha}, & \text{for any complex number } \alpha \neq 1 \end{cases}.$$

This is often referred to as the *finite sum formula.*

(b) Show that if $|\alpha| < 1$, then

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.$$

This is often referred to as the *infinite sum formula.*

(c) Show also if $|\alpha| < 1$, then

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2}.$$

(d) Evaluate

$$\sum_{n=k}^{\infty} \alpha^n,$$

assuming that $|\alpha| < 1$.

1.55. Using the results from Problem 1.54, evaluate each of the following sums and express your answer in Cartesian (rectangular) form:

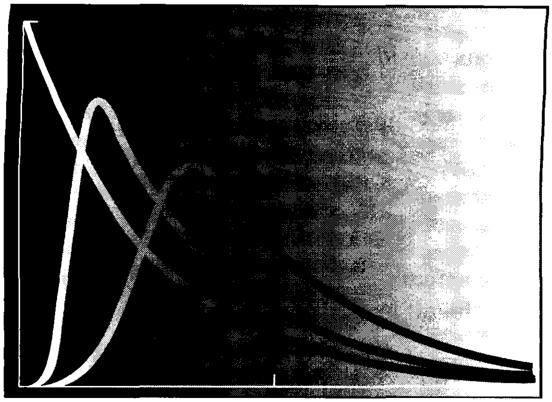
(a) $\sum_{n=0}^{9} e^{j\pi n/2}$ **(b)** $\sum_{n=-2}^{7} e^{j\pi n/2}$
(c) $\sum_{n=0}^{\infty} (\frac{1}{2})^n e^{j\pi n/2}$ **(d)** $\sum_{n=2}^{\infty} (\frac{1}{2})^n e^{j\pi n/2}$
(e) $\sum_{n=0}^{9} \cos(\frac{\pi}{2} n)$ **(f)** $\sum_{n=0}^{\infty} (\frac{1}{2})^n \cos(\frac{\pi}{2} n)$

1.56. Evaluate each of the following integrals, and express your answer in Cartesian (rectangular) form:

(a) $\int_0^4 e^{j\pi t/2} dt$ **(b)** $\int_0^6 e^{j\pi t/2} dt$
(c) $\int_2^8 e^{j\pi t/2} dt$ **(d)** $\int_0^\infty e^{-(1+j)t} dt$
(e) $\int_0^\infty e^{-t} \cos(t) dt$ **(f)** $\int_0^\infty e^{-2t} \sin(3t) dt$

2

LINEAR TIME-INVARIANT SYSTEMS



2.0 INTRODUCTION

In Section 1.6 we introduced and discussed a number of basic system properties. Two of these, linearity and time invariance, play a fundamental role in signal and system analysis for two major reasons. First, many physical processes possess these properties and thus can be modeled as linear time-invariant (LTI) systems. In addition, LTI systems can be analyzed in considerable detail, providing both insight into their properties and a set of powerful tools that form the core of signal and system analysis.

A principal objective of this book is to develop an understanding of these properties and tools and to provide an introduction to several of the very important applications in which the tools are used. In this chapter, we begin the development by deriving and examining a fundamental and extremely useful representation for LTI systems and by introducing an important class of these systems.

One of the primary reasons LTI systems are amenable to analysis is that any such system possesses the superposition property described in Section 1.6.6. As a consequence, if we can represent the input to an LTI system in terms of a linear combination of a set of basic signals, we can then use superposition to compute the output of the system in terms of its responses to these basic signals.

As we will see in the following sections, one of the important characteristics of the unit impulse, both in discrete time and in continuous time, is that very general signals can be represented as linear combinations of delayed impulses. This fact, together with the properties of superposition and time invariance, will allow us to develop a complete characterization of any LTI system in terms of its response to a unit impulse. Such a representation, referred to as the convolution sum in the discrete-time case and the convolution integral in continuous time, provides considerable analytical convenience in dealing with LTI systems. Following our development of the convolution sum and the convolution integral we use these characterizations to examine some of the other properties of LTI systems. We then consider the class of continuous-time systems described by linear constant-coefficient differential equations and its discrete-time counterpart, the class of systems described by linear constant-coefficient difference equations. We will return to examine these two very important classes of systems on a number of occasions in subsequent chapters. Finally, we will take another look at the continuous-time unit impulse function and a number of other signals that are closely related to it in order to provide some additional insight into these idealized signals and, in particular, to their use and interpretation in the context of analyzing LTI systems.

2.1 DISCRETE-TIME LTI SYSTEMS: THE CONVOLUTION SUM

2.1.1 The Representation of Discrete-Time Signals in Terms of Impulses

The key idea in visualizing how the discrete-time unit impulse can be used to construct any discrete-time signal is to think of a discrete-time signal as a sequence of individual impulses. To see how this intuitive picture can be turned into a mathematical representation, consider the signal $x[n]$ depicted in Figure 2.1(a). In the remaining parts of this figure, we have depicted five time-shifted, scaled unit impulse sequences, where the scaling on each impulse equals the value of $x[n]$ at the particular instant the unit sample occurs. For example,

$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases},$$

$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases},$$

$$x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}.$$

Therefore, the sum of the five sequences in the figure equals $x[n]$ for $-2 \leq n \leq 2$. More generally, by including additional shifted, scaled impulses, we can write

$$\begin{aligned} x[n] = \ldots &+ x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] \\ &+ x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots. \end{aligned} \tag{2.1}$$

For any value of n, only one of the terms on the right-hand side of eq. (2.1) is nonzero, and the scaling associated with that term is precisely $x[n]$. Writing this summation in a more compact form, we have

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]. \tag{2.2}$$

This corresponds to the representation of an arbitrary sequence as a linear combination of shifted unit impulses $\delta[n-k]$, where the weights in this linear combination are $x[k]$. As an example, consider $x[n] = u[n]$, the unit step. In this case, since $u[k] = 0$ for $k < 0$

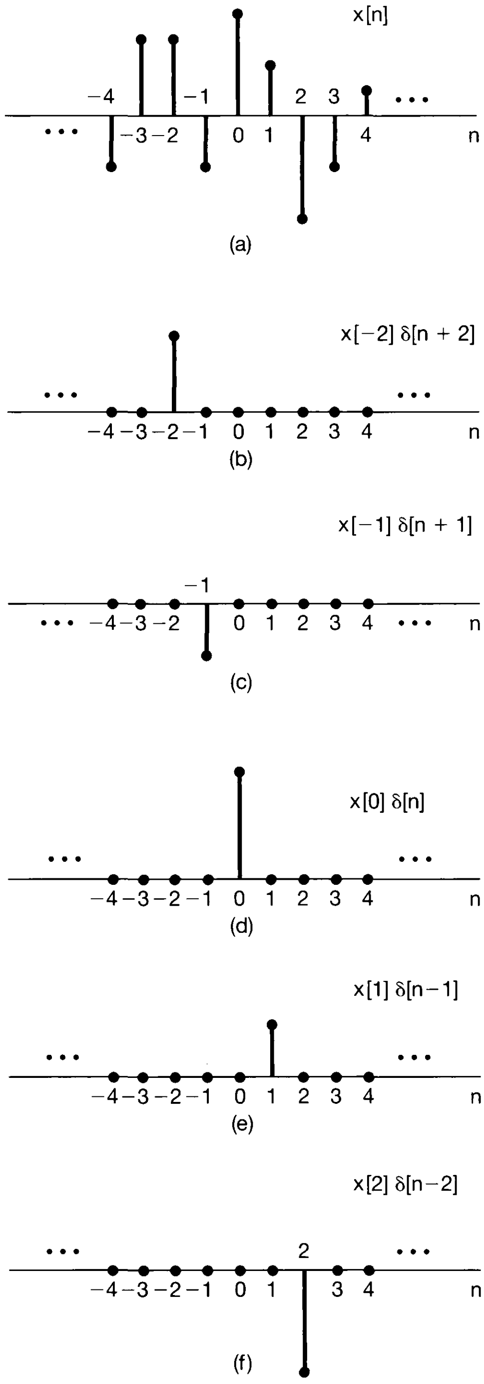


Figure 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

and $u[k] = 1$ for $k \geq 0$, eq. (2.2) becomes

$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k],$$

which is identical to the expression we derived in Section 1.4. [See eq. (1.67).]

Equation (2.2) is called the *sifting property* of the discrete-time unit impulse. Because the sequence $\delta[n-k]$ is nonzero only when $k = n$, the summation on the right-hand side of eq. (2.2) "sifts" through the sequence of values $x[k]$ and preserves only the value corresponding to $k = n$. In the next subsection, we will exploit this representation of discrete-time signals in order to develop the convolution-sum representation for a discrete-time LTI system.

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

The importance of the sifting property of eqs. (2.1) and (2.2) lies in the fact that it represents $x[n]$ as a superposition of scaled versions of a very simple set of elementary functions, namely, shifted unit impulses $\delta[n-k]$, each of which is nonzero (with value 1) at a single point in time specified by the corresponding value of k. The response of a linear system to $x[n]$ will be the superposition of the scaled responses of the system to each of these shifted impulses. Moreover, the property of time invariance tells us that the responses of a time-invariant system to the time-shifted unit impulses are simply time-shifted versions of one another. The convolution-sum representation for discrete-time systems that are *both* linear and time invariant results from putting these two basic facts together.

More specifically, consider the response of a linear (but possibly time-varying) system to an arbitrary input $x[n]$. We can represent the input through eq. (2.2) as a linear combination of shifted unit impulses. Let $h_k[n]$ denote the response of the linear system to the shifted unit impulse $\delta[n-k]$. Then, from the superposition property for a linear system [eqs. (1.123) and (1.124)], the response $y[n]$ of the linear system to the input $x[n]$ in eq. (2.2) is simply the weighted linear combination of these basic responses. That is, with the input $x[n]$ to a linear system expressed in the form of eq. (2.2), the output $y[n]$ can be expressed as

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]. \tag{2.3}$$

Thus, according to eq. (2.3), if we know the response of a linear system to the set of shifted unit impulses, we can construct the response to an arbitrary input. An interpretation of eq. (2.3) is illustrated in Figure 2.2. The signal $x[n]$ is applied as the input to a linear system whose responses $h_{-1}[n]$, $h_0[n]$, and $h_1[n]$ to the signals $\delta[n+1]$, $\delta[n]$, and $\delta[n-1]$, respectively, are depicted in Figure 2.2(b). Since $x[n]$ can be written as a linear combination of $\delta[n+1]$, $\delta[n]$, and $\delta[n-1]$, superposition allows us to write the response to $x[n]$ as a linear combination of the responses to the individual shifted impulses. The individual shifted and scaled impulses that constitute $x[n]$ are illustrated on the left-hand side of Figure 2.2(c), while the responses to these component signals are pictured on the right-hand side. In Figure 2.2(d) we have depicted the actual input $x[n]$, which is the sum of the components on the left side of Figure 2.2(c) and the actual output $y[n]$, which, by

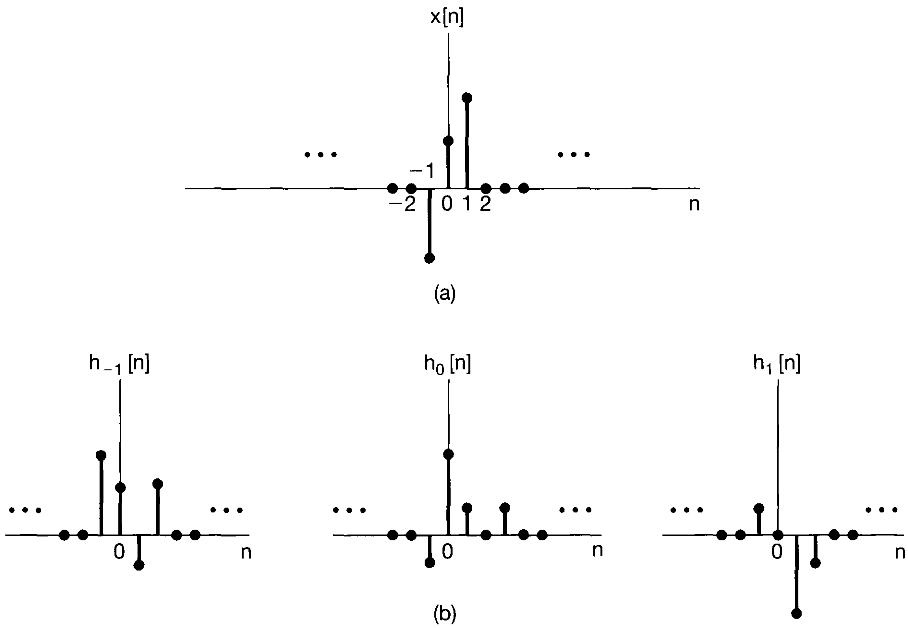


Figure 2.2 Graphical interpretation of the response of a discrete-time linear system as expressed in eq. (2.3).

superposition, is the sum of the components on the right side of Figure 2.2(c). Thus, the response at time n of a linear system is simply the superposition of the responses due to the input value at each point in time.

In general, of course, the responses $h_k[n]$ need not be related to each other for different values of k. However, if the linear system is also *time invariant*, then these responses to time-shifted unit impulses are all time-shifted versions of each other. Specifically, since $\delta[n-k]$ is a time-shifted version of $\delta[n]$, the response $h_k[n]$ is a time-shifted version of $h_0[n]$; i.e.,

$$h_k[n] = h_0[n-k]. \tag{2.4}$$

For notational convenience, we will drop the subscript on $h_0[n]$ and define the *unit impulse (sample) response*

$$h[n] = h_0[n]. \tag{2.5}$$

That is, $h[n]$ is the output of the LTI system when $\delta[n]$ is the input. Then for an LTI system, eq. (2.3) becomes

$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].} \tag{2.6}$$

This result is referred to as the *convolution sum* or *superposition sum*, and the operation on the right-hand side of eq. (2.6) is known as the *convolution* of the sequences $x[n]$ and $h[n]$. We will represent the operation of convolution symbolically as

$$y[n] = x[n] * h[n]. \tag{2.7}$$

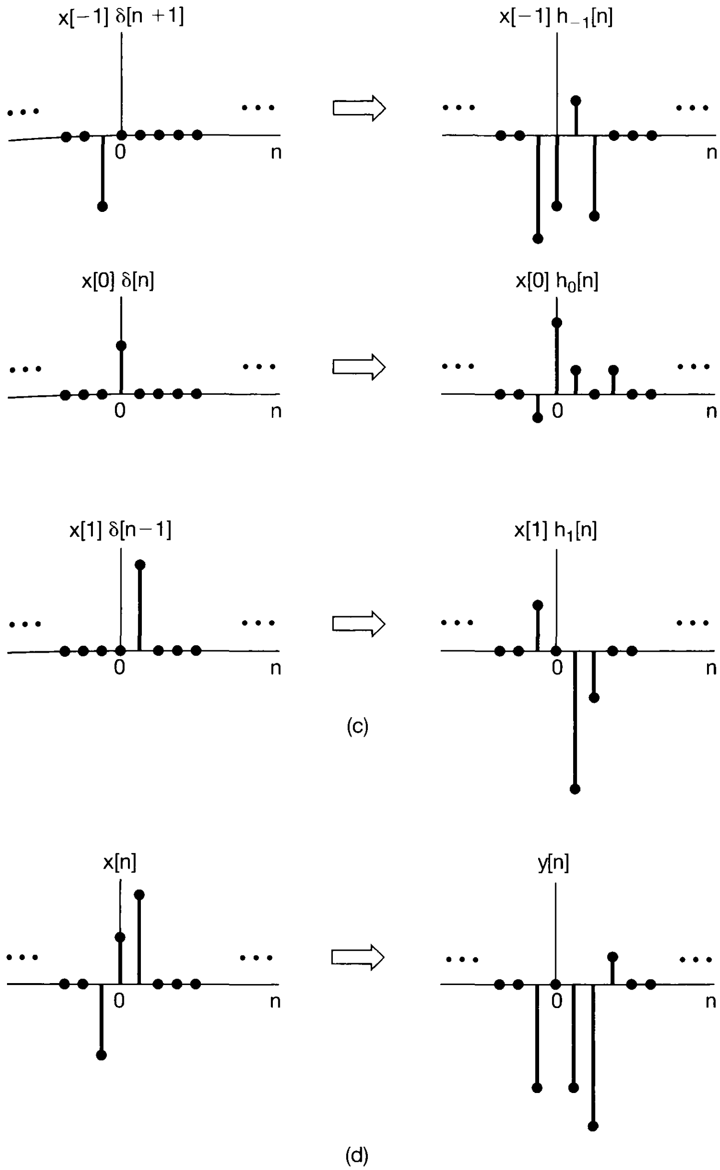


Figure 2.2 *Continued*

Note that eq. (2.6) expresses the response of an LTI system to an arbitrary input in terms of the system's response to the unit impulse. From this, we see that an LTI system is completely characterized by its response to a single signal, namely, its response to the unit impulse.

The interpretation of eq. (2.6) is similar to the one we gave for eq. (2.3), where, in the case of an LTI system, the response due to the input $x[k]$ applied at time k is $x[k]h[n-k]$; i.e., it is a shifted and scaled version (an "echo") of $h[n]$. As before, the actual output is the superposition of all these responses.

Example 2.1

Consider an LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated in Figure 2.3(a). For this case, since only $x[0]$ and $x[1]$ are nonzero, eq. (2.6) simplifies to the expression

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]. \tag{2.8}$$

The sequences $0.5h[n]$ and $2h[n-1]$ are the two echoes of the impulse response needed for the superposition involved in generating $y[n]$. These echoes are displayed in Figure 2.3(b). By summing the two echoes for each value of n, we obtain $y[n]$, which is shown in Figure 2.3(c).

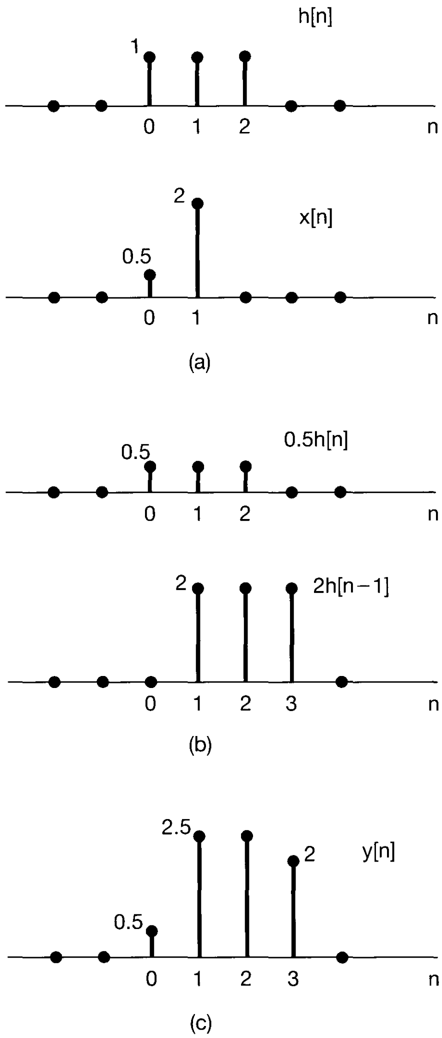


Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

By considering the effect of the superposition sum on each individual output sample, we obtain another very useful way to visualize the calculation of $y[n]$ using the convolution sum. In particular, consider the evaluation of the output value at some specific time n. A particularly convenient way of displaying this calculation graphically begins with the two signals $x[k]$ and $h[n-k]$ viewed as functions of k. Multiplying these two functions, we obtain a sequence $g[k] = x[k]h[n-k]$, which, at each time k, is seen to represent the contribution of $x[k]$ to the output at time n. We conclude that summing all the samples in the sequence of $g[k]$ yields the output value at the selected time n. Thus, to calculate $y[n]$ for all values of n requires repeating this procedure for each value of n. Fortunately, changing the value of n has a very simple graphical interpretation for the two signals $x[k]$ and $h[n-k]$, viewed as functions of k. The following examples illustrate this and the use of the aforementioned viewpoint in evaluating convolution sums.

Example 2.2

Let us consider again the convolution problem encountered in Example 2.1. The sequence $x[k]$ is shown in Figure 2.4(a), while the sequence $h[n-k]$, for n fixed and viewed as a function of k, is shown in Figure 2.4(b) for several different values of n. In sketching these sequences, we have used the fact that $h[n-k]$ (viewed as a function of k with n fixed) is a time-reversed and shifted version of the impulse response $h[k]$. In particular, as k increases, the argument $n-k$ decreases, explaining the need to perform a time reversal of $h[k]$. Knowing this, then in order to sketch the signal $h[n-k]$, we need only determine its value for some particular value of k. For example, the argument $n-k$ will equal 0 at the value $k = n$. Thus, if we sketch the signal $h[-k]$, we can obtain the signal $h[n-k]$ simply by shifting to the right (by n) if n is positive or to the left if n is negative. The result for our example for values of $n < 0$, $n = 0, 1, 2, 3$, and $n > 3$ are shown in Figure 2.4(b).

Having sketched $x[k]$ and $h[n-k]$ for any particular value of n, we multiply these two signals and sum over all values of k. For our example, for $n < 0$, we see from Figure 2.4 that $x[k]h[n-k] = 0$ for all k, since the nonzero values of $x[k]$ and $h[n-k]$ do not overlap. Consequently, $y[n] = 0$ for $n < 0$. For $n = 0$, since the product of the sequence $x[k]$ with the sequence $h[0-k]$ has only one nonzero sample with the value 0.5, we conclude that

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5. \tag{2.9}$$

The product of the sequence $x[k]$ with the sequence $h[1-k]$ has two nonzero samples, which may be summed to obtain

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5. \tag{2.10}$$

Similarly,

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5, \tag{2.11}$$

and

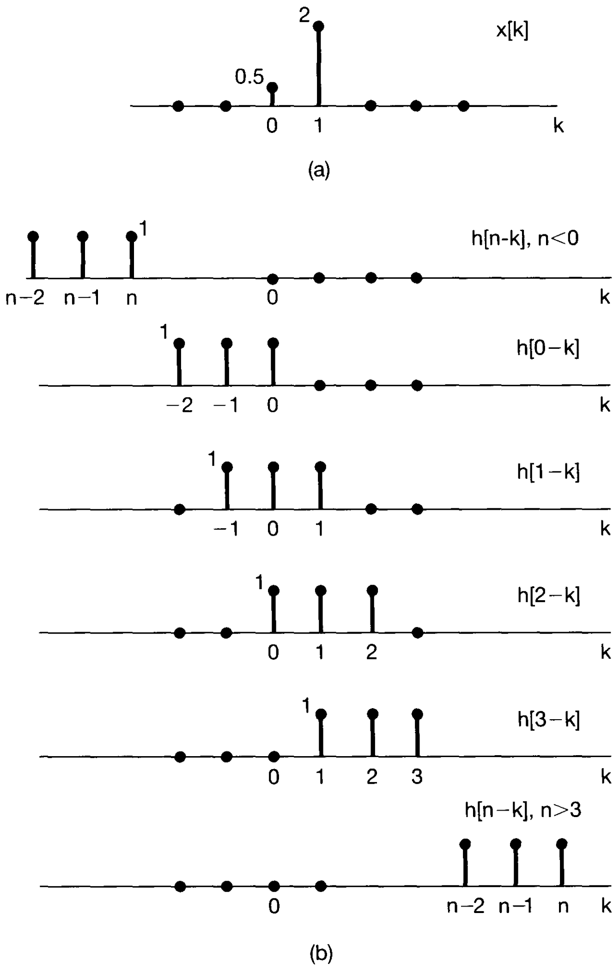


Figure 2.4 Interpretation of eq. (2.6) for the signals $h[n]$ and $x[n]$ in Figure 2.3; (a) the signal $x[k]$ and (b) the signal $h[n-k]$ (as a function of k with n fixed) for several values of n ($n < 0$; $n = 0, 1, 2, 3$; $n > 3$). Each of these signals is obtained by reflection and shifting of the unit impulse response $h[k]$. The response $y[n]$ for each value of n is obtained by multiplying the signals $x[k]$ and $h[n-k]$ in (a) and (b) and then summing the products over all values of k. The calculation for this example is carried out in detail in Example 2.2.

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0. \tag{2.12}$$

Finally, for $n > 3$, the product $x[k]h[n-k]$ is zero for all k, from which we conclude that $y[n] = 0$ for $n > 3$. The resulting output values agree with those obtained in Example 2.1.

Example 2.3

Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \alpha^n u[n],$$
$$h[n] = u[n],$$

with $0 < \alpha < 1$. These signals are illustrated in Figure 2.5. Also, to help us in visualizing and calculating the convolution of the signals, in Figure 2.6 we have depicted the signal $x[k]$ followed by $h[-k]$, $h[-1-k]$, and $h[1-k]$ (that is, $h[n-k]$ for $n = 0, -1$, and $+1$) and, finally, $h[n-k]$ for an arbitrary positive value of n and an arbitrary negative value of n. From this figure, we note that for $n < 0$, there is no overlap between the nonzero points in $x[k]$ and $h[n-k]$. Thus, for $n < 0$, $x[k]h[n-k] = 0$ for all values of k, and hence, from eq. (2.6), we see that $y[n] = 0$, $n < 0$. For $n \geq 0$,

$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}.$$

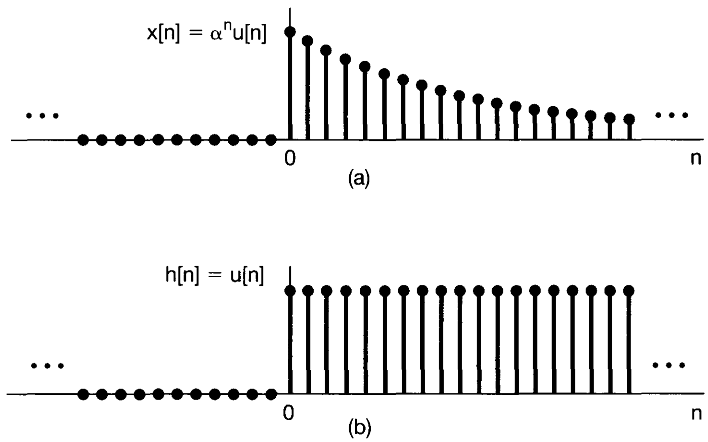


Figure 2.5 The signals $x[n]$ and $h[n]$ in Example 2.3.

Thus, for $n \geq 0$,

$$y[n] = \sum_{k=0}^{n} \alpha^k,$$

and using the result of Problem 1.54 we can write this as

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad \text{for } n \geq 0. \tag{2.13}$$

Thus, for all n,

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)u[n].$$

The signal $y[n]$ is sketched in Figure 2.7.

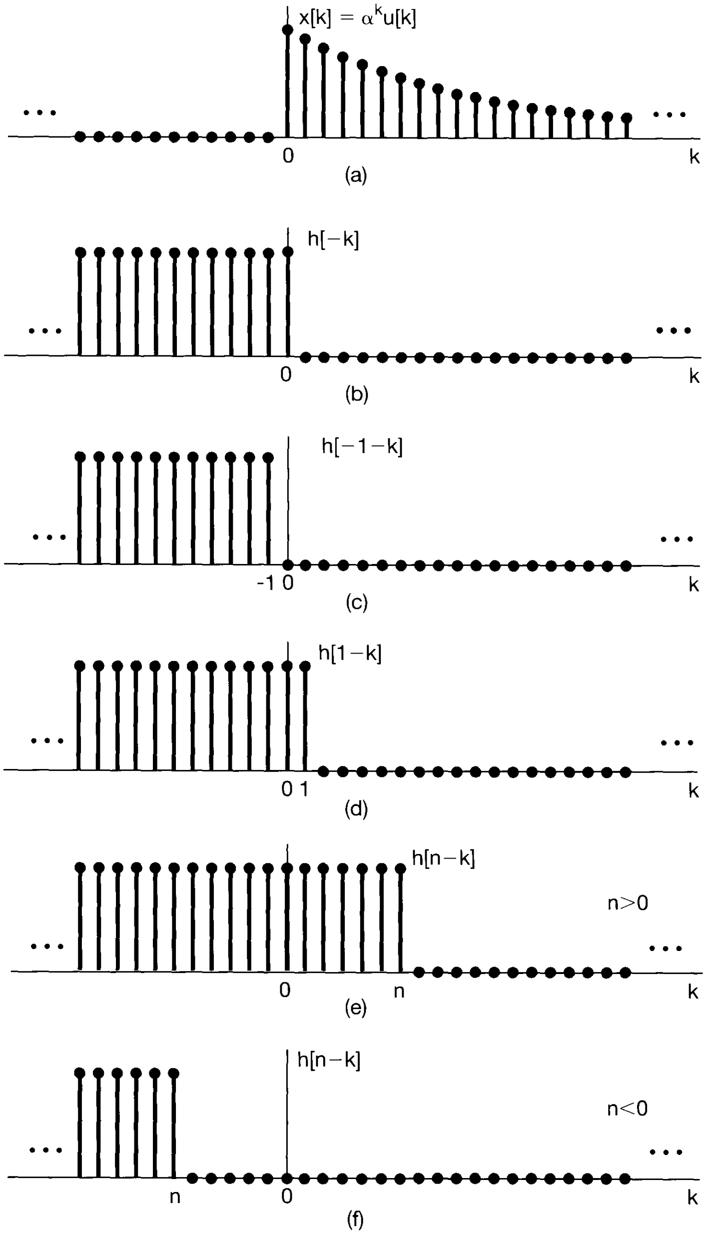


Figure 2.6 Graphical interpretation of the calculation of the convolution sum for Example 2.3.

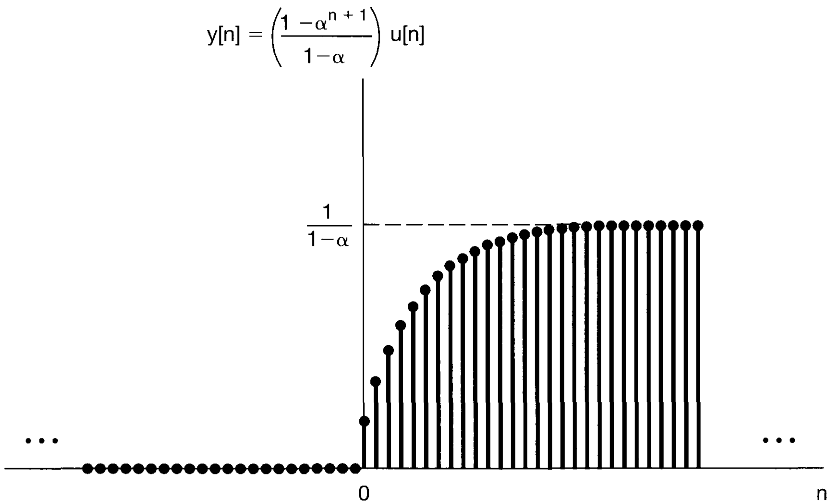


Figure 2.7 Output for Example 2.3.

The operation of convolution is sometimes described in terms of "sliding" the sequence $h[n-k]$ past $x[k]$. For example, suppose we have evaluated $y[n]$ for some particular value of n, say, $n = n_0$. That is, we have sketched the signal $h[n_0 - k]$, multiplied it by the signal $x[k]$, and summed the result over all values of k. To evaluate $y[n]$ at the next value of n—i.e., $n = n_0 + 1$—we need to sketch the signal $h[(n_0+1)-k]$. However, we can do this simply by taking the signal $h[n_0 - k]$ and shifting it to the right by one point. For each successive value of n, we continue this process of shifting $h[n-k]$ to the right by one point, multiplying by $x[k]$, and summing the result over k.

Example 2.4

As a further example, consider the two sequences

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}.$$

These signals are depicted in Figure 2.8 for a positive value of $\alpha > 1$. In order to calculate the convolution of the two signals, it is convenient to consider five separate intervals for n. This is illustrated in Figure 2.9.

Interval 1. For $n < 0$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and consequently, $y[n] = 0$.

Interval 2. For $0 \le n \le 4$,

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}.$$

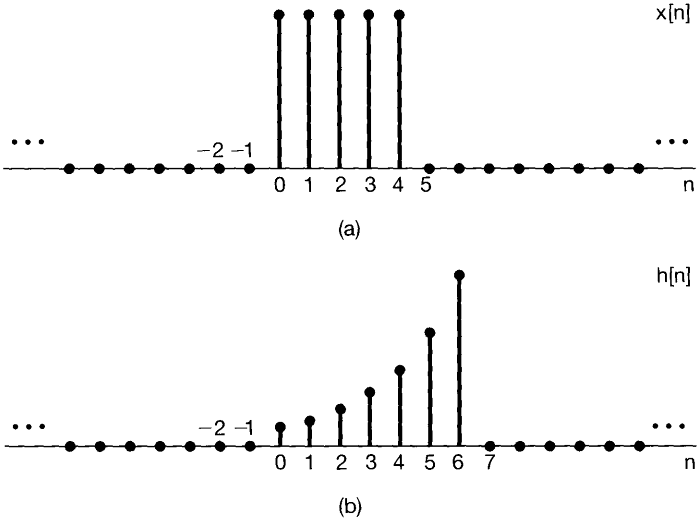


Figure 2.8 The signals to be convolved in Example 2.4.

Thus, in this interval,

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}. \tag{2.14}$$

We can evaluate this sum using the finite sum formula, eq. (2.13). Specifically, changing the variable of summation in eq. (2.14) from k to $r = n - k$, we obtain

$$y[n] = \sum_{r=0}^{n} \alpha^{r} = \frac{1-\alpha^{n+1}}{1-\alpha}.$$

Interval 3. For $n > 4$ but $n - 6 \leq 0$ (i.e., $4 < n \leq 6$),

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}.$$

Thus, in this interval,

$$y[n] = \sum_{k=0}^{4} \alpha^{n-k}. \tag{2.15}$$

Once again, we can use the geometric sum formula in eq. (2.13) to evaluate eq. (2.15). Specifically, factoring out the constant factor of α^n from the summation in eq. (2.15) yields

$$y[n] = \alpha^{n} \sum_{k=0}^{4} (\alpha^{-1})^{k} = \alpha^{n} \frac{1-(\alpha^{-1})^{5}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}. \tag{2.16}$$

Interval 4. For $n > 6$ but $n - 6 \leq 4$ (i.e., for $6 < n \leq 10$),

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases},$$

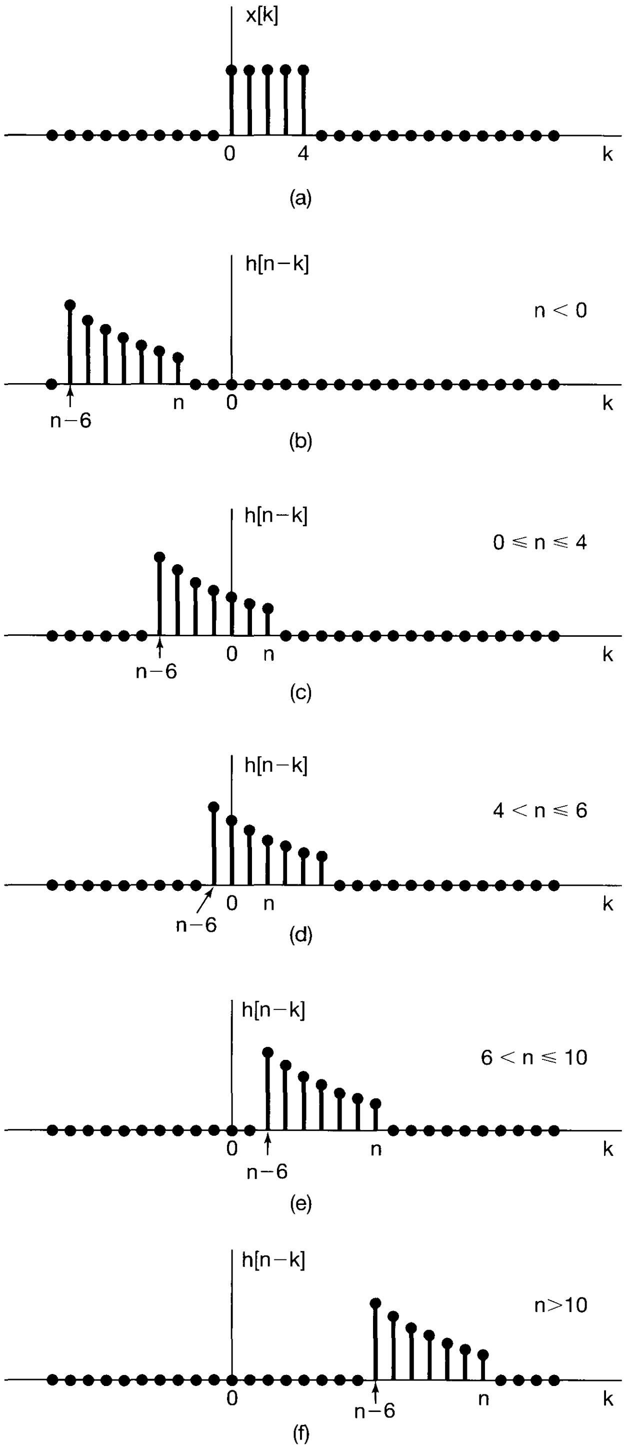


Figure 2.9 Graphical interpretation of the convolution performed in Example 2.4.

so that

$$y[n] = \sum_{k=n-6}^{4} \alpha^{n-k}.$$

We can again use eq. (2.13) to evaluate this summation. Letting $r = k-n+6$, we obtain

$$y[n] = \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^6 \sum_{r=0}^{10-n} (\alpha^{-1})^r = \alpha^6 \frac{1-\alpha^{n-11}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^7}{1-\alpha}.$$

Interval 5. For $n - 6 > 4$, or equivalently, $n > 10$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and hence,

$$y[n] = 0.$$

Summarizing, then, we obtain

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \leq n \leq 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \leq 6 \\ \dfrac{\alpha^{n-4}-\alpha^7}{1-\alpha}, & 6 < n \leq 10 \\ 0, & 10 < n \end{cases},$$

which is pictured in Figure 2.10.

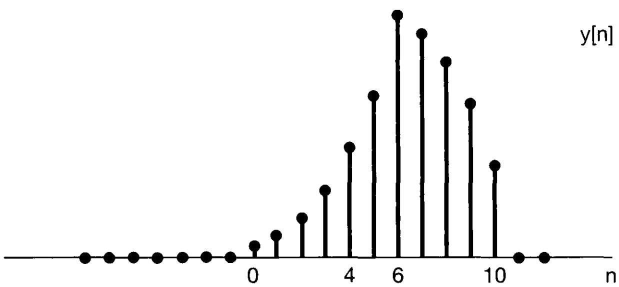


Figure 2.10 Result of performing the convolution in Example 2.4.

Example 2.5

Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ specified as follows:

$$x[n] = 2^n u[-n], \tag{2.17}$$

$$h[n] = u[n]. \tag{2.18}$$

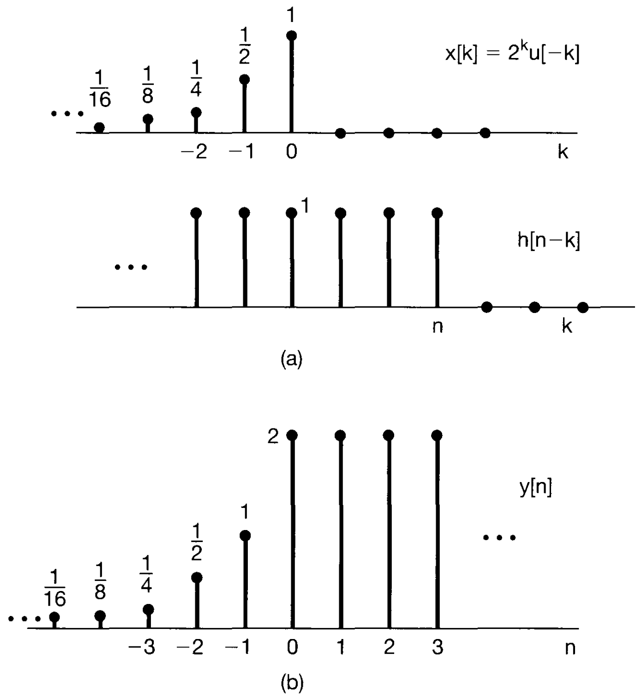


Figure 2.11 (a) The sequences $x[k]$ and $h[n-k]$ for the convolution problem considered in Example 2.5; (b) the resulting output signal $y[n]$.

The sequences $x[k]$ and $h[n-k]$ are plotted as functions of k in Figure 2.11(a). Note that $x[k]$ is zero for $k > 0$ and $h[n-k]$ is zero for $k > n$. We also observe that, regardless of the value of n, the sequence $x[k]h[n-k]$ always has nonzero samples along the k-axis. When $n \geq 0$, $x[k]h[n-k]$ has nonzero samples in the interval $k \leq 0$. It follows that, for $n \geq 0$,

$$y[n] = \sum_{k=-\infty}^{0} x[k]h[n-k] = \sum_{k=-\infty}^{0} 2^k. \tag{2.19}$$

To evaluate the infinite sum in eq. (2.19), we may use the *infinite sum formula*,

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}, \quad 0 < |\alpha| < 1. \tag{2.20}$$

Changing the variable of summation in eq. (2.19) from k to $r = -k$, we obtain

$$\sum_{k=-\infty}^{0} 2^k = \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-(1/2)} = 2. \tag{2.21}$$

Thus, $y[n]$ takes on a constant value of 2 for $n \geq 0$.

When $n < 0$, $x[k]h[n-k]$ has nonzero samples for $k \le n$. It follows that, for $n < 0$,

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=-\infty}^{n} 2^k. \tag{2.22}$$

By performing a change of variable $l = -k$ and then $m = l + n$, we can again make use of the infinite sum formula, eq. (2.20), to evaluate the sum in eq. (2.22). The result is the following for $n < 0$:

$$y[n] = \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n} = \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1}. \tag{2.23}$$

The complete sequence of $y[n]$ is sketched in Figure 2.11(b).

These examples illustrate the usefulness of visualizing the calculation of the convolution sum graphically. Moreover, in addition to providing a useful way in which to calculate the response of an LTI system, the convolution sum also provides an extremely useful representation for LTI systems that allows us to examine their properties in great detail. In particular, in Section 2.3 we will describe some of the properties of convolution and will also examine some of the system properties introduced in the previous chapter in order to see how these properties can be characterized for LTI systems.

2.2 CONTINUOUS-TIME LTI SYSTEMS: THE CONVOLUTION INTEGRAL

In analogy with the results derived and discussed in the preceding section, the goal of this section is to obtain a complete characterization of a continuous-time LTI system in terms of its unit impulse response. In discrete time, the key to our developing the convolution sum was the sifting property of the discrete-time unit impulse—that is, the mathematical representation of a signal as the superposition of scaled and shifted unit impulse functions. Intuitively, then, we can think of the discrete-time system as responding to a sequence of individual impulses. In continuous time, of course, we do not have a discrete sequence of input values. Nevertheless, as we discussed in Section 1.4.2, if we think of the unit impulse as the idealization of a pulse which is so short that its duration is inconsequential for any real, physical system, we can develop a representation for arbitrary continuous-time signals in terms of these idealized pulses with vanishingly small duration, or equivalently, impulses. This representation is developed in the next subsection, and, following that, we will proceed very much as in Section 2.1 to develop the convolution integral representation for continuous-time LTI systems.

2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

To develop the continuous-time counterpart of the discrete-time sifting property in eq. (2.2), we begin by considering a pulse or "staircase" approximation, $\hat{x}(t)$, to a continuous-time signal $x(t)$, as illustrated in Figure 2.12(a). In a manner similar to that

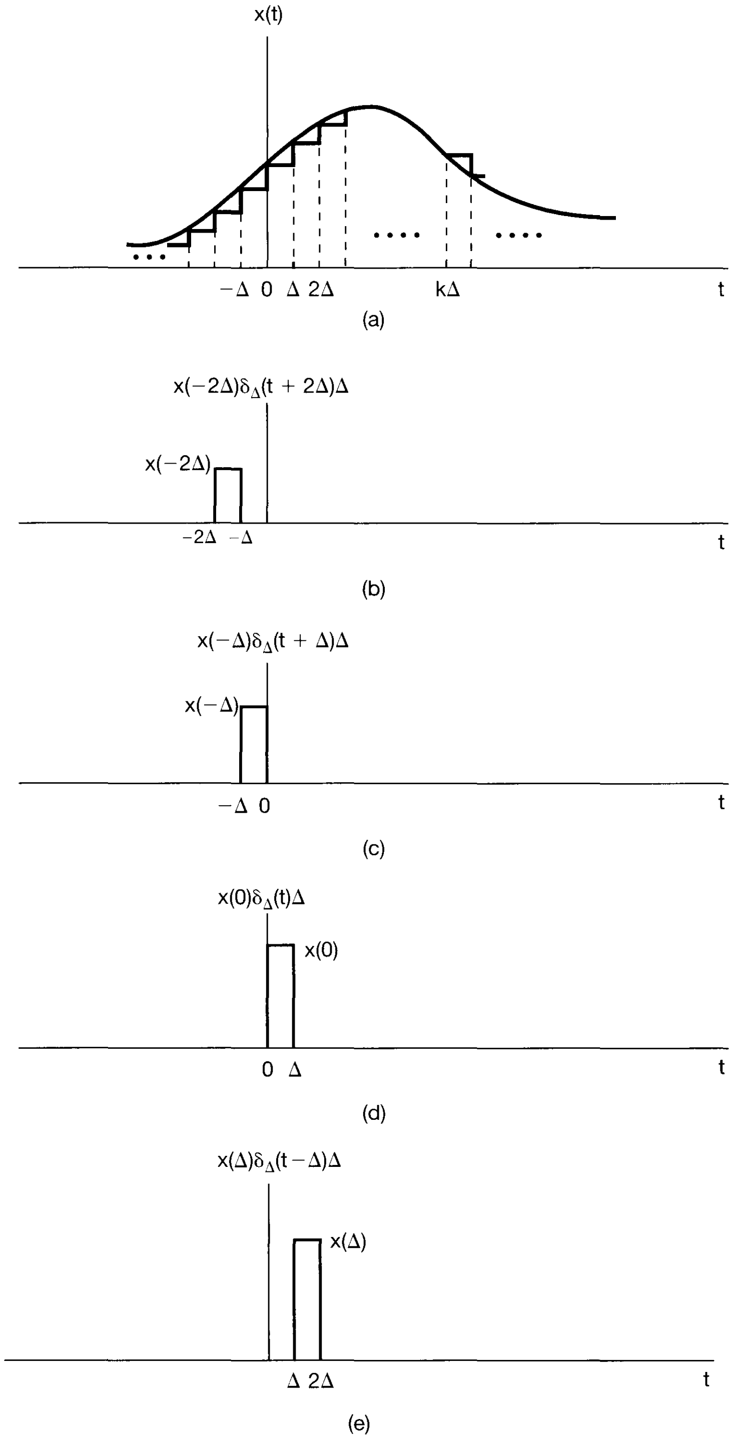


Figure 2.12 Staircase approximation to a continuous-time signal.

employed in the discrete-time case, this approximation can be expressed as a linear combination of delayed pulses, as illustrated in Figure 2.12(a)–(e). If we define

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t < \Delta \\ 0, & \text{otherwise} \end{cases}, \tag{2.24}$$

then, since $\Delta\delta_\Delta(t)$ has unit amplitude, we have the expression

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta. \tag{2.25}$$

From Figure 2.12, we see that, as in the discrete-time case [eq. (2.2)], for any value of t, only one term in the summation on the right-hand side of eq. (2.25) is nonzero.

As we let Δ approach 0, the approximation $\hat{x}(t)$ becomes better and better, and in the limit equals $x(t)$. Therefore,

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta. \tag{2.26}$$

Also, as $\Delta \to 0$, the summation in eq. (2.26) approaches an integral. This can be seen by considering the graphical interpretation of the equation, illustrated in Figure 2.13. Here, we have illustrated the signals $x(\tau)$, $\delta_\Delta(t - \tau)$, and their product. We have also indicated a shaded region whose area approaches the area under $x(\tau)\delta_\Delta(t - \tau)$ as $\Delta \to 0$. Note that the shaded region has an area equal to $x(m\Delta)$ where $t - \Delta < m\Delta < t$. Furthermore, for this value of t, only the term with $k = m$ is nonzero in the summation in eq. (2.26), and thus, the right-hand side of this equation also equals $x(m\Delta)$. Consequently, it follows from eq. (2.26) and from the preceding argument that $x(t)$ equals the limit as $\Delta \to 0$ of the area under $x(\tau)\delta_\Delta(t - \tau)$. Moreover, from eq. (1.74), we know that the limit as $\Delta \to 0$ of $\delta_\Delta(t)$ is the unit impulse function $\delta(t)$. Consequently,

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau. \tag{2.27}$$

As in discrete time, we refer to eq. (2.27) as the *sifting property* of the continuous-time impulse. We note that, for the specific example of $x(t) = u(t)$, eq. (2.27) becomes

$$u(t) = \int_{-\infty}^{+\infty} u(\tau)\delta(t - \tau)d\tau = \int_{0}^{\infty} \delta(t - \tau)d\tau, \tag{2.28}$$

since $u(\tau) = 0$ for $\tau < 0$ and $u(\tau) = 1$ for $\tau > 0$. Equation (2.28) is identical to eq. (1.75), derived in Section 1.4.2.

Once again, eq. (2.27) should be viewed as an idealization in the sense that, for Δ "small enough," the approximation of $x(t)$ in eq. (2.25) is essentially exact for any practical purpose. Equation (2.27) then simply represents an idealization of eq. (2.25) by taking Δ to be vanishingly small. Note also that we could have derived eq. (2.27) directly by using several of the basic properties of the unit impulse that we derived in Section 1.4.2.

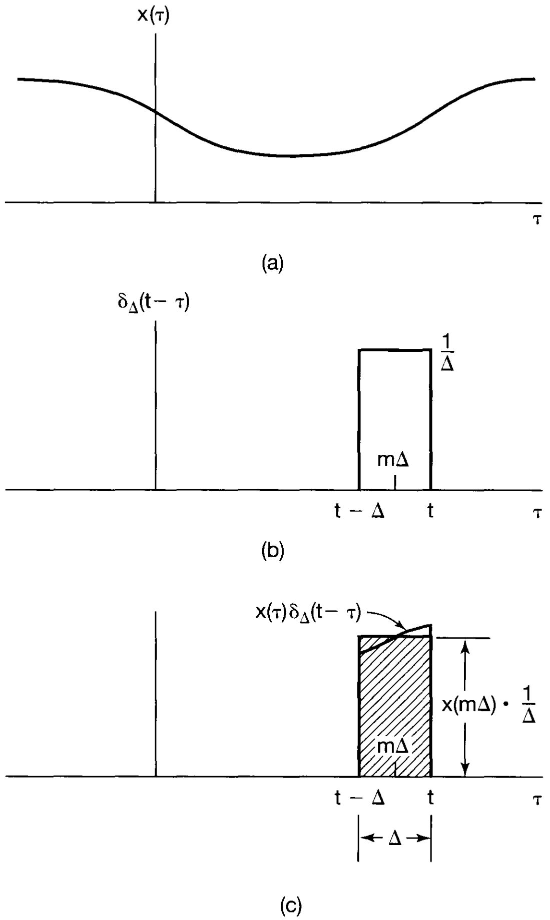


Figure 2.13 Graphical interpretation of eq. (2.26).

Specifically, as illustrated in Figure 2.14(b), the signal $\delta(t - \tau)$ (viewed as a function of τ with t fixed) is a unit impulse located at $\tau = t$. Thus, as shown in Figure 2.14(c), the signal $x(\tau)\delta(t - \tau)$ (once again viewed as a function of τ) equals $x(t)\delta(t - \tau)$ [i.e., it is a scaled impulse at $\tau = t$ with an area equal to the value of $x(t)$]. Consequently, the integral of this signal from $\tau = -\infty$ to $\tau = +\infty$ equals $x(t)$; that is,

$$\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{+\infty} \delta(t-\tau)d\tau = x(t).$$

Although this derivation follows directly from Section 1.4.2, we have included the derivation given in eqs. (2.24)–(2.27) to stress the similarities with the discrete-time case and, in particular, to emphasize the interpretation of eq. (2.27) as representing the signal $x(t)$ as a "sum" (more precisely, an integral) of weighted, shifted impulses.

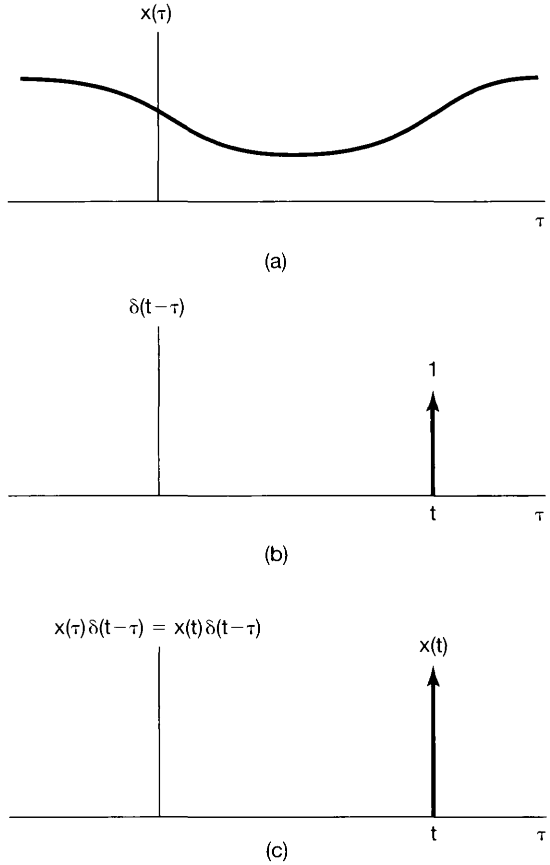


Figure 2.14 (a) Arbitrary signal $x(\tau)$; (b) impulse $\delta(t-\tau)$ as a function of τ with t fixed; (c) product of these two signals.

2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

As in the discrete-time case, the representation developed in the preceding section provides us with a way in which to view an arbitrary continuous-time signal as the superposition of scaled and shifted pulses. In particular, the approximate representation in eq. (2.25) represents the signal $\hat{x}(t)$ as a sum of scaled and shifted versions of the basic pulse signal $\delta_\Delta(t)$. Consequently, the response $\hat{y}(t)$ of a linear system to this signal will be the superposition of the responses to the scaled and shifted versions of $\delta_\Delta(t)$. Specifically, let us define $\hat{h}_{k\Delta}(t)$ as the response of an LTI system to the input $\delta_\Delta(t - k\Delta)$. Then, from eq. (2.25) and the superposition property, for continuous-time linear systems, we see that

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta. \tag{2.29}$$

The interpretation of eq. (2.29) is similar to that for eq. (2.3) in discrete time. In particular, consider Figure 2.15, which is the continuous-time counterpart of Figure 2.2. In

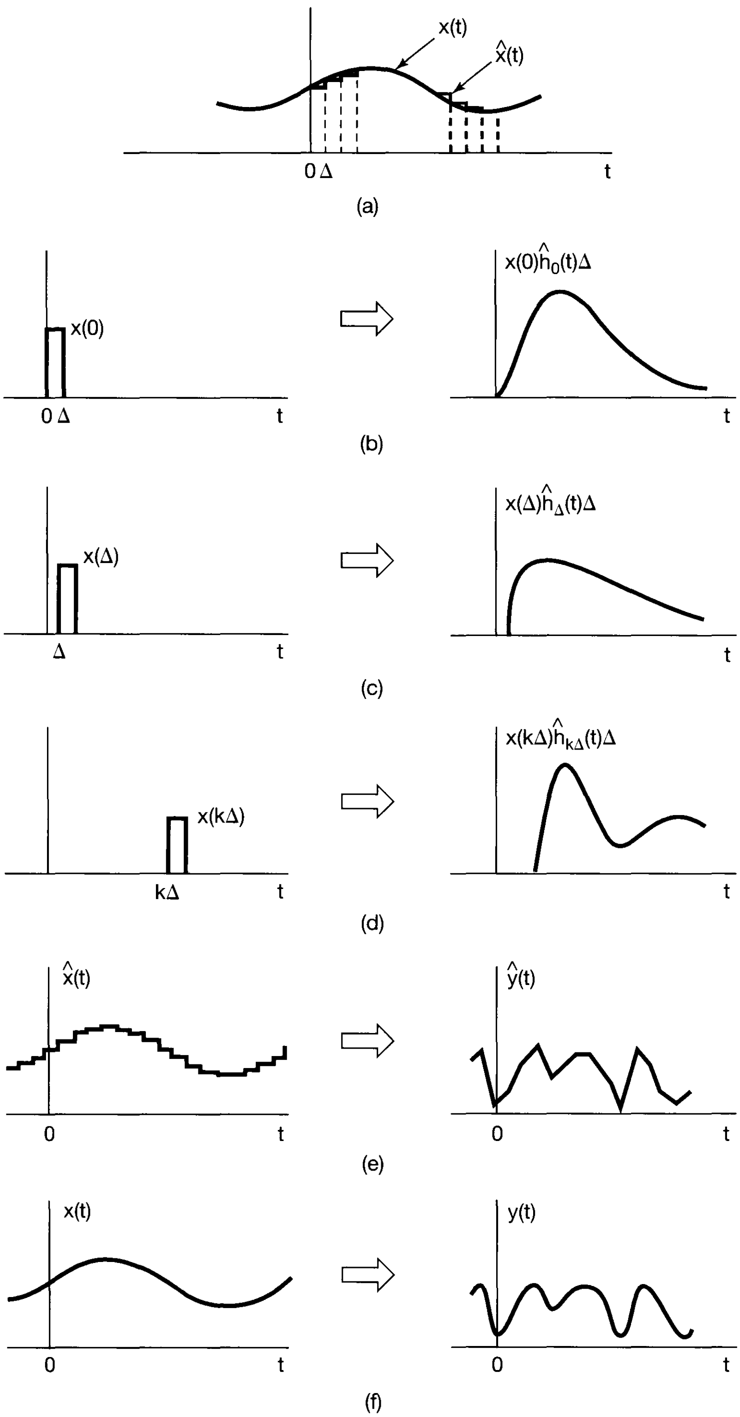


Figure 2.15 Graphical interpretation of the response of a continuous-time linear system as expressed in eqs. (2.29) and (2.30).

Figure 2.15(a) we have depicted the input $x(t)$ and its approximation $\hat{x}(t)$, while in Figure 2.15(b)–(d), we have shown the responses of the system to three of the weighted pulses in the expression for $\hat{x}(t)$. Then the output $\hat{y}(t)$ corresponding to $\hat{x}(t)$ is the superposition of all of these responses, as indicated in Figure 2.15(e).

What remains, then, is to consider what happens as Δ becomes vanishingly small—i.e., as $\Delta \rightarrow 0$. In particular, with $x(t)$ as expressed in eq. (2.26), $\hat{x}(t)$ becomes an increasingly good approximation to $x(t)$, and in fact, the two coincide as $\Delta \rightarrow 0$. Consequently, the response to $\hat{x}(t)$, namely, $\hat{y}(t)$ in eq. (2.29), must converge to $y(t)$, the response to the actual input $x(t)$, as illustrated in Figure 2.15(f). Furthermore, as we have said, for Δ "small enough," the duration of the pulse $\delta_\Delta(t - k\Delta)$ is of no significance, in that, as far as the system is concerned, the response to this pulse is essentially the same as the response to a unit impulse at the same point in time. That is, since the pulse $\delta_\Delta(t - k\Delta)$ corresponds to a shifted unit impulse as $\Delta \rightarrow 0$, the response $\hat{h}_{k\Delta}(t)$ to this input pulse becomes the response to an impulse in the limit. Therefore, if we let $h_\tau(t)$ denote the response at time t to a unit impulse $\delta(t - \tau)$ located at time τ, then

$$y(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta. \tag{2.30}$$

As $\Delta \rightarrow 0$, the summation on the right-hand side becomes an integral, as can be seen graphically in Figure 2.16. Specifically, in Figure 2.16 the shaded rectangle represents one term in the summation on the right-hand side of eq. (2.30) and as $\Delta \rightarrow 0$ the summation approaches the area under $x(\tau)h_\tau(t)$ viewed as a function of τ. Therefore,

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau. \tag{2.31}$$

The interpretation of eq. (2.31) is analogous to the one for eq. (2.29). As we showed in Section 2.2.1, any input $x(t)$ can be represented as

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau.$$

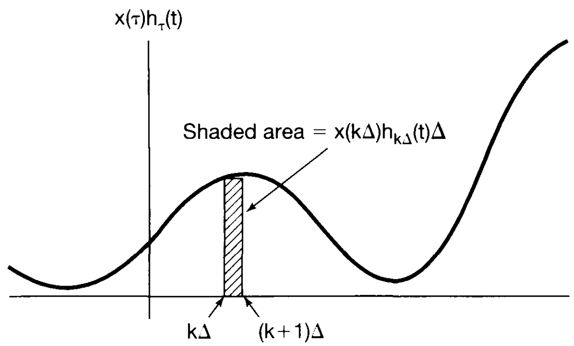


Figure 2.16 Graphical illustration of eqs. (2.30) and (2.31).

That is, we can intuitively think of $x(t)$ as a "sum" of weighted shifted impulses, where the weight on the impulse $\delta(t-\tau)$ is $x(\tau)d\tau$. With this interpretation, eq. (2.31) represents the superposition of the responses to each of these inputs, and by linearity, the weight on the response $h_\tau(t)$ to the shifted impulse $\delta(t-\tau)$ is also $x(\tau)d\tau$.

Equation (2.31) represents the general form of the response of a linear system in continuous time. If, in addition to being linear, the system is also time invariant, then $h_\tau(t) = h_0(t-\tau)$; i.e., the response of an LTI system to the unit impulse $\delta(t-\tau)$, which is shifted by τ seconds from the origin, is a similarly shifted version of the response to the unit impulse function $\delta(t)$. Again, for notational convenience, we will drop the subscript and define the *unit impulse response* $h(t)$ as

$$h(t) = h_0(t); \tag{2.32}$$

i.e., $h(t)$ is the response to $\delta(t)$. In this case, eq. (2.31) becomes

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.} \tag{2.33}$$

Equation (2.33), referred to as the *convolution integral* or the *superposition integral*, is the continuous-time counterpart of the convolution sum of eq. (2.6) and corresponds to the representation of a continuous-time LTI system in terms of its response to a unit impulse. The convolution of two signals $x(t)$ and $h(t)$ will be represented symbolically as

$$y(t) = x(t) * h(t). \tag{2.34}$$

While we have chosen to use the same symbol $*$ to denote both discrete-time and continuous-time convolution, the context will generally be sufficient to distinguish the two cases.

As in discrete time, we see that a continuous-time LTI system is completely characterized by its impulse response—i.e., by its response to a single elementary signal, the unit impulse $\delta(t)$. In the next section, we explore the implications of this as we examine a number of the properties of convolution and of LTI systems in both continuous time and discrete time.

The procedure for evaluating the convolution integral is quite similar to that for its discrete-time counterpart, the convolution sum. Specifically, in eq. (2.33) we see that, for any value of t, the output $y(t)$ is a weighted integral of the input, where the weight on $x(\tau)$ is $h(t-\tau)$. To evaluate this integral for a specific value of t, we first obtain the signal $h(t-\tau)$ (regarded as a function of τ with t fixed) from $h(\tau)$ by a reflection about the origin and a shift to the right by t if $t > 0$ or a shift to the left by $|t|$ for $t < 0$. We next multiply together the signals $x(\tau)$ and $h(t-\tau)$, and $y(t)$ is obtained by integrating the resulting product from $\tau = -\infty$ to $\tau = +\infty$. To illustrate the evaluation of the convolution integral, let us consider several examples.

Example 2.6

Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

$$x(t) = e^{-at}u(t), \quad a > 0$$

and

$$h(t) = u(t).$$

In Figure 2.17, we have depicted the functions $h(\tau)$, $x(\tau)$, and $h(t - \tau)$ for a negative value of t and for a positive value of t. From this figure, we see that for $t < 0$, the product of $x(\tau)$ and $h(t - \tau)$ is zero, and consequently, $y(t)$ is zero. For $t > 0$,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}.$$

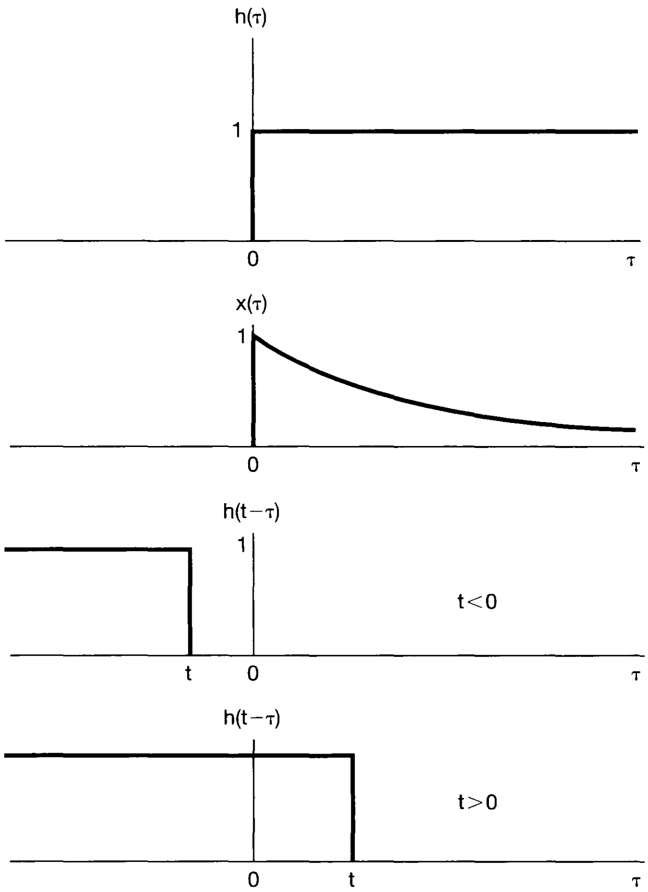


Figure 2.17 Calculation of the convolution integral for Example 2.6.

From this expression, we can compute $y(t)$ for $t > 0$:

$$y(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a}e^{-a\tau}\Big|_0^t$$
$$= \frac{1}{a}(1 - e^{-at}).$$

Thus, for all t, $y(t)$ is

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t),$$

which is shown in Figure 2.18.

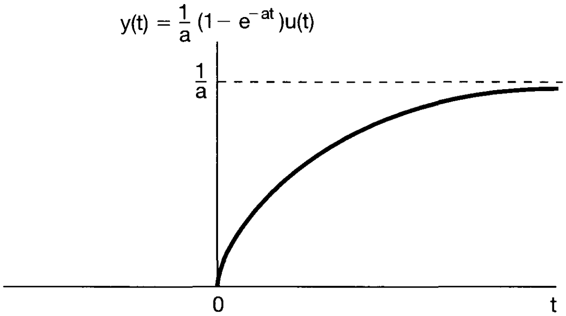


Figure 2.18 Response of the system in Example 2.6 with impulse response $h(t) = u(t)$ to the input $x(t) = e^{-at}u(t)$.

Example 2.7

Consider the convolution of the following two signals:

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases},$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}.$$

As in Example 2.4 for discrete-time convolution, it is convenient to consider the evaluation of $y(t)$ in separate intervals. In Figure 2.19, we have sketched $x(\tau)$ and have illustrated $h(t-\tau)$ in each of the intervals of interest. For $t < 0$ and for $t > 3T$, $x(\tau)h(t-\tau) = 0$ for all values of τ, and consequently, $y(t) = 0$. For the other intervals, the product $x(\tau)h(t-\tau)$ is as indicated in Figure 2.20. Thus, for these three intervals, the integration can be carried out graphically, with the result that

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases},$$

which is depicted in Figure 2.21.

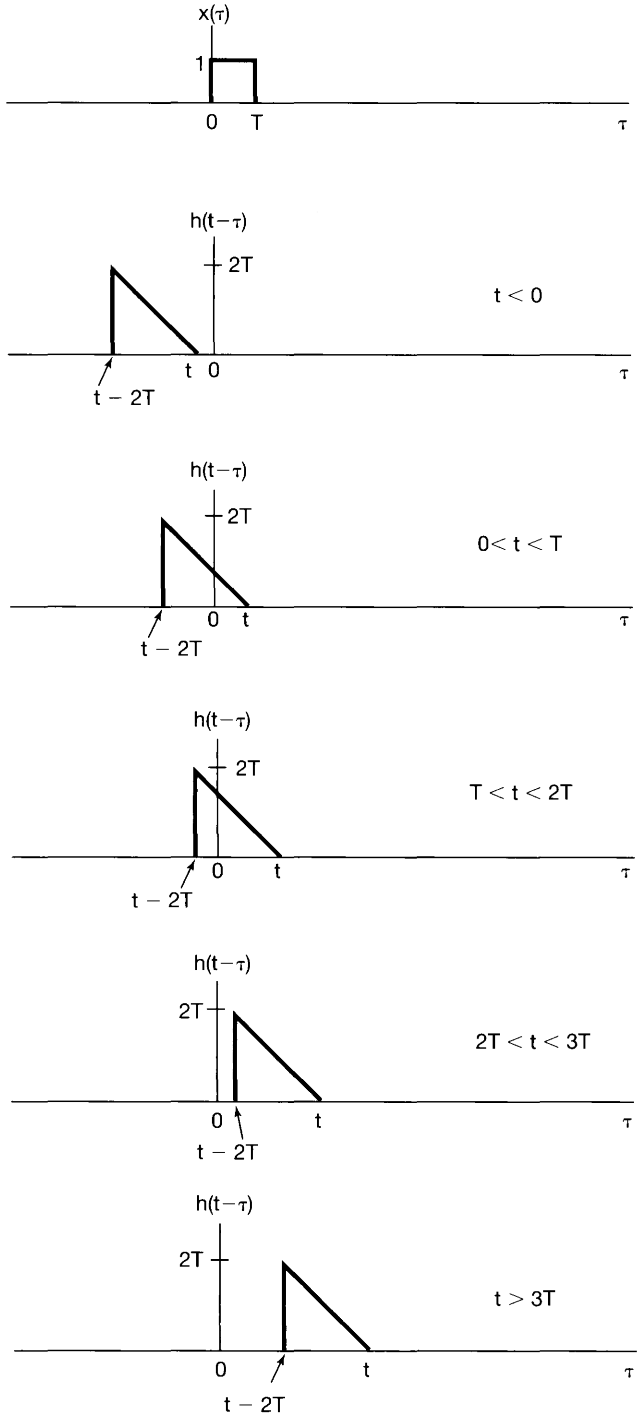


Figure 2.19 Signals $x(\tau)$ and $h(t-\tau)$ for different values of t for Example 2.7.

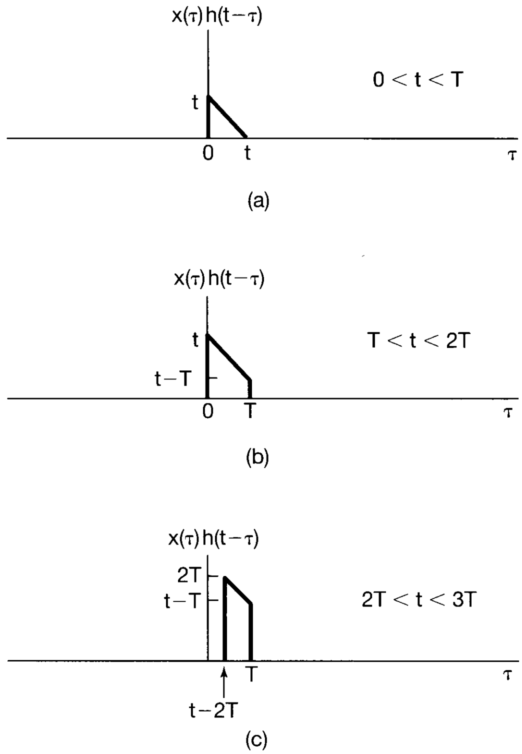


Figure 2.20 Product $x(\tau)h(t-\tau)$ for Example 2.7 for the three ranges of values of t for which this product is not identically zero. (See Figure 2.19.)

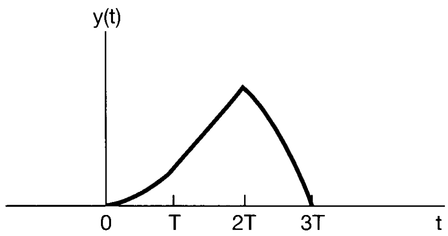


Figure 2.21 Signal $y(t) = x(t) * h(t)$ for Example 2.7.

Example 2.8

Let $y(t)$ denote the convolution of the following two signals:

$$x(t) = e^{2t}u(-t), \tag{2.35}$$

$$h(t) = u(t-3). \tag{2.36}$$

The signals $x(\tau)$ and $h(t-\tau)$ are plotted as functions of τ in Figure 2.22(a). We first observe that these two signals have regions of nonzero overlap, regardless of the value

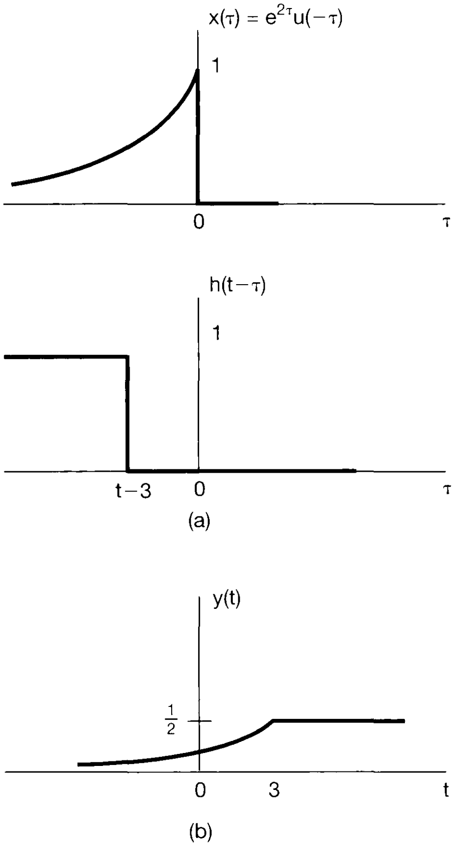


Figure 2.22 The convolution problem considered in Example 2.8.

of t. When $t - 3 \leq 0$, the product of $x(\tau)$ and $h(t - \tau)$ is nonzero for $-\infty < \tau < t - 3$, and the convolution integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2} e^{2(t-3)}. \tag{2.37}$$

For $t-3 \geq 0$, the product $x(\tau)h(t-\tau)$ is nonzero for $-\infty < \tau < 0$, so that the convolution integral is

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \frac{1}{2}. \tag{2.38}$$

The resulting signal $y(t)$ is plotted in Figure 2.22(b).

As these examples and those presented in Section 2.1 illustrate, the graphical interpretation of continuous-time and discrete-time convolution is of considerable value in visualizing the evaluation of convolution integrals and sums.

2.3 PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS

In the preceding two sections, we developed the extremely important representations of continuous-time and discrete-time LTI systems in terms of their unit impulse responses. In discrete time the representation takes the form of the convolution sum, while its continuous-time counterpart is the convolution integral, both of which we repeat here for convenience:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n] \tag{2.39}$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t) \tag{2.40}$$

As we have pointed out, one consequence of these representations is that the characteristics of an LTI system are completely determined by its impulse response. It is important to emphasize that this property holds in general *only* for LTI systems. In particular, as illustrated in the following example, the unit impulse response of a nonlinear system does *not* completely characterize the behavior of the system.

Example 2.9

Consider a discrete-time system with unit impulse response

$$h[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases}. \tag{2.41}$$

If the system is LTI, then eq. (2.41) completely determines its input-output behavior. In particular, by substituting eq. (2.41) into the convolution sum, eq. (2.39), we find the following explicit equation describing how the input and output of this LTI system are related:

$$y[n] = x[n] + x[n-1]. \tag{2.42}$$

On the other hand, there are *many* nonlinear systems with the same response—i.e., that given in eq. (2.41)—to the input $\delta[n]$. For example, both of the following systems have this property:

$$y[n] = (x[n] + x[n-1])^2,$$
$$y[n] = \max(x[n], x[n-1]).$$

Consequently, if the system is nonlinear it is not completely characterized by the impulse response in eq. (2.41).

The preceding example illustrates the fact that LTI systems have a number of properties not possessed by other systems, beginning with the very special representations that they have in terms of convolution sums and integrals. In the remainder of this section, we explore some of the most basic and important of these properties.

2.3.1 The Commutative Property

A basic property of convolution in both continuous and discrete time is that it is a *commutative* operation. That is, in discrete time

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k], \tag{2.43}$$

and in continuous time

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau. \tag{2.44}$$

These expressions can be verified in a straightforward manner by means of a substitution of variables in eqs. (2.39) and (2.40). For example, in the discrete-time case, if we let $r = n - k$ or, equivalently, $k = n - r$, eq. (2.39) becomes

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{r=-\infty}^{+\infty} x[n-r]h[r] = h[n] * x[n]. \tag{2.45}$$

With this substitution of variables, the roles of $x[n]$ and $h[n]$ are interchanged. According to eq. (2.45), the output of an LTI system with input $x[n]$ and unit impulse response $h[n]$ is identical to the output of an LTI system with input $h[n]$ and unit impulse response $x[n]$. For example, we could have calculated the convolution in Example 2.4 by first reflecting and shifting $x[k]$, then multiplying the signals $x[n-k]$ and $h[k]$, and finally summing the products for all values of k.

Similarly, eq. (2.44) can be verified by a change of variables, and the implications of this result in continuous time are the same: The output of an LTI system with input $x(t)$ and unit impulse response $h(t)$ is identical to the output of an LTI system with input $h(t)$ and unit impulse response $x(t)$. Thus, we could have calculated the convolution in Example 2.7 by reflecting and shifting $x(t)$, multiplying the signals $x(t-\tau)$ and $h(\tau)$, and integrating over $-\infty < \tau < +\infty$. In specific cases, one of the two forms for computing convolutions [i.e., eq. (2.39) or (2.43) in discrete time and eq. (2.40) or (2.44) in continuous time] may be easier to visualize, but both forms always result in the same answer.

2.3.2 The Distributive Property

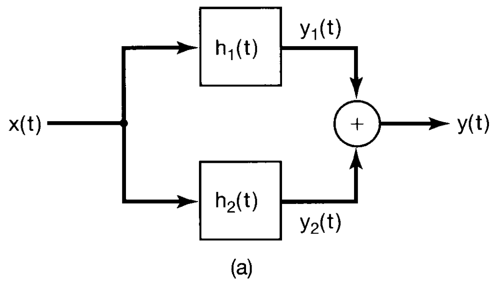
Another basic property of convolution is the *distributive* property. Specifically, convolution distributes over addition, so that in discrete time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n], \tag{2.46}$$

and in continuous time

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t). \tag{2.47}$$

This property can be verified in a straightforward manner.



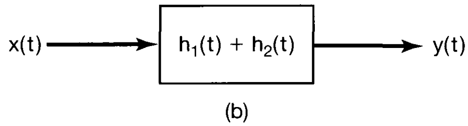


Figure 2.23 Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

The distributive property has a useful interpretation in terms of system interconnections. Consider two continuous-time LTI systems in parallel, as indicated in Figure 2.23(a). The systems shown in the block diagram are LTI systems with the indicated unit impulse responses. This pictorial representation is a particularly convenient way in which to denote LTI systems in block diagrams, and it also reemphasizes the fact that the impulse response of an LTI system completely characterizes its behavior.

The two systems, with impulse responses $h_1(t)$ and $h_2(t)$, have identical inputs, and their outputs are added. Since

$$y_1(t) = x(t) * h_1(t)$$

and

$$y_2(t) = x(t) * h_2(t),$$

the system of Figure 2.23(a) has output

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t), \tag{2.48}$$

corresponding to the right-hand side of eq. (2.47). The system of Figure 2.23(b) has output

$$y(t) = x(t) * [h_1(t) + h_2(t)], \tag{2.49}$$

corresponding to the left-hand side of eq. (2.47). Applying eq. (2.47) to eq. (2.49) and comparing the result with eq. (2.48), we see that the systems in Figures 2.23(a) and (b) are identical.

There is an identical interpretation in discrete time, in which each of the signals in Figure 2.23 is replaced by a discrete-time counterpart (i.e., $x(t)$, $h_1(t)$, $h_2(t)$, $y_1(t)$, $y_2(t)$, and $y(t)$ are replaced by $x[n]$, $h_1[n]$, $h_2[n]$, $y_1[n]$, $y_2[n]$, and $y[n]$, respectively). In summary, then, by virtue of the distributive property of convolution, a parallel combination of LTI systems can be replaced by a single LTI system whose unit impulse response is the sum of the individual unit impulse responses in the parallel combination.

Also, as a consequence of both the commutative and distributive properties, we have

$$[x_1[n] + x_2[n]] * h[n] = x_1[n] * h[n] + x_2[n] * h[n] \tag{2.50}$$

and

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t), \tag{2.51}$$

which simply state that the response of an LTI system to the sum of two inputs must equal the sum of the responses to these signals individually.

As illustrated in the next example, the distributive property of convolution can also be exploited to break a complicated convolution into several simpler ones.

Example 2.10

Let $y[n]$ denote the convolution of the following two sequences:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n], \tag{2.52}$$

$$h[n] = u[n]. \tag{2.53}$$

Note that the sequence $x[n]$ is nonzero along the entire time axis. Direct evaluation of such a convolution is somewhat tedious. Instead, we may use the distributive property to express $y[n]$ as the sum of the results of two simpler convolution problems. In particular, if we let $x_1[n] = (1/2)^n u[n]$ and $x_2[n] = 2^n u[-n]$, it follows that

$$y[n] = (x_1[n] + x_2[n]) * h[n]. \tag{2.54}$$

Using the distributive property of convolution, we may rewrite eq. (2.54) as

$$y[n] = y_1[n] + y_2[n], \tag{2.55}$$

where

$$y_1[n] = x_1[n] * h[n] \tag{2.56}$$

and

$$y_2[n] = x_2[n] * h[n]. \tag{2.57}$$

The convolution in eq. (2.56) for $y_1[n]$ can be obtained from Example 2.3 (with $\alpha = 1/2$), while $y_2[n]$ was evaluated in Example 2.5. Their sum is $y[n]$, which is shown in Figure 2.24.

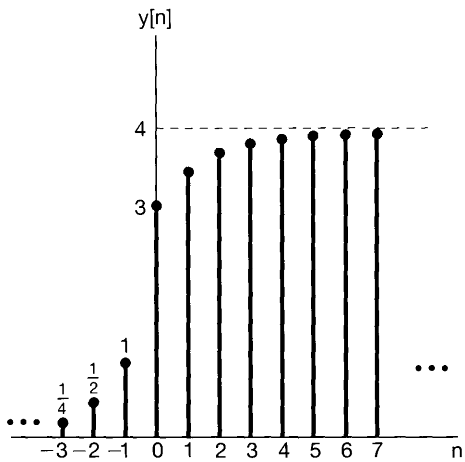


Figure 2.24 The signal $y[n] = x[n] * h[n]$ for Example 2.10.

2.3.3 The Associative Property

Another important and useful property of convolution is that it is *associative*. That is, in discrete time

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n], \tag{2.58}$$

and in continuous time

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t). \tag{2.59}$$

This property is proven by straightforward manipulations of the summations and integrals involved. Examples verifying it are given in Problem 2.43.

As a consequence of the associative property, the expressions

$$y[n] = x[n] * h_1[n] * h_2[n] \tag{2.60}$$

and

$$y(t) = x(t) * h_1(t) * h_2(t) \tag{2.61}$$

are unambiguous. That is, according to eqs. (2.58) and (2.59), it does not matter in which order we convolve these signals.

An interpretation of the associative property is illustrated for discrete-time systems in Figures 2.25(a) and (b). In Figure 2.25(a),

$$\begin{aligned} y[n] &= w[n] * h_2[n] \\ &= (x[n] * h_1[n]) * h_2[n]. \end{aligned}$$

In Figure 2.25(b),

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * (h_1[n] * h_2[n]). \end{aligned}$$

According to the associative property, the series interconnection of the two systems in Figure 2.25(a) is equivalent to the single system in Figure 2.25(b). This can be generalized to an arbitrary number of LTI systems in cascade, and the analogous interpretation and conclusion also hold in continuous time.

By using the commutative property together with the associative property, we find another very important property of LTI systems. Specifically, from Figures 2.25(a) and (b), we can conclude that the impulse response of the cascade of two LTI systems is the convolution of their individual impulse responses. Since convolution is commutative, we can compute this convolution of $h_1[n]$ and $h_2[n]$ in either order. Thus, Figures 2.25(b) and (c) are equivalent, and from the associative property, these are in turn equivalent to the system of Figure 2.25(d), which we note is a cascade combination of two systems as in Figure 2.25(a), but with the order of the cascade reversed. Consequently, the unit impulse response of a cascade of two LTI systems does not depend on the order in which they are cascaded. In fact, this holds for an arbitrary number of LTI systems in cascade: The order in which they are cascaded does not matter as far as the overall system impulse response is concerned. The same conclusions hold in continuous time as well.

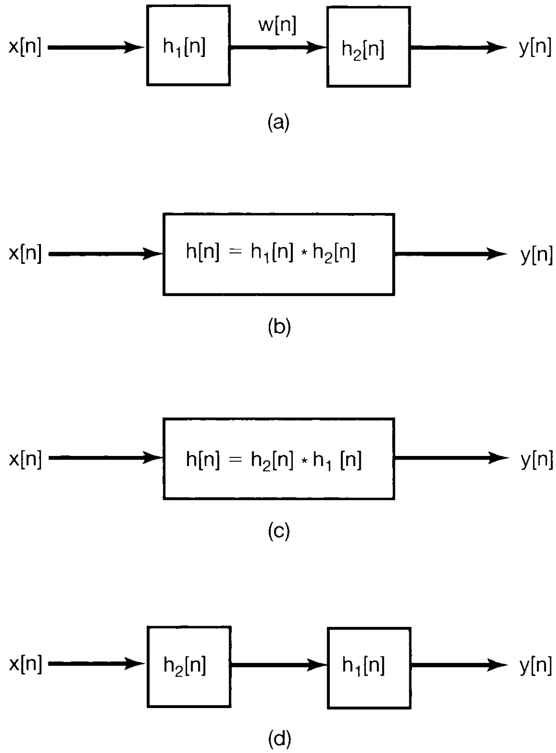


Figure 2.25 Associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI systems.

It is important to emphasize that the behavior of LTI systems in cascade—and, in particular, the fact that the overall system response does not depend upon the order of the systems in the cascade—is very special to such systems. In contrast, the order in which nonlinear systems are cascaded cannot be changed, in general, without changing the overall response. For instance, if we have two memoryless systems, one being multiplication by 2 and the other squaring the input, then if we multiply first and square second, we obtain

$$y[n] = 4x^2[n].$$

However, if we multiply by 2 after squaring, we have

$$y[n] = 2x^2[n].$$

Thus, being able to interchange the order of systems in a cascade is a characteristic particular to LTI systems. In fact, as shown in Problem 2.51, we need both linearity *and* time invariance in order for this property to be true in general.

2.3.4 LTI Systems with and without Memory

As specified in Section 1.6.1, a system is memoryless if its output at any time depends only on the value of the input at that same time. From eq. (2.39), we see that the only way that this can be true for a discrete-time LTI system is if $h[n] = 0$ for $n \neq 0$. In this case the impulse response has the form

$$h[n] = K\delta[n], \tag{2.62}$$

where $K = h[0]$ is a constant, and the convolution sum reduces to the relation

$$y[n] = Kx[n]. \tag{2.63}$$

If a discrete-time LTI system has an impulse response $h[n]$ that is not identically zero for $n \neq 0$, then the system has memory. An example of an LTI system with memory is the system given by eq. (2.42). The impulse response for this system, given in eq. (2.41), is nonzero for $n = 1$.

From eq. (2.40), we can deduce similar properties for continuous-time LTI systems with and without memory. In particular, a continuous-time LTI system is memoryless if $h(t) = 0$ for $t \neq 0$, and such a memoryless LTI system has the form

$$y(t) = Kx(t) \tag{2.64}$$

for some constant K and has the impulse response

$$h(t) = K\delta(t). \tag{2.65}$$

Note that if $K = 1$ in eqs. (2.62) and (2.65), then these systems become identity systems, with output equal to the input and with unit impulse response equal to the unit impulse. In this case, the convolution sum and integral formulas imply that

$$x[n] = x[n] * \delta[n]$$

and

$$x(t) = x(t) * \delta(t),$$

which reduce to the sifting properties of the discrete-time and continuous-time unit impulses:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau.$$

2.3.5 Invertibility of LTI Systems

Consider a continuous-time LTI system with impulse response $h(t)$. Based on the discussion in Section 1.6.2, this system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system. Furthermore, if an LTI system is invertible, then it has an LTI inverse. (See Problem 2.50.) Therefore, we have the picture shown in Figure 2.26. We are given a system with impulse response $h(t)$. The inverse system, with impulse response $h_1(t)$, results in $w(t) = x(t)$—such that the series interconnection in Figure 2.26(a) is identical to the

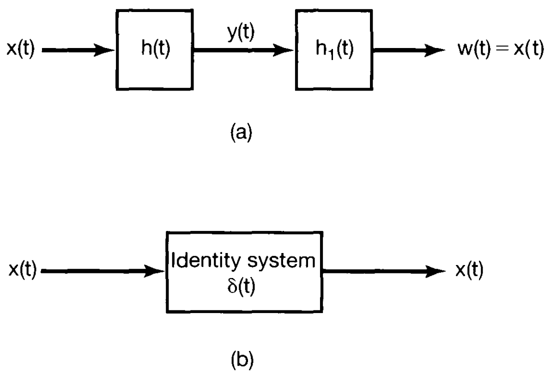


Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

identity system in Figure 2.26(b). Since the overall impulse response in Figure 2.26(a) is $h(t) * h_1(t)$, we have the condition that $h_1(t)$ must satisfy for it to be the impulse response of the inverse system, namely,

$$h(t) * h_1(t) = \delta(t). \tag{2.66}$$

Similarly, in discrete time, the impulse response $h_1[n]$ of the inverse system for an LTI system with impulse response $h[n]$ must satisfy

$$h[n] * h_1[n] = \delta[n]. \tag{2.67}$$

The following two examples illustrate invertibility and the construction of an inverse system.

Example 2.11

Consider the LTI system consisting of a pure time shift

$$y(t) = x(t - t_0). \tag{2.68}$$

Such a system is a *delay* if $t_0 > 0$ and an *advance* if $t_0 < 0$. For example, if $t_0 > 0$, then the output at time t equals the value of the input at the earlier time $t - t_0$. If $t_0 = 0$, the system in eq. (2.68) is the identity system and thus is memoryless. For any other value of t_0, this system has memory, as it responds to the value of the input at a time other than the current time.

The impulse response for the system can be obtained from eq. (2.68) by taking the input equal to $\delta(t)$, i.e.,

$$h(t) = \delta(t - t_0). \tag{2.69}$$

Therefore,

$$x(t - t_0) = x(t) * \delta(t - t_0). \tag{2.70}$$

That is, the convolution of a signal with a shifted impulse simply shifts the signal.

To recover the input from the output, i.e., to invert the system, all that is required is to shift the output back. The system with this compensating time shift is then the inverse system. That is, if we take

$$h_1(t) = \delta(t + t_0),$$

then

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t).$$

Similarly, a pure time shift in discrete time has the unit impulse response $\delta[n-n_0]$, so that convolving a signal with a shifted impulse is the same as shifting the signal. Furthermore, the inverse of the LTI system with impulse response $\delta[n - n_0]$ is the LTI system that shifts the signal in the opposite direction by the same amount—i.e., the LTI system with impulse response $\delta[n + n_0]$.

Example 2.12

Consider an LTI system with impulse response

$$h[n] = u[n]. \tag{2.71}$$

Using the convolution sum, we can calculate the response of this system to an arbitrary input:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]u[n - k]. \tag{2.72}$$

Since $u[n - k]$ is 0 for $n - k < 0$ and 1 for $n - k \geq 0$, eq. (2.72) becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]. \tag{2.73}$$

That is, this system, which we first encountered in Section 1.6.1 [see eq. (1.92)], is a summer or accumulator that computes the running sum of all the values of the input up to the present time. As we saw in Section 1.6.2, such a system is invertible, and its inverse, as given by eq. (1.99), is

$$y[n] = x[n] - x[n - 1], \tag{2.74}$$

which is simply a *first difference* operation. Choosing $x[n] = \delta[n]$, we find that the impulse response of the inverse system is

$$h_1[n] = \delta[n] - \delta[n - 1]. \tag{2.75}$$

As a check that $h[n]$ in eq. (2.71) and $h_1[n]$ in eq. (2.75) are indeed the impulse responses of LTI systems that are inverses of each other, we can verify eq. (2.67) by direct calculation:

$$\begin{aligned} h[n] * h_1[n] &= u[n] * \{\delta[n] - \delta[n - 1]\} \\ &= u[n] * \delta[n] - u[n] * \delta[n - 1] \\ &= u[n] - u[n - 1] \\ &= \delta[n]. \end{aligned} \tag{2.76}$$

2.3.6 Causality for LTI Systems

In Section 1.6.3, we introduced the property of causality: The output of a causal system depends only on the present and past values of the input to the system. By using the convolution sum and integral, we can relate this property to a corresponding property of the impulse response of an LTI system. Specifically, in order for a discrete-time LTI system to be causal, $y[n]$ must not depend on $x[k]$ for $k > n$. From eq. (2.39), we see that for this to be true, all of the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k > n$ must be zero. This then requires that the impulse response of a causal discrete-time LTI system satisfy the condition

$$h[n] = 0 \quad \text{for } n < 0. \tag{2.77}$$

According to eq. (2.77), the impulse response of a causal LTI system must be zero before the impulse occurs, which is consistent with the intuitive concept of causality. More generally, as shown in Problem 1.44, causality for a linear system is equivalent to the condition of *initial rest*; i.e., if the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time. It is important to emphasize that the equivalence of causality and the condition of initial rest applies only to linear systems. For example, as discussed in Section 1.6.6, the system $y[n] = 2x[n] + 3$ is not linear. However, it is causal and, in fact, memoryless. On the other hand, if $x[n] = 0$, $y[n] = 3 \neq 0$, so it does not satisfy the condition of initial rest.

For a causal discrete-time LTI system, the condition in eq. (2.77) implies that the convolution sum representation in eq. (2.39) becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k], \tag{2.78}$$

and the alternative equivalent form, eq. (2.43), becomes

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]. \tag{2.79}$$

Similarly, a continuous-time LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0, \tag{2.80}$$

and in this case the convolution integral is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau. \tag{2.81}$$

Both the accumulator ($h[n] = u[n]$) and its inverse ($h[n] = \delta[n] - \delta[n-1]$), described in Example 2.12, satisfy eq. (2.77) and therefore are causal. The pure time shift with impulse response $h(t) = \delta(t-t_0)$ is causal for $t_0 \geq 0$ (when the time shift is a delay), but is noncausal for $t_0 < 0$ (in which case the time shift is an advance, so that the output anticipates future values of the input).

Finally, while causality is a property of systems, it is common terminology to refer to a signal as being causal if it is zero for $n < 0$ or $t < 0$. The motivation for this terminology comes from eqs. (2.77) and (2.80): Causality of an LTI system is equivalent to its impulse response being a causal signal.

2.3.7 Stability for LTI Systems

Recall from Section 1.6.4 that a system is *stable* if every bounded input produces a bounded output. In order to determine conditions under which LTI systems are stable, consider an input $x[n]$ that is bounded in magnitude:

$$|x[n]| < B \quad \text{for all } n. \tag{2.82}$$

Suppose that we apply this input to an LTI system with unit impulse response $h[n]$. Then, using the convolution sum, we obtain an expression for the magnitude of the output:

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right|. \tag{2.83}$$

Since the magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers, it follows from eq. (2.83) that

$$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|. \tag{2.84}$$

From eq. (2.82), $|x[n-k]| < B$ for all values of k and n. Together with eq. (2.84), this implies that

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| \quad \text{for all } n. \tag{2.85}$$

From eq. (2.85), we can conclude that if the impulse response is *absolutely summable*, that is, if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty, \tag{2.86}$$

then $y[n]$ is bounded in magnitude, and hence, the system is stable. Therefore, eq. (2.86) is a sufficient condition to guarantee the stability of a discrete-time LTI system. In fact, this condition is also a necessary condition, since, as shown in Problem 2.49, if eq. (2.86) is not satisfied, there are bounded inputs that result in unbounded outputs. Thus, the stability of a discrete-time LTI system is completely equivalent to eq. (2.86).

In continuous time, we obtain an analogous characterization of stability in terms of the impulse response of an LTI system. Specifically, if $|x(t)| < B$ for all t, then, in analogy with eqs. (2.83)–(2.85), it follows that

$$\begin{aligned}|y(t)| &= \left|\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau\right| \\ &\le \int_{-\infty}^{+\infty} |h(\tau)||x(t-\tau)|d\tau \\ &\le B\int_{-\infty}^{+\infty} |h(\tau)|d\tau.\end{aligned}$$

Therefore, the system is stable if the impulse response is *absolutely integrable*, i.e., if

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty. \tag{2.87}$$

As in discrete time, if eq. (2.87) is not satisfied, there are bounded inputs that produce unbounded outputs; therefore, the stability of a continuous-time LTI system is equivalent to eq. (2.87). The use of eqs (2.86) and (2.87) to test for stability is illustrated in the next two examples.

Example 2.13

Consider a system that is a pure time shift in either continuous time or discrete time. Then, in discrete time

$$\sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} |\delta[n-n_0]| = 1, \tag{2.88}$$

while in continuous time

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau = \int_{-\infty}^{+\infty} |\delta(\tau-t_0)|d\tau = 1, \tag{2.89}$$

and we conclude that both of these systems are stable. This should not be surprising, since if a signal is bounded in magnitude, so is any time-shifted version of that signal.

Now consider the accumulator described in Example 2.12. As we discussed in Section 1.6.4, this is an unstable system, since, if we apply a constant input to an accumulator, the output grows without bound. That this system is unstable can also be seen from the fact that its impulse response $u[n]$ is not absolutely summable:

$$\sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=0}^{\infty} u[n] = \infty.$$

Similarly, consider the integrator, the continuous-time counterpart of the accumulator:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau. \tag{2.90}$$

This is an unstable system for precisely the same reason as that given for the accumulator; i.e., a constant input gives rise to an output that grows without bound. The impulse response for the integrator can be found by letting $x(t) = \delta(t)$, in which case

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$

and

$$\int_{-\infty}^{+\infty} |u(\tau)|d\tau = \int_{0}^{+\infty} d\tau = \infty.$$

Since the impulse response is not absolutely integrable, the system is not stable.

2.3.8 The Unit Step Response of an LTI System

Up to now, we have seen that the representation of an LTI system in terms of its unit impulse response allows us to obtain very explicit characterizations of system properties. Specifically, since $h[n]$ or $h(t)$ completely determines the behavior of an LTI system, we have been able to relate system properties such as stability and causality to properties of the impulse response.

There is another signal that is also used quite often in describing the behavior of LTI systems: the *unit step response*, $s[n]$ or $s(t)$, corresponding to the output when $x[n] = u[n]$ or $x(t) = u(t)$. We will find it useful on occasion to refer to the step response, and therefore, it is worthwhile relating it to the impulse response. From the convolution-sum representation, the step response of a discrete-time LTI system is the convolution of the unit step with the impulse response; that is,

$$s[n] = u[n] * h[n].$$

However, by the commutative property of convolution, $s[n] = h[n] * u[n]$, and therefore, $s[n]$ can be viewed as the response to the input $h[n]$ of a discrete-time LTI system with unit impulse response $u[n]$. As we have seen in Example 2.12, $u[n]$ is the unit impulse response of the accumulator. Therefore,

$$s[n] = \sum_{k=-\infty}^{n} h[k]. \tag{2.91}$$

From this equation and from Example 2.12, it is clear that $h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1]. \tag{2.92}$$

That is, the step response of a discrete-time LTI system is the running sum of its impulse response [eq. (2.91)]. Conversely, the impulse response of a discrete-time LTI system is the first difference of its step response [eq. (2.92)].

Similarly, in continuous time, the step response of an LTI system with impulse response $h(t)$ is given by $s(t) = u(t) * h(t)$, which also equals the response of an integrator [with impulse response $u(t)$] to the input $h(t)$. That is, the unit step response of a continuous-time LTI system is the running integral of its impulse response, or

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \tag{2.93}$$

and from eq. (2.93), the unit impulse response is the first derivative of the unit step response,[1] or

$$h(t) = \frac{ds(t)}{dt} = s'(t). \tag{2.94}$$

Therefore, in both continuous and discrete time, the unit step response can also be used to characterize an LTI system, since we can calculate the unit impulse response from it. In Problem 2.45, expressions analogous to the convolution sum and convolution integral are derived for the representations of an LTI system in terms of its unit step response.

2.4 CAUSAL LTI SYSTEMS DESCRIBED BY DIFFERENTIAL AND DIFFERENCE EQUATIONS

An extremely important class of continuous-time systems is that for which the input and output are related through a *linear constant-coefficient differential equation*. Equations of this type arise in the description of a wide variety of systems and physical phenomena. For example, as we illustrated in Chapter 1, the response of the *RC* circuit in Figure 1.1 and the motion of a vehicle subject to acceleration inputs and frictional forces, as depicted in Figure 1.2, can both be described through linear constant-coefficient differential equations. Similar differential equations arise in the description of mechanical systems containing restoring and damping forces, in the kinetics of chemical reactions, and in many other contexts as well.

Correspondingly, an important class of discrete-time systems is that for which the input and output are related through a *linear constant-coefficient difference equation*. Equations of this type are used to describe the sequential behavior of many different processes. For instance, in Example 1.10 we saw how difference equations arise in describing the accumulation of savings in a bank account, and in Example 1.11 we saw how they can be used to describe a digital simulation of a continuous-time system described by a differential equation. Difference equations also arise quite frequently in the specification of discrete-time systems designed to perform particular operations on the input signal. For example, the system that calculates the difference between successive input values, as in eq. (1.99), and the system described by eq. (1.104) that computes the average value of the input over an interval are described by difference equations.

Throughout this book, there will be many occasions in which we will consider and examine systems described by linear constant-coefficient differential and difference equations. In this section we take a first look at these systems to introduce some of the basic ideas involved in solving differential and difference equations and to uncover and explore some of the properties of systems described by such equations. In subsequent chapters, we develop additional tools for the analysis of signals and systems that will add considerably both to our ability to analyze systems described by such equations and to our understanding of their characteristics and behavior.

[1]Throughout this book, we will use both the notations indicated in eq. (2.94) to denote first derivatives. Analogous notation will also be used for higher derivatives.

2.4.1 Linear Constant-Coefficient Differential Equations

To introduce some of the important ideas concerning systems specified by linear constant-coefficient differential equations, let us consider a first-order differential equation as in eq. (1.85), viz.,

$$\frac{dy(t)}{dt} + 2y(t) = x(t), \tag{2.95}$$

where $y(t)$ denotes the output of the system and $x(t)$ is the input. For example, comparing eq. (2.95) to the differential equation (1.84) for the velocity of a vehicle subject to applied and frictional forces, we see that eq. (2.95) would correspond exactly to this system if $y(t)$ were identified with the vehicle's velocity $v(t)$, if $x(t)$ were taken as the applied force $f(t)$, and if the parameters in eq. (1.84) were normalized in units such that $b/m = 2$ and $1/m = 1$.

A very important point about differential equations such as eq. (2.95) is that they provide an *implicit* specification of the system. That is, they describe a relationship between the input and the output, rather than an explicit expression for the system output as a function of the input. In order to obtain an explicit expression, we must solve the differential equation. To find a solution, we need more information than that provided by the differential equation alone. For example, to determine the speed of an automobile at the end of a 10-second interval when it has been subjected to a constant acceleration of 1 m/sec^2 for 10 seconds, we would also need to know how fast the vehicle was moving at the *start* of the interval. Similarly, if we are told that a constant source voltage of 1 volt is applied to the *RC* circuit in Figure 1.1 for 10 seconds, we cannot determine what the capacitor voltage is at the end of that interval without also knowing what the initial capacitor voltage is.

More generally, to solve a differential equation, we must specify one or more auxiliary conditions, and once these are specified, we can then, in principle, obtain an explicit expression for the output in terms of the input. In other words, a differential equation such as eq. (2.95) describes a constraint between the input and the output of a system, but to characterize the system completely, we must also specify auxiliary conditions. Different choices for these auxiliary conditions then lead to different relationships between the input and the output. For the most part, in this book we will focus on the use of differential equations to describe causal LTI systems, and for such systems the auxiliary conditions take a particular, simple form. To illustrate this and to uncover some of the basic properties of the solutions to differential equations, let us take a look at the solution of eq. (2.95) for a specific input signal $x(t)$.[2]

[2]Our discussion of the solution of linear constant-coefficient differential equations is brief, since we assume that the reader has some familiarity with this material. For review, we recommend a text on the solution of ordinary differential equations, such as *Ordinary Differential Equations* (3rd ed.), by G. Birkhoff and G.-C. Rota (New York: John Wiley and Sons, 1978), or *Elementary Differential Equations* (3rd ed.), by W.E. Boyce and R.C. DiPrima (New York: John Wiley and Sons, 1977). There are also numerous texts that discuss differential equations in the context of circuit theory. See, for example, *Basic Circuit Theory*, by L.O. Chua, C.A. Desoer, and E.S. Kuh (New York: McGraw-Hill Book Company, 1987). As mentioned in the text, in the following chapters we present other very useful methods for solving linear differential equations that will be sufficient for our purposes. In addition, a number of exercises involving the solution of differential equations are included in the problems at the end of the chapter.

Example 2.14

Consider the solution of eq. (2.95) when the input signal is

$$x(t) = Ke^{3t}u(t), \tag{2.96}$$

where K is a real number.

The complete solution to eq. (2.96) consists of the sum of a *particular solution,* $y_p(t)$, and a *homogeneous solution,* $y_h(t)$, i.e.,

$$y(t) = y_p(t) + y_h(t), \tag{2.97}$$

where the particular solution satisfies eq. (2.95) and $y_h(t)$ is a solution of the homogeneous differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 0. \tag{2.98}$$

A common method for finding the particular solution for an exponential input signal as in eq. (2.96) is to look for a so-called *forced response*—i.e., a signal of the same form as the input. With regard to eq. (2.95), since $x(t) = Ke^{3t}$ for $t > 0$, we hypothesize a solution for $t > 0$ of the form

$$y_p(t) = Ye^{3t}, \tag{2.99}$$

where Y is a number that we must determine. Substituting eqs. (2.96) and (2.99) into eq. (2.95) for $t > 0$ yields

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t}. \tag{2.100}$$

Canceling the factor e^{3t} from both sides of eq. (2.100), we obtain

$$3Y + 2Y = K, \tag{2.101}$$

or

$$Y = \frac{K}{5}, \tag{2.102}$$

so that

$$y_p(t) = \frac{K}{5}e^{3t}, \quad t > 0. \tag{2.103}$$

In order to determine $y_h(t)$, we hypothesize a solution of the form

$$y_h(t) = Ae^{st}. \tag{2.104}$$

Substituting this into eq. (2.98) gives

$$Ase^{st} + 2Ae^{st} = Ae^{st}(s + 2) = 0. \tag{2.105}$$

From this equation, we see that we must take $s = -2$ and that Ae^{-2t} is a solution to eq. (2.98) for *any* choice of A. Utilizing this fact and eq. (2.103) in eq. (2.97), we find that the solution of the differential equation for $t > 0$ is

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \quad t > 0. \tag{2.106}$$

As noted earlier, the differential equation (2.95) by itself does not specify uniquely the response $y(t)$ to the input $x(t)$ in eq. (2.96). In particular, the constant A in eq. (2.106) has not yet been determined. In order for the value of A to be determined, we need to specify an auxiliary condition in addition to the differential equation (2.95). As explored in Problem 2.34, different choices for this auxiliary condition lead to different solutions $y(t)$ and, consequently, to different relationships between the input and the output. As we have indicated, for the most part in this book we focus on differential and difference equations used to describe systems that are LTI and causal, and in this case the auxiliary condition takes the form of the condition of initial rest. That is, as shown in Problem 1.44, for a causal LTI system, if $x(t) = 0$ for $t < t_0$, then $y(t)$ must also equal 0 for $t < t_0$. From eq. (2.96), we see that for our example $x(t) = 0$ for $t < 0$, and thus, the condition of initial rest implies that $y(t) = 0$ for $t < 0$. Evaluating eq. (2.106) at $t = 0$ and setting $y(0) = 0$ yields

$$0 = A + \frac{K}{5},$$

or

$$A = -\frac{K}{5}.$$

Thus, for $t > 0$,

$$y(t) = \frac{K}{5}\left[e^{3t} - e^{-2t}\right], \tag{2.107}$$

while for $t < 0$, $y(t) = 0$, because of the condition of initial rest. Combining these two cases, we obtain the full solution

$$y(t) = \frac{K}{5}\left[e^{3t} - e^{-2t}\right]u(t). \tag{2.108}$$

Example 2.14 illustrates several very important points concerning linear constant-coefficient differential equations and the systems they represent. First, the response to an input $x(t)$ will generally consist of the sum of a particular solution to the differential equation and a homogeneous solution—i.e., a solution to the differential equation with the input set to zero. The homogeneous solution is often referred to as the *natural response* of the system. The natural responses of simple electrical circuits and mechanical systems are explored in Problems 2.61 and 2.62.

In Example 2.14 we also saw that, in order to determine completely the relationship between the input and the output of a system described by a differential equation such as eq. (2.95), we must specify auxiliary conditions. An implication of this fact, which is illustrated in Problem 2.34, is that different choices of auxiliary conditions lead to different relationships between the input and the output. As we illustrated in the example, for the most part we will use the condition of initial rest for systems described by differential equations. In the example, since the input was 0 for $t < 0$, the condition of initial rest implied the initial condition $y(0) = 0$. As we have stated, and as illustrated in

Problem 2.33, under the condition of initial rest the system described by eq. (2.95) is LTI and causal.[3] For example, if we multiply the input in eq. (2.96) by 2, the resulting output would be twice the output in eq. (2.108).

It is important to emphasize that the condition of initial rest does not specify a zero initial condition at a fixed point in time, but rather adjusts this point in time so that the response is zero *until* the input becomes nonzero. Thus, if $x(t) = 0$ for $t \leq t_0$ for the causal LTI system described by eq. (2.95), then $y(t) = 0$ for $t \leq t_0$, and we would use the initial condition $y(t_0) = 0$ to solve for the output for $t > t_0$. As a physical example, consider again the circuit in Figure 1.1, also discussed in Example 1.8. Initial rest for this example corresponds to the statement that, until we connect a nonzero voltage source to the circuit, the capacitor voltage is zero. Thus, if we begin to use the circuit at noon today, the initial capacitor voltage as we connect the voltage source at noon today is zero. Similarly, if we begin to use the circuit at noon tomorrow instead, the initial capacitor voltage as we connect the voltage source at noon tomorrow is zero.

This example also provides us with some intuition as to why the condition of initial rest makes a system described by a linear constant-coefficient differential equation time invariant. For example, if we perform an experiment on the circuit, starting from initial rest, then, assuming that the coefficients R and C don't change over time, we would expect to get the same results whether we ran the experiment today or tomorrow. That is, if we perform identical experiments on the two days, where the circuit starts from initial rest at noon on each day, then we would expect to see identical responses—i.e., responses that are simply time-shifted by one day with respect to each other.

While we have used the first-order differential equation (2.95) as the vehicle for the discussion of these issues, the same ideas extend directly to systems described by higher order differential equations. A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{2.109}$$

The order refers to the highest derivative of the output $y(t)$ appearing in the equation. In the case when $N = 0$, eq. (2.109) reduces to

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{2.110}$$

In this case, $y(t)$ is an explicit function of the input $x(t)$ and its derivatives. For $N \geq 1$, eq. (2.109) specifies the output implicitly in terms of the input. In this case, the analysis of the equation proceeds just as in our discussion of the first-order differential equation in Example 2.14. The solution $y(t)$ consists of two parts—a particular solution to eq. (2.109)

[3]In fact, as is also shown in Problem 2.34, if the initial condition for eq. (2.95) is nonzero, the resulting system is incrementally linear. That is, the overall response can be viewed, much as in Figure 1.48, as the superposition of the response to the initial conditions alone (with input set to 0) and the response to the input with an initial condition of 0 (i.e., the response of the causal LTI system described by eq. (2.95)).

plus a solution to the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0. \tag{2.111}$$

The solutions to this equation are referred to as the *natural responses* of the system.

As in the first-order case, the differential equation (2.109) does not completely specify the output in terms of the input, and we need to identify auxiliary conditions to determine completely the input-output relationship for the system. Once again, different choices for these auxiliary conditions result in different input-output relationships, but for the most part, in this book we will use the condition of initial rest when dealing with systems described by differential equations. That is, if $x(t) = 0$ for $t \leq t_0$, we assume that $y(t) = 0$ for $t \leq t_0$, and therefore, the response for $t > t_0$ can be calculated from the differential equation (2.109) with the initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \ldots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0. \tag{2.112}$$

Under the condition of initial rest, the system described by eq. (2.109) is causal and LTI. Given the initial conditions in eq. (2.112), the output $y(t)$ can, in principle, be determined by solving the differential equation in the manner used in Example 2.14 and further illustrated in several problems at the end of the chapter. However, in Chapters 4 and 9 we will develop some tools for the analysis of continuous-time LTI systems that greatly facilitate the solution of differential equations and, in particular, provide us with powerful methods for analyzing and characterizing the properties of systems described by such equations.

2.4.2 Linear Constant-Coefficient Difference Equations

The discrete-time counterpart of eq. (2.109) is the Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{2.113}$$

An equation of this type can be solved in a manner exactly analogous to that for differential equations. (See Problem 2.32.)[4] Specifically, the solution $y[n]$ can be written as the sum of a particular solution to eq. (2.113) and a solution to the homogeneous equation

$$\sum_{k=0}^{N} a_k y[n-k] = 0. \tag{2.114}$$

[4]For a detailed treatment of the methods for solving linear constant-coefficient difference equations, see *Finite Difference Equations,* by H. Levy and F. Lessman (New York: Macmillan, Inc., 1961), or *Finite Difference Equations and Simulations* (Englewood Cliffs, NJ: Prentice-Hall, 1968) by F. B. Hildebrand. In Chapter 6, we present another method for solving difference equations that greatly facilitates the analysis of linear time-invariant systems that are so described. In addition, we refer the reader to the problems at the end of this chapter that deal with the solution of difference equations.

The solutions to this homogeneous equation are often referred to as the natural responses of the system described by eq. (2.113).

As in the continuous-time case, eq. (2.113) does not completely specify the output in terms of the input. To do this, we must also specify some auxiliary conditions. While there are many possible choices for auxiliary conditions, leading to different input-output relationships, we will focus for the most part on the condition of initial rest—i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$ as well. With initial rest, the system described by eq. (2.113) is LTI and causal.

Although all of these properties can be developed following an approach that directly parallels our discussion for differential equations, the discrete-time case offers an alternative path. This stems from the observation that eq. (2.113) can be rearranged in the form

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}. \tag{2.115}$$

Equation (2.115) directly expresses the output at time n in terms of previous values of the input and output. From this, we can immediately see the need for auxiliary conditions. In order to calculate $y[n]$, we need to know $y[n-1], \ldots, y[n-N]$. Therefore, if we are given the input for all n and a set of auxiliary conditions such as $y[-N], y[-N+1], \ldots, y[-1]$, eq. (2.115) can be solved for successive values of $y[n]$.

An equation of the form of eq. (2.113) or eq. (2.115) is called a *recursive equation,* since it specifies a recursive procedure for determining the output in terms of the input and previous outputs. In the special case when $N = 0$, eq. (2.115) reduces to

$$y[n] = \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k]. \tag{2.116}$$

This is the discrete-time counterpart of the continuous-time system given in eq. (2.110). Here, $y[n]$ is an explicit function of the present and previous values of the input. For this reason, eq. (2.116) is often called a *nonrecursive equation,* since we do not recursively use previously computed values of the output to compute the present value of the output. Therefore, just as in the case of the system given in eq. (2.110), we do not need auxiliary conditions in order to determine $y[n]$. Furthermore, eq. (2.116) describes an LTI system, and by direct computation, the impulse response of this system is found to be

$$h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}. \tag{2.117}$$

That is, eq. (2.116) is nothing more than the convolution sum. Note that the impulse response for it has finite duration; that is, it is nonzero only over a finite time interval. Because of this property, the system specified by eq. (2.116) is often called a *finite impulse response (FIR) system.*

Although we do not require auxiliary conditions for the case of $N = 0$, such conditions are needed for the recursive case when $N \ge 1$. To illustrate the solution of such an equation, and to gain some insight into the behavior and properties of recursive difference equations, let us examine the following simple example:

Example 2.15

Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]. \tag{2.118}$$

Eq. (2.118) can also be expressed in the form

$$y[n] = x[n] + \frac{1}{2}y[n-1], \tag{2.119}$$

highlighting the fact that we need the previous value of the output, $y[n-1]$, to calculate the current value. Thus, to begin the recursion, we need an initial condition.

For example, suppose that we impose the condition of initial rest and consider the input

$$x[n] = K\delta[n]. \tag{2.120}$$

In this case, since $x[n] = 0$ for $n \leq -1$, the condition of initial rest implies that $y[n] = 0$ for $n \leq -1$, so that we have as an initial condition $y[-1] = 0$. Starting from this initial condition, we can solve for successive values of $y[n]$ for $n \geq 0$ as follows:

$$y[0] = x[0] + \frac{1}{2}y[-1] = K, \tag{2.121}$$

$$y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}K, \tag{2.122}$$

$$y[2] = x[2] + \frac{1}{2}y[1] = \left(\frac{1}{2}\right)^2 K, \tag{2.123}$$

$$\vdots$$

$$y[n] = x[n] + \frac{1}{2}y[n-1] = \left(\frac{1}{2}\right)^n K. \tag{2.124}$$

Since the system specified by eq. (2.118) and the condition of initial rest is LTI, its input-output behavior is completely characterized by its impulse response. Setting $K = 1$, we see that the impulse response for the system considered in this example is

$$h[n] = \left(\frac{1}{2}\right)^n u[n]. \tag{2.125}$$

Note that the causal LTI system in Example 2.15 has an impulse response of infinite duration. In fact, if $N \geq 1$ in eq. (2.113), so that the difference equation is recursive, it is usually the case that the LTI system corresponding to this equation together with the condition of initial rest will have an impulse response of infinite duration. Such systems are commonly referred to as *infinite impulse response (IIR) systems.*

As we have indicated, for the most part we will use recursive difference equations in the context of describing and analyzing systems that are linear, time-invariant, and causal, and consequently, we will usually make the assumption of initial rest. In Chapters 5 and 10 we will develop tools for the analysis of discrete-time systems that will provide us with very useful and efficient methods for solving linear constant-coefficient difference equations and for analyzing the properties of the systems that they describe.

2.4.3 Block Diagram Representations of First-Order Systems Described by Differential and Difference Equations

An important property of systems described by linear constant-coefficient difference and differential equations is that they can be represented in very simple and natural ways in terms of block diagram interconnections of elementary operations. This is significant for a number of reasons. One is that it provides a pictorial representation which can add to our understanding of the behavior and properties of these systems. In addition, such representations can be of considerable value for the simulation or implementation of the systems. For example, the block diagram representation to be introduced in this section for continuous-time systems is the basis for early analog computer simulations of systems described by differential equations, and it can also be directly translated into a program for the simulation of such a system on a digital computer. In addition, the corresponding representation for discrete-time difference equations suggests simple and efficient ways in which the systems that the equations describe can be implemented in digital hardware. In this section, we illustrate the basic ideas behind these block diagram representations by constructing them for the causal first-order systems introduced in Examples 1.8–1.11. In Problems 2.57–2.60 and Chapters 9 and 10, we consider block diagrams for systems described by other, more complex differential and difference equations.

We begin with the discrete-time case and, in particular, the causal system described by the first-order difference equation

$$y[n] + ay[n-1] = bx[n]. \tag{2.126}$$

To develop a block diagram representation of this system, note that the evaluation of eq. (2.126) requires three basic operations: addition, multiplication by a coefficient, and delay (to capture the relationship between $y[n]$ and $y[n-1]$). Thus, let us define three basic network elements, as indicated in Figure 2.27. To see how these basic elements can be used to represent the causal system described by eq. (2.126), we rewrite this equation in the form that directly suggests a recursive algorithm for computing successive values of the output $y[n]$:

$$y[n] = -ay[n-1] + bx[n]. \tag{2.127}$$

This algorithm is represented pictorially in Figure 2.28, which is an example of a feedback system, since the output is fed back through a delay and a multiplication by a coefficient and is then added to $bx[n]$. The presence of feedback is a direct consequence of the recursive nature of eq. (2.127).

The block diagram in Figure 2.28 makes clear the required memory in this system and the consequent need for initial conditions. In particular, a delay corresponds to a memory element, as the element must retain the previous value of its input. Thus, the initial value of this memory element serves as a necessary initial condition for the recursive calculation specified pictorially in Figure 2.28 and mathematically in eq. (2.127). Of course, if the system described by eq. (2.126) is initially at rest, the initial value stored in the memory element is zero.

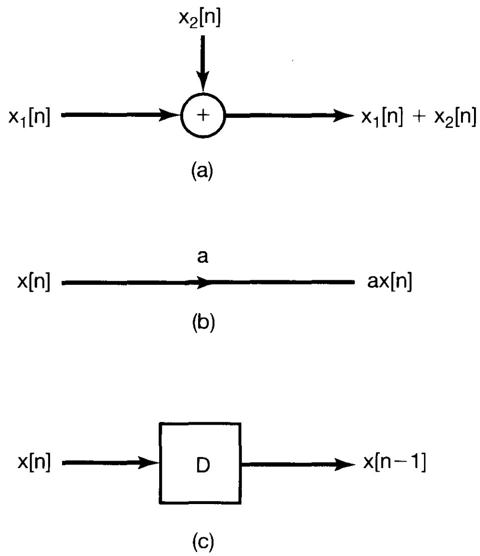


Figure 2.27 Basic elements for the block diagram representation of the causal system described by eq. (2.126): (a) an adder; (b) multiplication by a coefficient; (c) a unit delay.

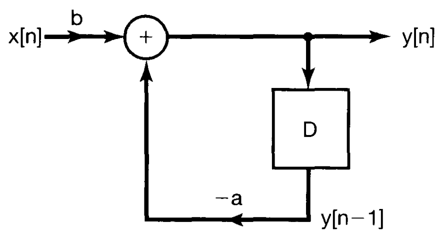


Figure 2.28 Block diagram representation for the causal discrete-time system described by eq. (2.126).

Consider next the causal continuous-time system described by a first-order differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t). \tag{2.128}$$

As a first attempt at defining a block diagram representation for this system, let us rewrite it as

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t). \tag{2.129}$$

The right-hand side of this equation involves three basic operations: addition, multiplication by a coefficient, and differentiation. Therefore, if we define the three basic network elements indicated in Figure 2.29, we can consider representing eq. (2.129) as an interconnection of these basic elements in a manner analogous to that used for the discrete-time system described previously, resulting in the block diagram of Figure 2.30.

While the latter figure is a valid representation of the causal system described by eq. (2.128), it is not the representation that is most frequently used or the representation that leads directly to practical implementations, since differentiators are both difficult to implement and extremely sensitive to errors and noise. An alternative implementation that

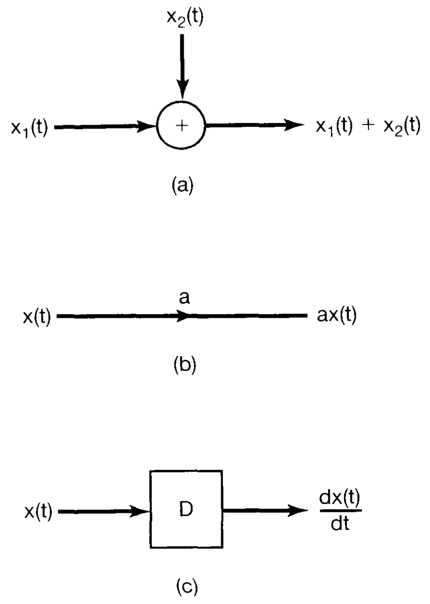


Figure 2.29 One possible set of basic elements for the block diagram representation of the continuous-time system described by eq. (2.128): (a) an adder; (b) multiplication by a coefficient; (c) a differentiator.

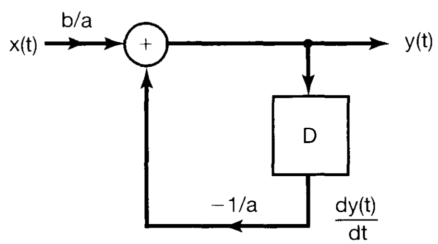


Figure 2.30 Block diagram representation for the system in eqs. (2.128) and (2.129), using adders, multiplications by coefficients, and differentiators.

is much more widely used can be obtained by first rewriting eq. (2.128) as

$$\frac{dy(t)}{dt} = bx(t) - ay(t) \tag{2.130}$$

and then integrating from $-\infty$ to t. Specifically, if we assume that in the system described by eq. (2.130) the value of $y(-\infty)$ is zero, then the integral of $dy(t)/dt$ from $-\infty$ to t is precisely $y(t)$. Consequently, we obtain the equation

$$y(t) = \int_{-\infty}^{t} \left[bx(\tau) - ay(\tau)\right] d\tau. \tag{2.131}$$

In this form, our system can be implemented using the adder and coefficient multiplier indicated in Figure 2.29, together with an *integrator*, as defined in Figure 2.31. Figure 2.32 is a block diagram representation for this system using these elements.

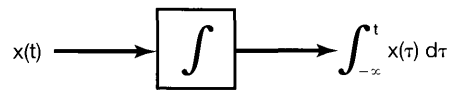


Figure 2.31 Pictorial representation of an integrator.

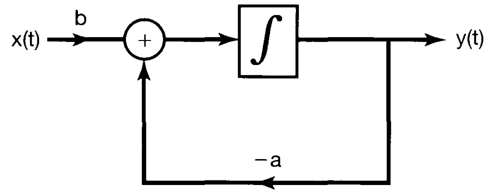


Figure 2.32 Block diagram representation for the system in eqs. (2.128) and (2.131), using adders, multiplications by coefficients, and integrators.

Since integrators can be readily implemented using operational amplifiers, representations such as that in Figure 2.32 lead directly to analog implementations, and indeed, this is the basis for both early analog computers and modern analog computation systems. Note that in the continuous-time case it is the integrator that represents the memory storage element of the system. This is perhaps more readily seen if we consider integrating eq. (2.130) from a finite point in time t_0, resulting in the expression

$$y(t) = y(t_0) + \int_{t_0}^{t} \left[bx(\tau) - ay(\tau)\right] d\tau. \tag{2.132}$$

Equation (2.132) makes clear the fact that the specification of $y(t)$ requires an initial condition, namely, the value of $y(t_0)$. It is precisely this value that the integrator stores at time t_0.

While we have illustrated block diagram constructions only for the simplest first-order differential and difference equations, such block diagrams can also be developed for higher order systems, providing both valuable intuition for and possible implementations of these systems. Examples of block diagrams for higher order systems can be found in Problems 2.58 and 2.60.

2.5 SINGULARITY FUNCTIONS

In this section, we take another look at the continuous-time unit impulse function in order to gain additional intuitions about this important idealized signal and to introduce a set of related signals known collectively as *singularity functions*. In particular, in Section 1.4.2 we suggested that a continuous-time unit impulse could be viewed as the idealization of a pulse that is "short enough" so that its shape and duration is of no practical consequence—i.e., so that as far as the response of any particular LTI system is concerned, all of the area under the pulse can be thought of as having been applied instantaneously. In this section, we would first like to provide a concrete example of what this means and then use the interpretation embodied within the example to show that the key to the use of unit impulses and other singularity functions is in the specification of how LTI systems respond to these idealized signals; i.e., the signals are in essence defined in terms of how they behave under convolution with other signals.

2.5.1 The Unit Impulse as an Idealized Short Pulse

From the sifting property, eq. (2.27), the unit impulse $\delta(t)$ is the impulse response of the identity system. That is,

$$x(t) = x(t) * \delta(t) \tag{2.133}$$

for any signal $x(t)$. Therefore, if we take $x(t) = \delta(t)$, we have

$$\delta(t) = \delta(t) * \delta(t). \tag{2.134}$$

Equation (2.134) is a basic property of the unit impulse, and it also has a significant implication for our interpretation of the unit impulse as an idealized pulse. For example, as in Section 1.4.2, suppose that we think of $\delta(t)$ as the limiting form of a rectangular pulse. Specifically, let $\delta_\Delta(t)$ correspond to the rectangular pulse defined in Figure 1.34, and let

$$r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t). \tag{2.135}$$

Then $r_\Delta(t)$ is as sketched in Figure 2.33. If we wish to interpret $\delta(t)$ as the limit as $\Delta \rightarrow 0$ of $\delta_\Delta(t)$, then, by virtue of eq. (2.134), the limit as $\Delta \rightarrow 0$ for $r_\Delta(t)$ must also be a unit impulse. In a similar manner, we can argue that the limits as $\Delta \rightarrow 0$ of $r_\Delta(t) * r_\Delta(t)$ or $r_\Delta(t) * \delta_\Delta(t)$ must be unit impulses, and so on. Thus, we see that for consistency, if we define the unit impulse as the limiting form of some signal, then in fact, there is an unlimited number of very dissimilar-looking signals, all of which behave like an impulse in the limit.

The key words in the preceding paragraph are "behave like an impulse," where, as we have indicated, what we mean by this is that the response of an LTI system to all of these signals is essentially identical, as long as the pulse is "short enough," i.e., Δ is "small enough." The following example illustrates this idea:

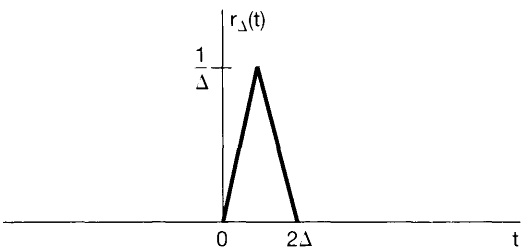


Figure 2.33 The signal $r_\Delta(t)$ defined in eq. (2.135).

Example 2.16

Consider the LTI system described by the first-order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t), \tag{2.136}$$

together with the condition of initial rest. Figure 2.34 depicts the response of this system to $\delta_\Delta(t)$, $r_\Delta(t)$, $r_\Delta(t) * \delta_\Delta(t)$, and $r_\Delta(t) * r_\Delta(t)$ for several values of Δ. For Δ large enough, the responses to these input signals differ noticeably. However, for Δ sufficiently small, the responses are essentially indistinguishable, so that all of the input signals "behave" in the same way. Furthermore, as suggested by the figure, the limiting form of all of these responses is precisely $e^{-2t}u(t)$. Since the limit of each of these signals as $\Delta \rightarrow 0$ is the unit impulse, we conclude that $e^{-2t}u(t)$ is the impulse response for this system.[5]

[5]In Chapters 4 and 9, we will describe much simpler ways to determine the impulse response of causal LTI systems described by linear constant-coefficient differential equations.

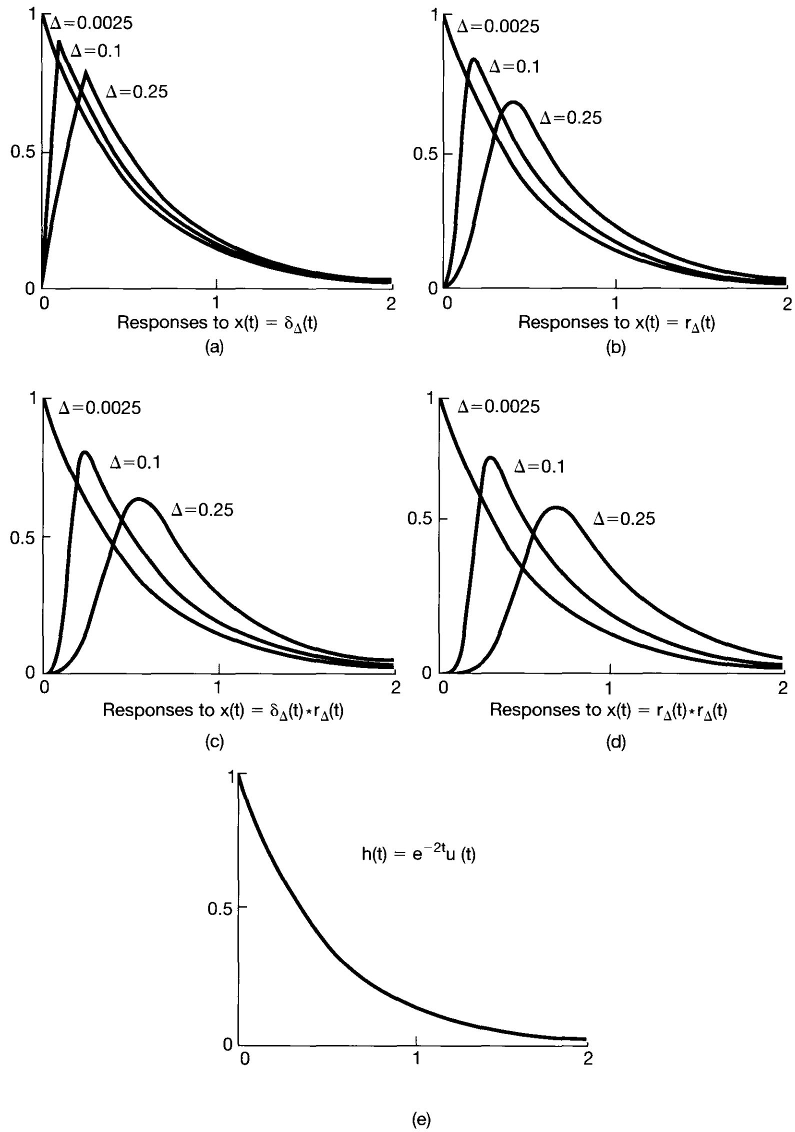


Figure 2.34 Interpretation of a unit impulse as the idealization of a pulse whose duration is "short enough" so that, as far as the response of an LTI system to this pulse is concerned, the pulse can be thought of as having been applied instantaneously: (a) responses of the causal LTI system described by eq. (2.136) to the input $\delta_\Delta(t)$ for $\Delta = 0.25$, 0.1, and 0.0025; (b) responses of the same system to $r_\Delta(t)$ for the same values of Δ; (c) responses to $\delta_\Delta(t) * r_\Delta(t)$; (d) responses to $r_\Delta(t) * r_\Delta(t)$; (e) the impulse response $h(t) = e^{-2t}u(t)$ for the system. Note that, for $\Delta = 0.25$, there are noticeable differences among the responses to these different signals; however, as Δ becomes smaller, the differences diminish, and all of the responses converge to the impulse response shown in (e).

One important point to be emphasized is that what we mean by "Δ small enough" depends on the particular LTI system to which the preceding pulses are applied. For example, in Figure 2.35, we have illustrated the responses to these pulses for different

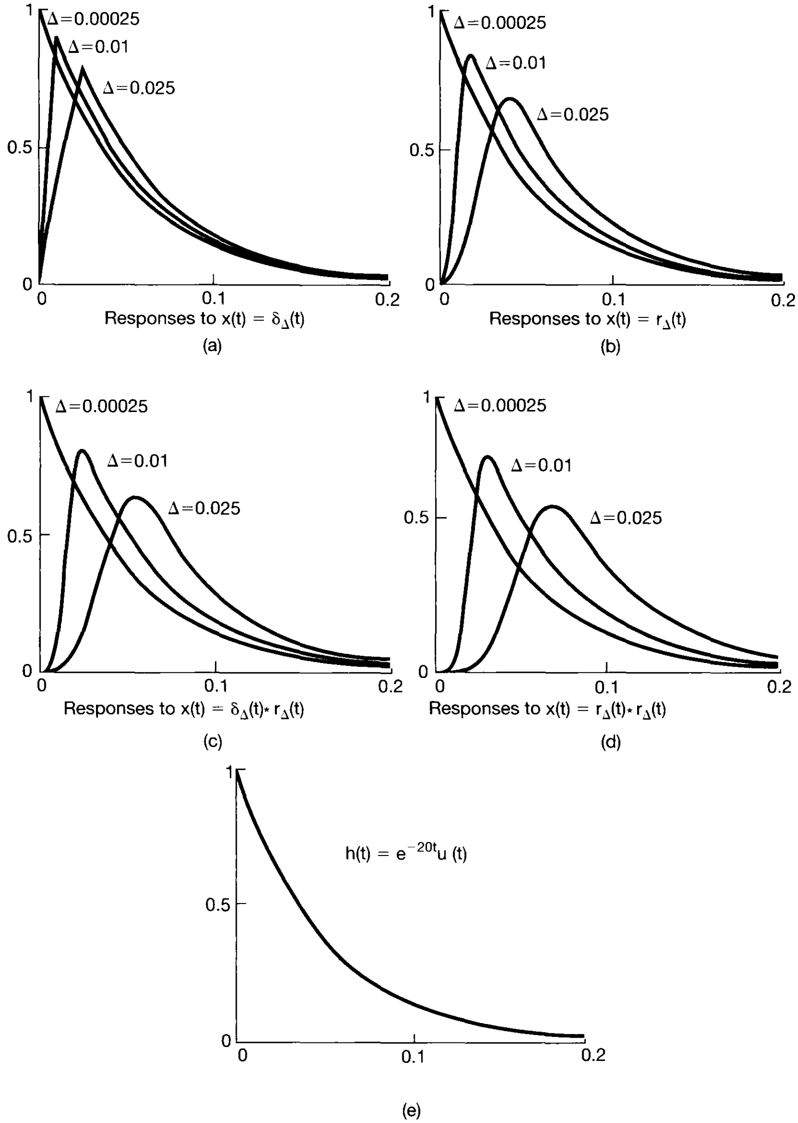


Figure 2.35 Finding a value of Δ that is "small enough" depends upon the system to which we are applying inputs: (a) responses of the causal LTI system described by eq. (2.137) to the input $\delta_\Delta(t)$ for $\Delta = 0.025$, 0.01, and 0.00025; (b) responses to $r_\Delta(t)$; (c) responses to $\delta_\Delta(t) * r_\Delta(t)$; (d) responses to $r_\Delta(t) * r_\Delta(t)$; (e) the impulse response $h(t) = e^{-20t}u(t)$ for the system. Comparing these responses to those in Figure 2.34, we see that we need to use a smaller value of Δ in this case before the duration and shape of the pulse are of no consequence.

values of Δ for the causal LTI system described by the first-order differential equation

$$\frac{dy(t)}{dt} + 20y(t) = x(t). \tag{2.137}$$

As seen in the figure, we need a smaller value of Δ in this case in order for the responses to be indistinguishable from each other and from the impulse response $h(t) = e^{-20t}u(t)$ for the system. Thus, while what we mean by "Δ small enough" is different for these two systems, we can find values of Δ small enough for both. The unit impulse is then the idealization of a short pulse whose duration is short enough for *all* systems.

2.5.2 Defining the Unit Impulse through Convolution

As the preceding example illustrates, for Δ small enough, the signals $\delta_\Delta(t)$, $r_\Delta(t)$, $r_\Delta(t) * \delta_\Delta(t)$, and $r_\Delta(t) * r_\Delta(t)$ all act like impulses when applied to an LTI system. In fact, there are many other signals for which this is true as well. What it suggests is that we should think of a unit impulse in terms of how an LTI system responds to it. While usually a function or signal is defined by what it is at each value of the independent variable, the primary importance of the unit impulse is not what it *is* at each value of t, but rather what it *does* under convolution. Thus, from the point of view of linear systems analysis, we may alternatively *define* the unit impulse as that signal which, when applied to an LTI system, yields the impulse response. That is, we define $\delta(t)$ as the signal for which

$$x(t) = x(t) * \delta(t) \tag{2.138}$$

for any $x(t)$. In this sense, signals, such as $\delta_\Delta(t)$, $r_\Delta(t)$, etc., which correspond to short pulses with vanishingly small duration as $\Delta \to 0$, all behave like a unit impulse in the limit because, if we replace $\delta(t)$ by any of these signals, then eq. (2.138) is satisfied in the limit.

All the properties of the unit impulse that we need can be obtained from the *operational definition* given by eq. (2.138). For example, if we let $x(t) = 1$ for all t, then

$$\begin{aligned} 1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t) &= \int_{-\infty}^{+\infty} \delta(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{+\infty} \delta(\tau)\,d\tau, \end{aligned}$$

so that the unit impulse has unit area.

It is sometimes useful to use another completely equivalent operational definition of $\delta(t)$. To obtain this alternative form, consider taking an arbitrary signal $g(t)$, reversing it in time to obtain $g(-t)$, and then convolving this with $\delta(t)$. Using eq. (2.138), we obtain

$$g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{+\infty} g(\tau - t)\,\delta(\tau)\,d\tau,$$

which, for $t = 0$, yields

$$g(0) = \int_{-\infty}^{+\infty} g(\tau)\,\delta(\tau)\,d\tau. \tag{2.139}$$

Therefore, the operational definition of $\delta(t)$ given by eq. (2.138) implies eq. (2.139). On the other hand, eq. (2.139) implies eq. (2.138). To see this, let $x(t)$ be a given signal, fix a time t, and define

$$g(\tau) = x(t - \tau).$$

Then, using eq. (2.139), we have

$$x(t) = g(0) = \int_{-\infty}^{+\infty} g(\tau)\,\delta(\tau)\,d\tau = \int_{-\infty}^{+\infty} x(t-\tau)\,\delta(\tau)\,d\tau,$$

which is precisely eq. (2.138). Therefore, eq. (2.139) is an equivalent operational definition of the unit impulse. That is, the unit impulse is the signal which, when multiplied by a signal $g(t)$ and then integrated from $-\infty$ to $+\infty$, produces the value $g(0)$.

Since we will be concerned principally with LTI systems, and thus with convolution, the characterization of $\delta(t)$ given in eq. (2.138) will be the one to which we will refer most often. However, eq. (2.139) is useful in determining some of the other properties of the unit impulse. For example, consider the signal $f(t)\,\delta(t)$, where $f(t)$ is another signal. Then, from eq. (2.139),

$$\int_{-\infty}^{+\infty} g(\tau) f(\tau)\,\delta(\tau)\,d\tau = g(0) f(0). \qquad (2.140)$$

On the other hand, if we consider the signal $f(0)\,\delta(t)$, we see that

$$\int_{-\infty}^{+\infty} g(\tau) f(0)\,\delta(\tau)\,d\tau = g(0) f(0). \qquad (2.141)$$

Comparing eqs. (2.140) and (2.141), we find that the two signals $f(t)\,\delta(t)$ and $f(0)\,\delta(t)$ behave identically when they are multiplied by any signal $g(t)$ and then integrated from $-\infty$ to $+\infty$. Consequently, using this form of the operational definition of signals, we conclude that

$$f(t)\,\delta(t) = f(0)\,\delta(t), \qquad (2.142)$$

which is a property that we derived by alternative means in Section 1.4.2. [See eq. (1.76).]

2.5.3 Unit Doublets and Other Singularity Functions

The unit impulse is one of a class of signals known as *singularity functions,* each of which can be defined operationally in terms of its behavior under convolution. Consider the LTI system for which the output is the derivative of the input, i.e.,

$$y(t) = \frac{dx(t)}{dt} \qquad (2.143)$$

The unit impulse response of this system is the derivative of the unit impulse, which is called the *unit doublet* $u_1(t)$. From the convolution representation for LTI systems, we have

$$\frac{dx(t)}{dt} = x(t) * u_1(t) \tag{2.144}$$

for any signal $x(t)$. Just as eq. (2.138) serves as the operational definition of $\delta(t)$, we will take eq. (2.144) as the operational definition of $u_1(t)$. Similarly, we can define $u_2(t)$, the second derivative of $\delta(t)$, as the impulse response of an LTI system that takes the second derivative of the input, i.e.,

$$\frac{d^2x(t)}{dt^2} = x(t) * u_2(t). \tag{2.145}$$

From eq. (2.144), we see that

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = x(t) * u_1(t) * u_1(t), \tag{2.146}$$

and therefore,

$$u_2(t) = u_1(t) * u_1(t). \tag{2.147}$$

In general, $u_k(t)$, $k > 0$, is the kth derivative of $\delta(t)$ and thus is the impulse response of a system that takes the kth derivative of the input. Since this system can be obtained as the cascade of k differentiators, we have

$$u_k(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{k \text{ times}}. \tag{2.148}$$

As with the unit impulse, each of these singularity functions has properties that can be derived from its operational definition. For example, if we consider the constant signal $x(t) = 1$, we find that

$$\begin{aligned} 0 = \frac{dx(t)}{dt} &= x(t) * u_1(t) = \int_{-\infty}^{+\infty} u_1(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{+\infty} u_1(\tau)\,d\tau, \end{aligned}$$

so that the unit doublet has zero area. Moreover, if we convolve the signal $g(-t)$ with $u_1(t)$, we obtain

$$\int_{-\infty}^{+\infty} g(\tau - t)u_1(\tau)\,d\tau = g(-t) * u_1(t) = \frac{dg(-t)}{dt} = -g'(-t),$$

which, for $t = 0$, yields

$$-g'(0) = \int_{-\infty}^{+\infty} g(\tau)u_1(\tau)\,d\tau. \tag{2.149}$$

In an analogous manner, we can derive related properties of $u_1(t)$ and higher order singularity functions, and several of these properties are considered in Problem 2.69.

As with the unit impulse, each of these singularity functions can be informally related to short pulses. For example, since the unit doublet is formally the derivative of the unit impulse, we can think of the doublet as the idealization of the derivative of a short pulse with unit area. For instance, consider the short pulse $\delta_\Delta(t)$ in Figure 1.34. This pulse behaves like an impulse as $\Delta \to 0$. Consequently, we would expect its derivative to behave like a doublet as $\Delta \to 0$. As verified in Problem 2.72, $d\delta_\Delta(t)/dt$ is as depicted in Figure 2.36: It consists of a unit impulse at $t = 0$ with area $+1/\Delta$, followed by a unit impulse of area $-1/\Delta$ at $t = \Delta$, i.e.,

$$\frac{d\delta_\Delta(t)}{dt} = \frac{1}{\Delta}[\delta(t) - \delta(t - \Delta)]. \tag{2.150}$$

Consequently, using the fact that $x(t) * \delta(t - t_0) = x(t - t_0)$ [see eq. (2.70)], we find that

$$x(t) * \frac{d\delta_\Delta(t)}{dt} = \frac{x(t) - x(t - \Delta)}{\Delta} \cong \frac{dx(t)}{dt}, \tag{2.151}$$

where the approximation becomes increasingly accurate as $\Delta \to 0$. Comparing eq. (2.151) with eq. (2.144), we see that $d\delta_\Delta(t)/dt$ does indeed behave like a unit doublet as $\Delta \to 0$.

In addition to singularity functions that are derivatives of different orders of the unit impulse, we can also define signals that represent successive integrals of the unit impulse function. As we saw in Example 2.13, the unit step is the impulse response of an integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau.$$

Therefore,

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau, \tag{2.152}$$

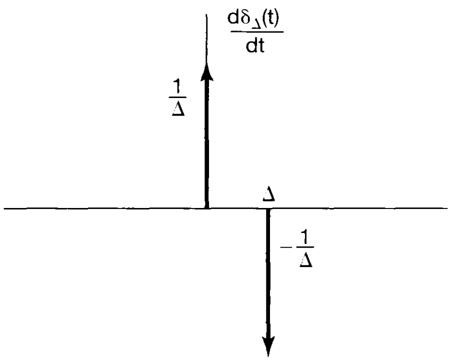


Figure 2.36 The derivative $d\delta_\Delta(t)/dt$ of the short rectangular pulse $\delta_\Delta(t)$ of Figure 1.34.

and we also have the following operational definition of $u(t)$:

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau. \tag{2.153}$$

Similarly, we can define the system that consists of a cascade of two integrators. Its impulse response is denoted by $u_{-2}(t)$, which is simply the convolution of $u(t)$, the impulse response of one integrator, with itself:

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau)\,d\tau. \tag{2.154}$$

Since $u(t)$ equals 0 for $t < 0$ and equals 1 for $t > 0$, it follows that

$$u_{-2}(t) = tu(t). \tag{2.155}$$

This signal, which is referred to as the *unit ramp function*, is shown in Figure 2.37. Also, we can obtain an operational definition for the behavior of $u_{-2}(t)$ under convolution from eqs. (2.153) and (2.154):

$$\begin{aligned} x(t) * u_{-2}(t) &= x(t) * u(t) * u(t) \\ &= \left(\int_{-\infty}^{t} x(\sigma)\,d\sigma\right) * u(t) \\ &= \int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} x(\sigma)\,d\sigma\right)d\tau. \end{aligned} \tag{2.156}$$

In an analogous fashion, we can define higher order integrals of $\delta(t)$ as the impulse responses of cascades of integrators:

$$u_{-k}(t) = \underbrace{u(t) * \cdots * u(t)}_{k \text{ times}} = \int_{-\infty}^{t} u_{-(k-1)}(\tau)\,d\tau. \tag{2.157}$$

The convolution of $x(t)$ with $u_{-3}(t), u_{-4}(t), \ldots$ generate correspondingly higher order integrals of $x(t)$. Also, note that the integrals in eq. (2.157) can be evaluated directly (see

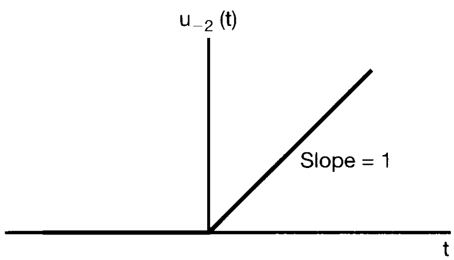


Figure 2.37 Unit ramp function.

Problem 2.73), as was done in eq. (2.155), to obtain

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!}u(t). \tag{2.158}$$

Thus, unlike the derivatives of $\delta(t)$, the successive integrals of the unit impulse are functions that can be defined for each value of t [eq. (2.158)], as well as by their behavior under convolution.

At times it will be worthwhile to use an alternative notation for $\delta(t)$ and $u(t)$, namely,

$$\delta(t) = u_0(t), \tag{2.159}$$

$$u(t) = u_{-1}(t). \tag{2.160}$$

With this notation, $u_k(t)$ for $k > 0$ denotes the impulse response of a cascade of k differentiators, $u_0(t)$ is the impulse response of the identity system, and, for $k < 0$, $u_k(t)$ is the impulse response of a cascade of $|k|$ integrators. Furthermore, since a differentiator is the inverse system of an integrator,

$$u(t) * u_1(t) = \delta(t),$$

or, in our alternative notation,

$$u_{-1}(t) * u_1(t) = u_0(t). \tag{2.161}$$

More generally, from eqs. (2.148), (2.157), and (2.161), we see that for any integers k and r,

$$u_k(t) * u_r(t) = u_{k+r}(t). \tag{2.162}$$

If k and r are both positive, eq. (2.162) states that a cascade of k differentiators followed by r more differentiators yields an output that is the $(k+r)$th derivative of the input. Similarly, if k is negative and r is negative, we have a cascade of $|k|$ integrators followed by another $|r|$ integrators. Also, if k is negative and r is positive, we have a cascade of $|k|$ integrators followed by r differentiators, and the overall system is equivalent to a cascade of $|k + r|$ integrators if $k+r < 0$, a cascade of $k+r$ differentiators if $k+r > 0$, or the identity system if $k + r = 0$. Therefore, by defining singularity functions in terms of their behavior under convolution, we obtain a characterization that allows us to manipulate them with relative ease and to interpret them directly in terms of their significance for LTI systems. Since this is our primary concern in the book, the operational definition for singularity functions that we have given in this section will suffice for our purposes.[6]

[6]As mentioned in Chapter 1, singularity functions have been heavily studied in the field of mathematics under the alternative names of *generalized functions* and *distribution theory*. The approach we have taken in this section is actually closely allied in spirit with the rigorous approach taken in the references given in footnote 3 of Section 1.4.

2.6 SUMMARY

In this chapter, we have developed important representations for LTI systems, both in discrete time and in continuous time. In discrete time we derived a representation of signals as weighted sums of shifted unit impulses, and we then used this to derive the convolution-sum representation for the response of a discrete-time LTI system. In continuous time we derived an analogous representation of continuous-time signals as weighted integrals of shifted unit impulses, and we used this to derive the convolution integral representation for continuous-time LTI systems. These representations are extremely important, as they allow us to compute the response of an LTI system to an arbitrary input in terms of the system's response to a unit impulse. Moreover, in Section 2.3 the convolution sum and integral provided us with a means of analyzing the properties of LTI systems and, in particular, of relating LTI system properties, including causality and stability, to corresponding properties of the unit impulse response. Also, in Section 2.5 we developed an interpretation of the continuous-time unit impulse and other related singularity functions in terms of their behavior under convolution. This interpretation is particularly useful in the analysis of LTI systems.

An important class of continuous-time systems consists of those described by linear constant-coefficient differential equations. Similarly, in discrete time, linear constant-coefficient difference equations play an equally important role. In Section 2.4, we examined simple examples of differential and difference equations and discussed some of the properties of systems described by these types of equations. In particular, systems described by linear constant-coefficient differential and difference equations together with the condition of initial rest are causal and LTI. In subsequent chapters, we will develop additional tools that greatly facilitate our ability to analyze such systems.

Chapter 2 Problems

The first section of problems belongs to the basic category, and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

Extension problems introduce applications, concepts, or methods beyond those presented in the text.

BASIC PROBLEMS WITH ANSWERS

2.1. Let

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3] \quad \text{and} \quad h[n] = 2\delta[n+1] + 2\delta[n-1].$$

Compute and plot each of the following convolutions:

(a) $y_1[n] = x[n] * h[n]$ **(b)** $y_2[n] = x[n+2] * h[n]$

(c) $y_3[n] = x[n] * h[n+2]$

2.2. Consider the signal

$$h[n] = \left(\frac{1}{2}\right)^{n-1} \{u[n+3] - u[n-10]\}.$$

Express A and B in terms of n so that the following equation holds:

$$h[n-k] = \begin{cases} (\frac{1}{2})^{n-k-1}, & A \leq k \leq B \\ 0, & \text{elsewhere} \end{cases}.$$

2.3. Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2],$$

$$h[n] = u[n+2].$$

Determine and plot the output $y[n] = x[n] * h[n]$.

2.4. Compute and plot $y[n] = x[n] * h[n]$, where

$$x[n] = \begin{cases} 1, & 3 \leq n \leq 8 \\ 0, & \text{otherwise} \end{cases},$$

$$h[n] = \begin{cases} 1, & 4 \leq n \leq 15 \\ 0, & \text{otherwise} \end{cases}.$$

2.5. Let

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 9 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad h[n] = \begin{cases} 1, & 0 \leq n \leq N \\ 0, & \text{elsewhere} \end{cases},$$

where $N \leq 9$ is an integer. Determine the value of N, given that $y[n] = x[n] * h[n]$ and

$$y[4] = 5, \quad y[14] = 0.$$

2.6. Compute and plot the convolution $y[n] = x[n] * h[n]$, where

$$x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1] \quad \text{and} \quad h[n] = u[n-1].$$

2.7. A linear system S has the relationship

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] g[n-2k]$$

between its input $x[n]$ and its output $y[n]$, where $g[n] = u[n] - u[n-4]$.

(a) Determine $y[n]$ when $x[n] = \delta[n-1]$.
(b) Determine $y[n]$ when $x[n] = \delta[n-2]$.
(c) Is S LTI?
(d) Determine $y[n]$ when $x[n] = u[n]$.

2.8. Determine and sketch the convolution of the following two signals:

$$x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{elsewhere} \end{cases},$$

$$h(t) = \delta(t+2) + 2\delta(t+1).$$

2.9. Let

$$h(t) = e^{2t}u(-t+4) + e^{-2t}u(t-5).$$

Determine A and B such that

$$h(t-\tau) = \begin{cases} e^{-2(t-\tau)}, & \tau < A \\ 0, & A < \tau < B \\ e^{2(t-\tau)}, & B < \tau \end{cases}.$$

2.10. Suppose that

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

and $h(t) = x(t/\alpha)$, where $0 < \alpha \le 1$.
(a) Determine and sketch $y(t) = x(t) * h(t)$.
(b) If $dy(t)/dt$ contains only three discontinuities, what is the value of α?

2.11. Let

$$x(t) = u(t-3) - u(t-5) \quad \text{and} \quad h(t) = e^{-3t}u(t).$$

(a) Compute $y(t) = x(t) * h(t)$.
(b) Compute $g(t) = (dx(t)/dt) * h(t)$.
(c) How is $g(t)$ related to $y(t)$?

2.12. Let

$$y(t) = e^{-t}u(t) * \sum_{k=-\infty}^{\infty} \delta(t-3k).$$

Show that $y(t) = Ae^{-t}$ for $0 \le t < 3$, and determine the value of A.

2.13. Consider a discrete-time system S_1 with impulse response

$$h[n] = \left(\frac{1}{5}\right)^n u[n].$$

(a) Find the integer A such that $h[n] - Ah[n-1] = \delta[n]$.
(b) Using the result from part (a), determine the impulse response $g[n]$ of an LTI system S_2 which is the inverse system of S_1.

2.14. Which of the following impulse responses correspond(s) to stable LTI systems?
(a) $h_1(t) = e^{-(1-2j)t}u(t)$ **(b)** $h_2(t) = e^{-t}\cos(2t)u(t)$

2.15. Which of the following impulse responses correspond(s) to stable LTI systems?
(a) $h_1[n] = n\cos(\frac{\pi}{4}n)u[n]$ **(b)** $h_2[n] = 3^n u[-n+10]$

2.16. For each of the following statements, determine whether it is true or false:
(a) If $x[n] = 0$ for $n < N_1$ and $h[n] = 0$ for $n < N_2$, then $x[n] * h[n] = 0$ for $n < N_1 + N_2$.
(b) If $y[n] = x[n] * h[n]$, then $y[n-1] = x[n-1] * h[n-1]$.
(c) If $y(t) = x(t) * h(t)$, then $y(-t) = x(-t) * h(-t)$.
(d) If $x(t) = 0$ for $t > T_1$ and $h(t) = 0$ for $t > T_2$, then $x(t) * h(t) = 0$ for $t > T_1 + T_2$.

2.17. Consider an LTI system whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d}{dt}y(t) + 4y(t) = x(t). \qquad \text{(P2.17–1)}$$

The system also satisfies the condition of initial rest.
(a) If $x(t) = e^{(-1+3j)t}u(t)$, what is $y(t)$?
(b) Note that $\mathcal{R}e\{x(t)\}$ will satisfy eq. (P2.17–1) with $\mathcal{R}e\{y(t)\}$. Determine the output $y(t)$ of the LTI system if

$$x(t) = e^{-t}\cos(3t)u(t).$$

2.18. Consider a causal LTI system whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] = \frac{1}{4}y[n-1] + x[n].$$

Determine $y[n]$ if $x[n] = \delta[n-1]$.

2.19. Consider the cascade of the following two systems S_1 and S_2, as depicted in Figure P2.19:

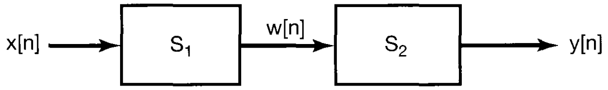


Figure P2.19

$$S_1 : \text{causal LTI,}$$
$$w[n] = \frac{1}{2}w[n-1] + x[n];$$
$$S_2 : \text{causal LTI,}$$
$$y[n] = \alpha y[n-1] + \beta w[n].$$

The difference equation relating $x[n]$ and $y[n]$ is:

$$y[n] = -\frac{1}{8}y[n-2] + \frac{3}{4}y[n-1] + x[n].$$

(a) Determine α and β.
(b) Show the impulse response of the cascade connection of S_1 and S_2.

2.20. Evaluate the following integrals:
(a) $\int_{-\infty}^{\infty} u_0(t)\cos(t)\,dt$
(b) $\int_0^5 \sin(2\pi t)\,\delta(t+3)\,dt$
(c) $\int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)\,d\tau$

BASIC PROBLEMS

2.21. Compute the convolution $y[n] = x[n] * h[n]$ of the following pairs of signals:

(a) $\left.\begin{array}{l} x[n] = \alpha^n u[n], \\ h[n] = \beta^n u[n], \end{array}\right\} \alpha \neq \beta$
(b) $x[n] = h[n] = \alpha^n u[n]$
(c) $x[n] = (-\frac{1}{2})^n u[n-4]$
$h[n] = 4^n u[2-n]$
(d) $x[n]$ and $h[n]$ are as in Figure P2.21.

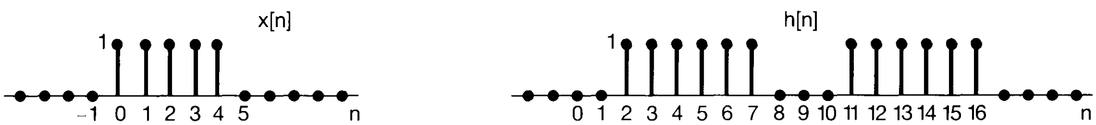


Figure P2.21

2.22. For each of the following pairs of waveforms, use the convolution integral to find the response $y(t)$ of the LTI system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.

(a) $\left.\begin{array}{l} x(t) = e^{-\alpha t}u(t) \\ h(t) = e^{-\beta t}u(t) \end{array}\right\}$ (Do this both when $\alpha \neq \beta$ and when $\alpha = \beta$.)

(b) $x(t) = u(t) - 2u(t-2) + u(t-5)$
$h(t) = e^{2t}u(1-t)$
(c) $x(t)$ and $h(t)$ are as in Figure P2.22(a).
(d) $x(t)$ and $h(t)$ are as in Figure P2.22(b).
(e) $x(t)$ and $h(t)$ are as in Figure P2.22(c).

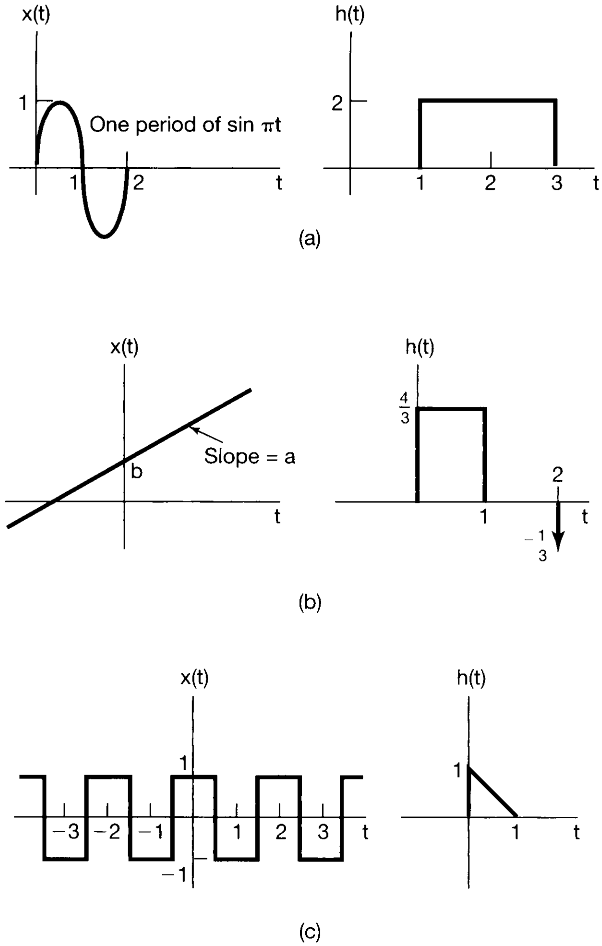


Figure P2.22

2.23. Let $h(t)$ be the triangular pulse shown in Figure P2.23(a), and let $x(t)$ be the impulse train depicted in Figure P2.23(b). That is,

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT).$$

Determine and sketch $y(t) = x(t) * h(t)$ for the following values of T:
(a) $T = 4$ **(b)** $T = 2$ **(c)** $T = 3/2$ **(d)** $T = 1$

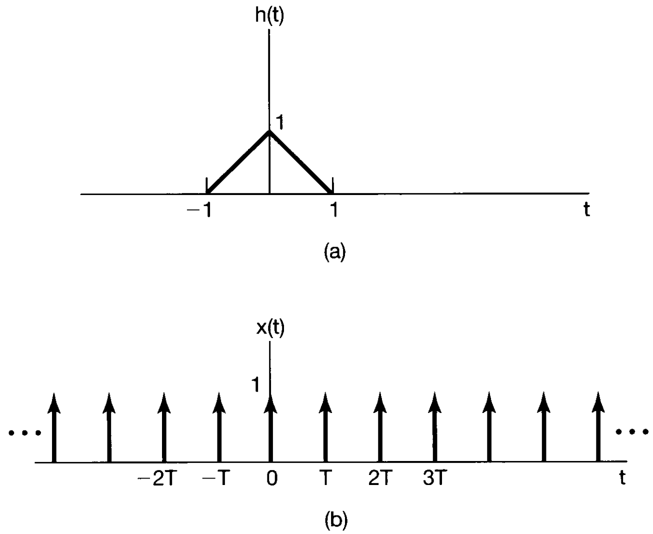


Figure P2.23

2.24. Consider the cascade interconnection of three causal LTI systems, illustrated in Figure P2.24(a). The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-2],$$

and the overall impulse response is as shown in Figure P2.24(b).

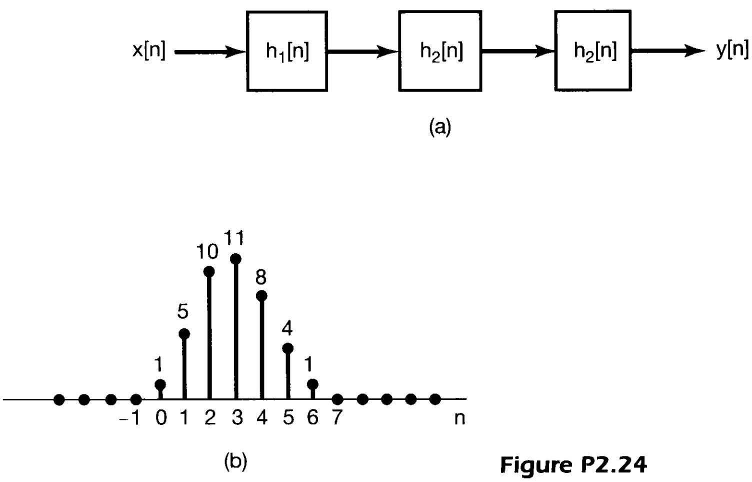


Figure P2.24

(a) Find the impulse response $h_1[n]$.
(b) Find the response of the overall system to the input

$$x[n] = \delta[n] - \delta[n-1].$$

2.25. Let the signal

$$y[n] = x[n] * h[n],$$

where

$$x[n] = 3^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$$

and

$$h[n] = \left(\frac{1}{4}\right)^n u[n+3].$$

(a) Determine $y[n]$ *without* utilizing the distributive property of convolution.
(b) Determine $y[n]$ *utilizing* the distributive property of convolution.

2.26. Consider the evaluation of

$$y[n] = x_1[n] * x_2[n] * x_3[n],$$

where $x_1[n] = (0.5)^n u[n]$, $x_2[n] = u[n+3]$, and $x_3[n] = \delta[n] - \delta[n-1]$.
(a) Evaluate the convolution $x_1[n] * x_2[n]$.
(b) Convolve the result of part (a) with $x_3[n]$ in order to evaluate $y[n]$.
(c) Evaluate the convolution $x_2[n] * x_3[n]$.
(d) Convolve the result of part (c) with $x_1[n]$ in order to evaluate $y[n]$.

2.27. We define the area under a continuous-time signal $v(t)$ as

$$A_v = \int_{-\infty}^{+\infty} v(t)\,dt.$$

Show that if $y(t) = x(t) * h(t)$, then

$$A_y = A_x A_h.$$

2.28. The following are the impulse responses of discrete-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.
(a) $h[n] = (\frac{1}{5})^n u[n]$
(b) $h[n] = (0.8)^n u[n+2]$
(c) $h[n] = (\frac{1}{2})^n u[-n]$
(d) $h[n] = (5)^n u[3-n]$
(e) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[n-1]$
(f) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[1-n]$
(g) $h[n] = n(\frac{1}{3})^n u[n-1]$

2.29. The following are the impulse responses of continuous-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.
(a) $h(t) = e^{-4t}u(t-2)$
(b) $h(t) = e^{-6t}u(3-t)$
(c) $h(t) = e^{-2t}u(t+50)$
(d) $h(t) = e^{2t}u(-1-t)$

(e) $h(t) = e^{-6|t|}$
(f) $h(t) = te^{-t}u(t)$
(g) $h(t) = (2e^{-t} - e^{(t-100)/100})u(t)$

2.30. Consider the first-order difference equation

$$y[n] + 2y[n-1] = x[n].$$

Assuming the condition of initial rest (i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$), find the impulse response of a system whose input and output are related by this difference equation. You may solve the problem by rearranging the difference equation so as to express $y[n]$ in terms of $y[n-1]$ and $x[n]$ and generating the values of $y[0],\ y[+1],\ y[+2], \ldots$ in that order.

2.31. Consider the LTI system initially at rest and described by the difference equation

$$y[n] + 2y[n-1] = x[n] + 2x[n-2].$$

Find the response of this system to the input depicted in Figure P2.31 by solving the difference equation recursively.

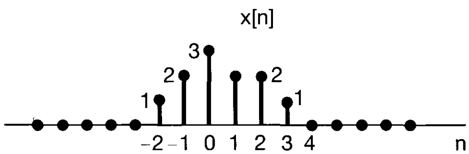


Figure P2.31

2.32. Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n], \tag{P2.32–1}$$

and suppose that

$$x[n] = \left(\frac{1}{3}\right)^n u[n]. \tag{P2.32–2}$$

Assume that the solution $y[n]$ consists of the sum of a particular solution $y_p[n]$ to eq. (P2.32–1) and a homogeneous solution $y_h[n]$ satisfying the equation

$$y_h[n] - \frac{1}{2}y_h[n-1] = 0.$$

(a) Verify that the homogeneous solution is given by

$$y_h[n] = A\left(\frac{1}{2}\right)^n$$

(b) Let us consider obtaining a particular solution $y_p[n]$ such that

$$y_p[n] - \frac{1}{2}y_p[n-1] = \left(\frac{1}{3}\right)^n u[n].$$

By assuming that $y_p[n]$ is of the form $B(\frac{1}{3})^n$ for $n \geq 0$, and substituting this in the above difference equation, determine the value of B.

(c) Suppose that the LTI system described by eq. (P2.32–1) and initially at rest has as its input the signal specified by eq. (P2.32–2). Since $x[n] = 0$ for $n < 0$, we have that $y[n] = 0$ for $n < 0$. Also, from parts (a) and (b) we have that $y[n]$ has the form

$$y[n] = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$$

for $n \geq 0$. In order to solve for the unknown constant A, we must specify a value for $y[n]$ for some $n \geq 0$. Use the condition of initial rest and eqs. (P2.32–1) and (P2.32–2) to determine $y[0]$. From this value determine the constant A. The result of this calculation yields the solution to the difference equation (P2.32–1) under the condition of initial rest, when the input is given by eq. (P2.32–2).

2.33. Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t). \qquad \text{(P2.33–1)}$$

The system also satisfies the condition of initial rest.

(a) (i) Determine the system output $y_1(t)$ when the input is $x_1(t) = e^{3t}u(t)$.

(ii) Determine the system output $y_2(t)$ when the input is $x_2(t) = e^{2t}u(t)$.

(iii) Determine the system output $y_3(t)$ when the input is $x_3(t) = \alpha e^{3t}u(t) + \beta e^{2t}u(t)$, where α and β are real numbers. Show that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

(iv) Now let $x_1(t)$ and $x_2(t)$ be arbitrary signals such that

$$x_1(t) = 0, \quad \text{for } t < t_1,$$
$$x_2(t) = 0, \quad \text{for } t < t_2.$$

Letting $y_1(t)$ be the system output for input $x_1(t)$, $y_2(t)$ be the system output for input $x_2(t)$, and $y_3(t)$ be the system output for $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, show that

$$y_3(t) = \alpha y_1(t) + \beta y_2(t).$$

We may therefore conclude that the system under consideration is linear.

(b) (i) Determine the system output $y_1(t)$ when the input is $x_1(t) = Ke^{2t}u(t)$.

(ii) Determine the system output $y_2(t)$ when the input is $x_2(t) = Ke^{2(t-T)}u(t-T)$. Show that $y_2(t) = y_1(t-T)$.

(iii) Now let $x_1(t)$ be an arbitrary signal such that $x_1(t) = 0$ for $t < t_0$. Letting $y_1(t)$ be the system output for input $x_1(t)$ and $y_2(t)$ be the system output for $x_2(t) = x_1(t-T)$, show that

$$y_2(t) = y_1(t-T).$$

We may therefore conclude that the system under consideration is time invariant. In conjunction with the result derived in part (a), we conclude that the given system is LTI. Since this system satisfies the condition of initial rest, it is causal as well.

2.34. The initial rest assumption corresponds to a zero-valued auxiliary condition being imposed at a time determined in accordance with the input signal. In this problem we show that if the auxiliary condition used is nonzero or if it is always applied at a fixed time (regardless of the input signal) the corresponding system cannot be LTI. Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation (P2.33–1).

(a) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not linear.
(b) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not time invariant.
(c) Given the auxiliary condition $y(1) = 1$, show that the system is incrementally linear.
(d) Given the auxiliary condition $y(1) = 0$, show that the system is linear but not time invariant.
(e) Given the auxiliary condition $y(0) + y(4) = 0$, show that the system is linear but not time invariant.

2.35. In the previous problem we saw that application of an auxiliary condition at a fixed time (regardless of the input signal) leads to the corresponding system being not time-invariant. In this problem, we explore the effect of fixed auxiliary conditions on the causality of a system. Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation (P2.33–1). Assume that the auxiliary condition associated with the differential equation is $y(0) = 0$. Determine the output of the system for each of the following two inputs:

(a) $x_1(t) = 0, \text{ for all } t$

(b) $x_2(t) = \begin{cases} 0, & t < -1 \\ 1, & t > -1 \end{cases}$

Observe that if $y_1(t)$ is the output for input $x_1(t)$ and $y_2(t)$ is the output for input $x_2(t)$, then $y_1(t)$ and $y_2(t)$ are not identical for $t < -1$, even though $x_1(t)$ and $x_2(t)$ are identical for $t < -1$. Use this observation as the basis of an argument to conclude that the given system is not causal.

2.36. Consider a discrete-time system whose input $x[n]$ and output $y[n]$ are related by

$$y[n] = \left(\frac{1}{2}\right)y[n-1] + x[n].$$

(a) Show that if this system satisfies the condition of initial rest (i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$), then it is linear and time invariant.
(b) Show that if this system does not satisfy the condition of initial rest, but instead uses the auxiliary condition $y[0] = 0$, it is not causal. [*Hint:* Use an approach similar to that used in Problem 2.35.]

2.37. Consider a system whose input and output are related by the first-order differential equation (P2.33–1). Assume that the system satisfies the condition of final rest [i. e., if $x(t) = 0$ for $t > t_0$, then $y(t) = 0$ for $t > t_0$]. Show that this system is *not* causal. [*Hint:* Consider two inputs to the system, $x_1(t) = 0$ and $x_2(t) = e^t(u(t) - u(t-1))$, which result in outputs $y_1(t)$ and $y_2(t)$, respectively. Then show that $y_1(t) \neq y_2(t)$ for $t < 0$.]

2.38. Draw block diagram representations for causal LTI systems described by the following difference equations:

(a) $y[n] = \frac{1}{3}y[n-1] + \frac{1}{2}x[n]$

(b) $y[n] = \frac{1}{3}y[n-1] + x[n-1]$

2.39. Draw block diagram representations for causal LTI systems described by the following differential equations:

(a) $y(t) = -(\frac{1}{2})dy(t)/dt + 4x(t)$

(b) $dy(t)/dt + 3y(t) = x(t)$

ADVANCED PROBLEMS

2.40. **(a)** Consider an LTI system with input and output related through the equation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau-2)\,d\tau.$$

What is the impulse response $h(t)$ for this system?

(b) Determine the response of the system when the input $x(t)$ is as shown in Figure P2.40.

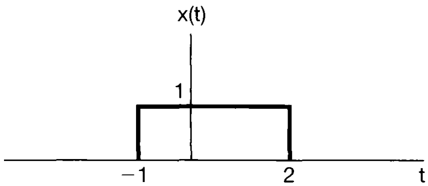


Figure P2.40

2.41. Consider the signal

$$x[n] = \alpha^n u[n].$$

(a) Sketch the signal $g[n] = x[n] - \alpha x[n-1]$.

(b) Use the result of part (a) in conjunction with properties of convolution in order to determine a sequence $h[n]$ such that

$$x[n] * h[n] = \left(\frac{1}{2}\right)^n \{u[n+2] - u[n-2]\}.$$

2.42. Suppose that the signal

$$x(t) = u(t+0.5) - u(t-0.5)$$

is convolved with the signal

$$h(t) = e^{j\omega_0 t}.$$

(a) Determine a value of ω_0 which ensures that

$$y(0) = 0,$$

where $y(t) = x(t) * h(t)$.

(b) Is your answer to the previous part unique?

2.43. One of the important properties of convolution, in both continuous and discrete time, is the associativity property. In this problem, we will check and illustrate this property.

(a) Prove the equality

$$[x(t) * h(t)] * g(t) = x(t) * [h(t) * g(t)] \qquad \text{(P2.43–1)}$$

by showing that both sides of eq. (P2.43–1) equal

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)h(\sigma)g(t-\tau-\sigma)\,d\tau\,d\sigma.$$

(b) Consider two LTI systems with the unit sample responses $h_1[n]$ and $h_2[n]$ shown in Figure P2.43(a). These two systems are cascaded as shown in Figure P2.43(b). Let $x[n] = u[n]$.

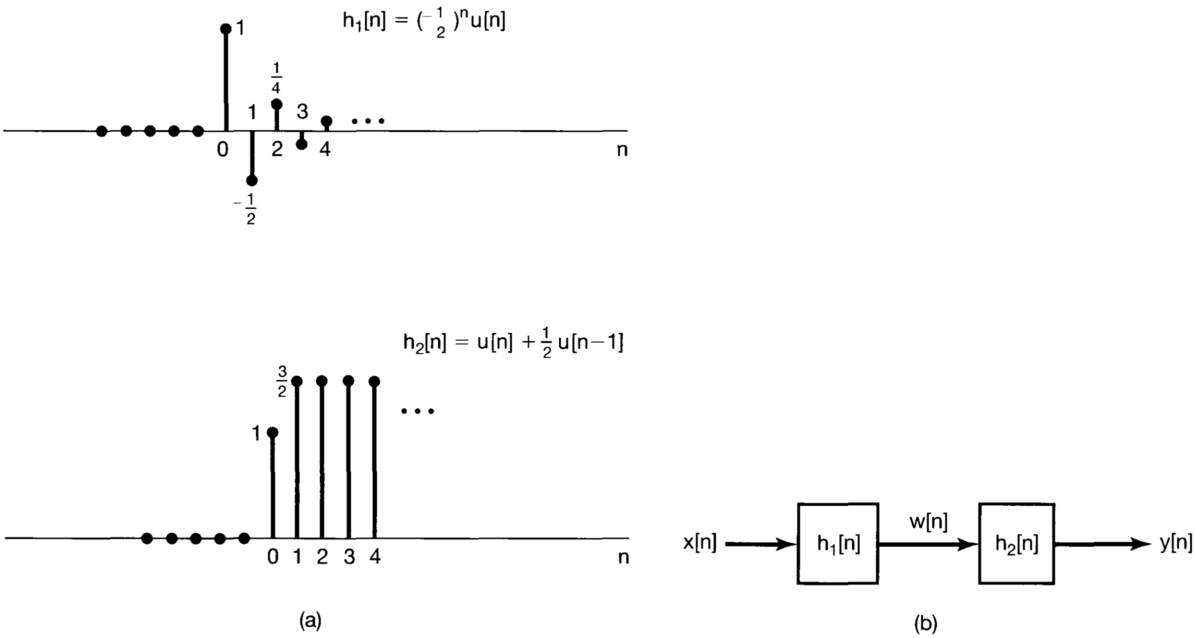


Figure P2.43

(i) Compute $y[n]$ by first computing $w[n] = x[n] * h_1[n]$ and then computing $y[n] = w[n] * h_2[n]$; that is, $y[n] = [x[n] * h_1[n]] * h_2[n]$.

(ii) Now find $y[n]$ by first convolving $h_1[n]$ and $h_2[n]$ to obtain $g[n] = h_1[n] * h_2[n]$ and then convolving $x[n]$ with $g[n]$ to obtain $y[n] = x[n] * [h_1[n] * h_2[n]]$.

The answers to (i) and (ii) should be identical, illustrating the associativity property of discrete-time convolution.

(c) Consider the cascade of two LTI systems as in Figure P2.43(b), where in this case

$$h_1[n] = \sin 8n$$

and

$$h_2[n] = a^n u[n], \quad |a| < 1,$$

and where the input is

$$x[n] = \delta[n] - a\delta[n-1].$$

Determine the output $y[n]$. (*Hint:* The use of the associative and commutative properties of convolution should greatly facilitate the solution.)

2.44. (a) If

$$x(t) = 0, \quad |t| > T_1,$$

and

$$h(t) = 0, \quad |t| > T_2,$$

then

$$x(t) * h(t) = 0, \quad |t| > T_3$$

for some positive number T_3. Express T_3 in terms of T_1 and T_2.

(b) A discrete-time LTI system has input $x[n]$, impulse response $h[n]$, and output $y[n]$. If $h[n]$ is known to be zero everywhere outside the interval $N_0 \leq n \leq N_1$ and $x[n]$ is known to be zero everywhere outside the interval $N_2 \leq n \leq N_3$, then the output $y[n]$ is constrained to be zero everywhere, except on some interval $N_4 \leq n \leq N_5$.

(i) Determine N_4 and N_5 in terms of N_0, N_1, N_2, and N_3.

(ii) If the interval $N_0 \leq n \leq N_1$ is of length M_h, $N_2 \leq n \leq N_3$ is of length M_x, and $N_4 \leq n \leq N_5$ is of length M_y, express M_y in terms of M_h and M_x.

(c) Consider a discrete-time LTI system with the property that if the input $x[n] = 0$ for all $n \geq 10$, then the output $y[n] = 0$ for all $n \geq 15$. What condition must $h[n]$, the impulse response of the system, satisfy for this to be true?

(d) Consider an LTI system with impulse response in Figure P2.44. Over what interval must we know $x(t)$ in order to determine $y(0)$?

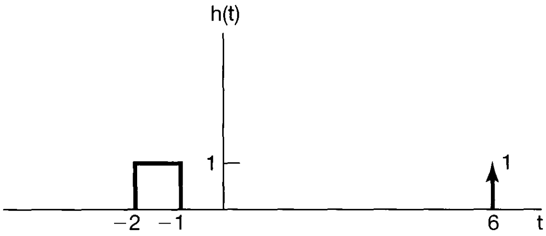


Figure P2.44

2.45. (a) Show that if the response of an LTI system to $x(t)$ is the output $y(t)$, then the response of the system to

$$x'(t) = \frac{dx(t)}{dt}$$

is $y'(t)$. Do this problem in three different ways:

(i) Directly from the properties of linearity and time invariance and the fact that

$$x'(t) = \lim_{h \to 0} \frac{x(t) - x(t-h)}{h}.$$

(ii) By differentiating the convolution integral.

(iii) By examining the system in Figure P2.45.

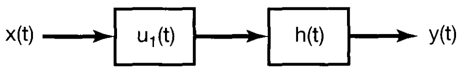


Figure P2.45

(b) Demonstrate the validity of the following relationships:

(i) $y'(t) = x(t) * h'(t)$

(ii) $y(t) = (\int_{-\infty}^{t} x(\tau)\, d\tau) * h'(t) = \int_{-\infty}^{t}[x'(\tau) * h(\tau)]d\tau = x'(t) * (\int_{-\infty}^{t} h(\tau)d\tau)$

[*Hint:* These are easily done using block diagrams as in (iii) of part (a) and the fact that $u_1(t) * u_{-1}(t) = \delta(t)$.]

(c) An LTI system has the response $y(t) = \sin \omega_0 t$ to input $x(t) = e^{-5t}u(t)$. Use the result of part (a) to aid in determining the impulse response of this system.

(d) Let $s(t)$ be the unit step response of a continuous-time LTI system. Use part (b) to deduce that the response $y(t)$ to the input $x(t)$ is

$$y(t) = \int_{-\infty}^{+\infty} x'(\tau) * s(t - \tau)\, d\tau. \qquad \text{(P2.45–1)}$$

Show also that

$$x(t) = \int_{-\infty}^{+\infty} x'(\tau) u(t - \tau)\, d\tau. \qquad \text{(P2.45–2)}$$

(e) Use eq. (P2.45–1) to determine the response of an LTI system with step response

$$s(t) = (e^{-3t} - 2e^{-2t} + 1)u(t)$$

to the input $x(t) = e^{t}u(t)$.

(f) Let $s[n]$ be the unit step response of a discrete-time LTI system. What are the discrete-time counterparts of eqs. (P2.45–1) and (P2.45–2)?

2.46. Consider an LTI system S and a signal $x(t) = 2e^{-3t}u(t-1)$. If

$$x(t) \longrightarrow y(t)$$

and

$$\frac{dx(t)}{dt} \longrightarrow -3y(t) + e^{-2t}u(t),$$

determine the impulse response $h(t)$ of S.

2.47. We are given a certain linear time-invariant system with impulse response $h_0(t)$. We are told that when the input is $x_0(t)$ the output is $y_0(t)$, which is sketched in Figure P2.47. We are then given the following set of inputs to linear time-invariant systems with the indicated impulse responses:

	Input $x(t)$	*Impulse response* $h(t)$
(a)	$x(t) = 2x_0(t)$	$h(t) = h_0(t)$
(b)	$x(t) = x_0(t) - x_0(t-2)$	$h(t) = h_0(t)$
(c)	$x(t) = x_0(t-2)$	$h(t) = h_0(t+1)$
(d)	$x(t) = x_0(-t)$	$h(t) = h_0(t)$
(e)	$x(t) = x_0(-t)$	$h(t) = h_0(-t)$
(f)	$x(t) = x_0'(t)$	$h(t) = h_0'(t)$

[Here $x_0'(t)$ and $h_0'(t)$ denote the first derivatives of $x_0(t)$ and $h_0(t)$, respectively.]

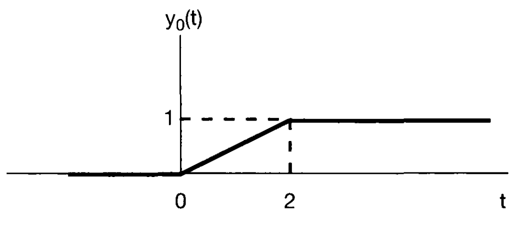


Figure P2.47

In each of these cases, determine whether or not we have enough information to determine the output $y(t)$ when the input is $x(t)$ and the system has impulse response $h(t)$. If it is possible to determine $y(t)$, provide an accurate sketch of it with numerical values clearly indicated on the graph.

2.48. Determine whether each of the following statements concerning LTI systems is true or false. Justify your answers.

(a) If $h(t)$ is the impulse response of an LTI system and $h(t)$ is periodic and nonzero, the system is unstable.

(b) The inverse of a causal LTI system is always causal.

(c) If $|h[n]| \leq K$ for each n, where K is a given number, then the LTI system with $h[n]$ as its impulse response is stable.

(d) If a discrete-time LTI system has an impulse response $h[n]$ of finite duration, the system is stable.

(e) If an LTI system is causal, it is stable.

(f) The cascade of a noncausal LTI system with a causal one is necessarily non-causal.

(g) A continuous-time LTI system is stable if and only if its step response $s(t)$ is absolutely integrable—that is, if and only if

$$\int_{-\infty}^{+\infty} |s(t)|\, dt < \infty.$$

(h) A discrete-time LTI system is causal if and only if its step response $s[n]$ is zero for $n < 0$.

2.49. In the text, we showed that if $h[n]$ is absolutely summable, i.e., if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty,$$

then the LTI system with impulse response $h[n]$ is stable. This means that absolute summability is a *sufficient* condition for stability. In this problem, we shall show that it is also a *necessary* condition. Consider an LTI system with impulse response $h[n]$ that is not absolutely summable; that is,

$$\sum_{k=-\infty}^{+\infty} |h[k]| = \infty.$$

(a) Suppose that the input to this system is

$$x[n] = \begin{cases} 0, & \text{if } h[-n] = 0 \\ \frac{h[-n]}{|h[-n]|}, & \text{if } h[-n] \neq 0 \end{cases}.$$

Does this input signal represent a bounded input? If so, what is the smallest number B such that

$$|x[n]| \leq B \text{ for all } n?$$

(b) Calculate the output at $n = 0$ for this particular choice of input. Does the result prove the contention that absolute summability is a necessary condition for stability?

(c) In a similar fashion, show that a continuous-time LTI system is stable if and only if its impulse response is absolutely integrable.

2.50. Consider the cascade of two systems shown in Figure P2.50. The first system, A, is known to be LTI. The second system, B, is known to be the inverse of system A. Let $y_1(t)$ denote the response of system A to $x_1(t)$, and let $y_2(t)$ denote the response of system A to $x_2(t)$.

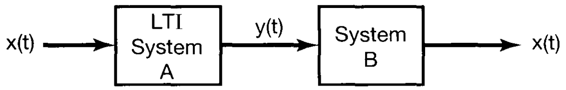


Figure P2.50

(a) What is the response of system B to the input $ay_1(t) + by_2(t)$, where a and b are constants?

(b) What is the response of system B to the input $y_1(t - \tau)$?

2.51. In the text, we saw that the overall input-output relationship of the cascade of two LTI systems does not depend on the order in which they are cascaded. This fact, known as the commutativity property, depends on both the linearity and the time invariance of both systems. In this problem, we illustrate the point.

(a) Consider two discrete-time systems A and B, where system A is an LTI system with unit sample response $h[n] = (1/2)^n u[n]$. System B, on the other hand, is linear but time varying. Specifically, if the input to system B is $w[n]$, its output is

$$z[n] = nw[n].$$

Show that the commutativity property does not hold for these two systems by computing the impulse responses of the cascade combinations in Figures P2.51(a) and P2.51(b), respectively.

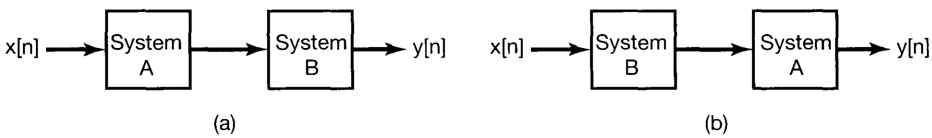


Figure P2.51

(b) Suppose that we replace system B in each of the interconnected systems of Figure P2.51 by the system with the following relationship between its input $w[n]$ and output $z[n]$:

$$z[n] = w[n] + 2.$$

Repeat the calculations of part (a) in this case.

2.52. Consider a discrete-time LTI system with unit sample response

$$h[n] = (n + 1)\alpha^n u[n],$$

where $|\alpha| < 1$. Show that the step response of this system is

$$s[n] = \left[\frac{1}{(\alpha - 1)^2} - \frac{\alpha}{(\alpha - 1)^2}\alpha^n + \frac{\alpha}{(\alpha - 1)}(n + 1)\alpha^n\right]u[n].$$

(*Hint:* Note that

$$\sum_{k=0}^{N}(k + 1)\alpha^k = \frac{d}{d\alpha}\sum_{k=0}^{N+1}\alpha^k.)$$

2.53. **(a)** Consider the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0. \qquad \text{(P2.53–1)}$$

Show that if s_0 is a solution of the equation

$$p(s) = \sum_{k=0}^{N} a_k s^k = 0, \qquad \text{(P2.53–2)}$$

then $Ae^{s_0 t}$ is a solution of eq. (P2.53–1), where A is an arbitrary complex constant.

(b) The polynomial $p(s)$ in eq. (P2.53–2) can be factored in terms of its roots $s_1, \ldots, s_r$ as

$$p(s) = a_N(s - s_1)^{\sigma_1}(s - s_2)^{\sigma_2}\ldots(s - s_r)^{\sigma_r},$$

where the s_i are the distinct solutions of eq. (P2.53–2) and the σ_i are their *multiplicities*—that is, the number of times each root appears as a solution of the equation. Note that

$$\sigma_1 + \sigma_2 + \ldots + \sigma_r = N.$$

In general, if $\sigma_i > 1$, then not only is $Ae^{s_i t}$ a solution of eq. (P2.53–1), but so is $At^j e^{s_i t}$, as long as j is an integer greater than or equal to zero and less than or equal to $\sigma_i - 1$. To illustrate this, show that if $\sigma_i = 2$, then $Ate^{s_i t}$ is a solution of eq. (P2.53–1). [*Hint:* Show that if s is an arbitrary complex number, then

$$\sum_{k=0}^{N} \frac{d^k(Ate^{st})}{dt^k} = Ap(s)te^{st} + A\frac{dp(s)}{ds}e^{st}.]$$

Thus, the most general solution of eq. (P2.53–1) is

$$\sum_{i=1}^{r}\sum_{j=0}^{\sigma_i-1} A_{ij}t^j e^{s_i t},$$

where the A_{ij} are arbitrary complex constants.

(c) Solve the following homogeneous differential equations with the specified auxiliary conditions:

(i) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0,\ y(0) = 0,\ \ y'(0) = 2$

(ii) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0,\ y(0) = 1,\ \ y'(0) = -1$

(iii) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0,\ y(0) = 0,\ \ y'(0) = 0$

(iv) $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 0,\ y(0) = 1,\ \ y'(0) = 1$

(v) $\frac{d^3y(t)}{dt^3} + \frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - y(t) = 0,\ y(0) = 1,\ \ y'(0) = 1,\ \ y''(0) = -2$

(vi) $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = 0,\ y(0) = 1,\ \ y'(0) = 1$

2.54. (a) Consider the homogeneous difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = 0, \tag{P2.54–1}$$

Show that if z_0 is a solution of the equation

$$\sum_{k=0}^{N} a_k z^{-k} = 0, \tag{P2.54–2}$$

then Az_0^n is a solution of eq. (P2.54–1), where A is an arbitrary constant.

(b) As it is more convenient for the moment to work with polynomials that have only nonnegative powers of z, consider the equation obtained by multiplying both sides of eq. (P2.54–2) by z^N:

$$p(z) = \sum_{k=0}^{N} a_k z^{N-k} = 0. \tag{P2.54–3}$$

The polynomial $p(z)$ can be factored as

$$p(z) = a_0(z - z_1)^{\sigma_1}\dots(z - z_r)^{\sigma_r},$$

where the $z_1, \dots, z_r$ are the distinct roots of $p(z)$.

Show that if $y[n] = nz^{n-1}$, then

$$\sum_{k=0}^{N} a_k y[n-k] = \frac{dp(z)}{dz}z^{n-N} + (n-N)p(z)z^{n-N-1}.$$

Use this fact to show that if $\sigma_i = 2$, then both Az_i^n and Bnz_i^{n-1} are solutions of eq. (P2.54–1), where A and B are arbitrary complex constants. More generally, one can use this same procedure to show that if $\sigma_i > 1$, then

$$A\frac{n!}{r!(n-r)!}z^{n-r}$$

is a solution of eq. (P2.54–1) for $r = 0, 1, \ldots, \sigma_i - 1$.[7]

(c) Solve the following homogeneous difference equations with the specified auxiliary conditions:

(i) $y[n] + \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 0;\ y[0] = 1,\ y[-1] = -6$

(ii) $y[n] - 2y[n-1] + y[n-2] = 0;\ y[0] = 1,\ y[1] = 0$

(iii) $y[n] - 2y[n-1] + y[n-2] = 0;\ y[0] = 1,\ y[10] = 21$

(iv) $y[n] - \frac{\sqrt{2}}{2}y[n-1] + \frac{1}{4}y[n-2] = 0;\ y[0] = 0,\ y[-1] = 1$

2.55. In the text we described one method for solving linear constant-coefficient difference equations, and another method for doing this was illustrated in Problem 2.30. If the assumption of initial rest is made so that the system described by the difference equation is LTI and causal, then, in principle, we can determine the unit impulse response $h[n]$ using either of these procedures. In Chapter 5, we describe another method that allows us to determine $h[n]$ in a more elegant way. In this problem we describe yet another approach, which basically shows that $h[n]$ can be determined by solving the homogeneous equation with appropriate initial conditions.

(a) Consider the system initially at rest and described by the equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]. \qquad \text{(P2.55–1)}$$

Assuming that $x[n] = \delta[n]$, what is $y[0]$? What equation does $h[n]$ satisfy for $n \geq 1$, and with what auxiliary condition? Solve this equation to obtain a closed-form expression for $h[n]$.

(b) Consider next the LTI system initially at rest and described by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1]. \qquad \text{(P2.55–2)}$$

This system is depicted in Figure P2.55(a) as a cascade of two LTI systems that are initially at rest. Because of the properties of LTI systems, we can reverse the order of the systems in the cascade to obtain an alternative representation of the same overall system, as illustrated in Figure P2.55(b). From this fact, use the result of part (a) to determine the impulse response for the system described by eq. (P2.55–2).

(c) Consider again the system of part (a), with $h[n]$ denoting its impulse response. Show, by verifying that eq. (P2.55–3) satisfies the difference equation (P2.55–1), that the response $y[n]$ to an arbitrary input $x[n]$ is in fact given by the convolution sum

$$y[n] = \sum_{m=-\infty}^{+\infty} h[n-m]x[m]. \qquad \text{(P2.55–3)}$$

[7]Here, we are using factorial notation—that is, $k! = k(k-1)(k-2)\ldots(2)(1)$, where $0!$ is defined to be 1.

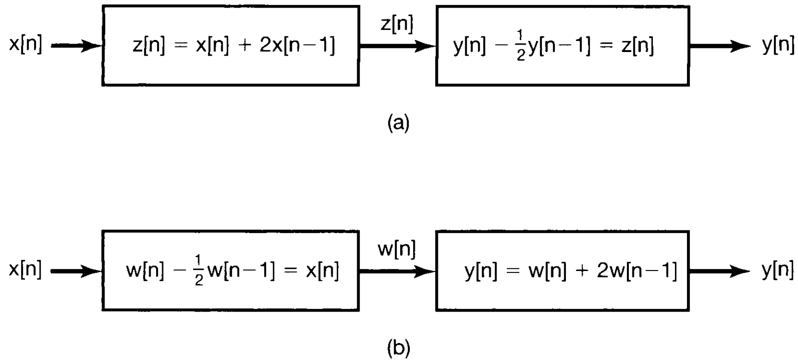


Figure P2.55

(d) Consider the LTI system initially at rest and described by the difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = x[n]. \qquad \text{(P2.55–4)}$$

Assuming that $a_0 \neq 0$, what is $y[0]$ if $x[n] = \delta[n]$? Using this result, specify the homogeneous equation and initial conditions that the impulse response of the system must satisfy.

Consider next the causal LTI system described by the difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \qquad \text{(P2.55–5)}$$

Express the impulse response of this system in terms of that for the LTI system described by eq. (P2.55–4).

(e) There is an alternative method for determining the impulse response of the LTI system described by eq. (P2.55–5). Specifically, given the condition of initial rest, i.e., in this case, $y[-N] = y[-N+1] = \ldots = y[-1] = 0$, solve eq. (P2.55–5) recursively when $x[n] = \delta[n]$ in order to determine $y[0], \ldots, y[M]$. What equation does $h[n]$ satisfy for $n \geq M$? What are the appropriate initial conditions for this equation?

(f) Using either of the methods outlined in parts (d) and (e), find the impulse responses of the causal LTI systems described by the following equations:

(i) $y[n] - y[n-2] = x[n]$
(ii) $y[n] - y[n-2] = x[n] + 2x[n-1]$
(iii) $y[n] - y[n-2] = 2x[n] - 3x[n-4]$
(iv) $y[n] - (\sqrt{3}/2)y[n-1] + \frac{1}{4}y[n-2] = x[n]$

2.56. In this problem, we consider a procedure that is the continuous-time counterpart of the technique developed in Problem 2.55. Again, we will see that the problem of determining the impulse response $h(t)$ for $t > 0$ for an LTI system initially at rest and described by a linear constant-coefficient differential equation reduces to the problem of solving the homogeneous equation with appropriate initial conditions.

(a) Consider the LTI system initially at rest and described by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t). \tag{P2.56–1}$$

Suppose that $x(t) = \delta(t)$. In order to determine the value of $y(t)$ *immediately after* the application of the unit impulse, consider integrating eq. (P2.56–1) from $t = 0^-$ to $t = 0^+$ (i.e., from "just before" to "just after" the application of the impulse). This yields

$$y(0^+) - y(0^-) + 2\int_{0^-}^{0^+} y(\tau)\,d\tau = \int_{0^-}^{0^+} \delta(\tau)\,d\tau = 1. \tag{P2.56–2}$$

Since the system is initially at rest and $x(t) = 0$ for $t < 0$, $y(0^-) = 0$. To satisfy eq. (P2.56–2) we must have $y(0^+) = 1$. Thus, since $x(t) = 0$ for $t > 0$, the impulse response of our system is the solution of the homogeneous differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 0$$

with initial condition

$$y(0^+) = 1.$$

Solve this differential equation to obtain the impulse response $h(t)$ for the system. Check your result by showing that

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)\,d\tau$$

satisfies eq. (P2.56–1) for any input $x(t)$.

(b) To generalize the preceding argument, consider an LTI system initially at rest and described by the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = x(t) \tag{P2.56–3}$$

with $x(t) = \delta(t)$. Assume the condition of initial rest, which, since $x(t) = 0$ for $t < 0$, implies that

$$y(0^-) = \frac{dy}{dt}(0^-) = \ldots = \frac{d^{N-1}y}{dt^{N-1}}(0^-) = 0. \tag{P2.56–4}$$

Integrate both sides of eq. (P2.56–3) once from $t = 0^-$ to $t = 0^+$, and use eq. (P2.56–4) and an argument similar to that used in part (a) to show that the resulting equation is satisfied with

$$y(0^+) = \frac{dy}{dt}(0^+) = \ldots = \frac{d^{N-2}y}{dt^{N-2}}(0^+) = 0 \qquad \text{(P2.56–5}a\text{)}$$

and

$$\frac{d^{N-1}y}{dt^{N-1}}(0^+) = \frac{1}{a_N}. \qquad \text{(P2.56–5}b\text{)}$$

Consequently, the system's impulse response for $t > 0$ can be obtained by solving the homogeneous equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

with initial conditions given by eqs. (P2.56–5).

(c) Consider now the causal LTI system described by the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \qquad \text{(P2.56–6)}$$

Express the impulse response of this system in terms of that for the system of part (b). (*Hint:* Examine Figure P2.56.)

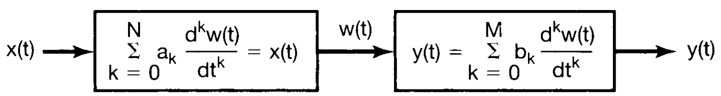


Figure P2.56

(d) Apply the procedures outlined in parts (b) and (c) to find the impulse responses for the LTI systems initially at rest and described by the following differential equations:

(i) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$

(ii) $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = x(t)$

(e) Use the results of parts (b) and (c) to deduce that if $M \geq N$ in eq. (P2.56–6), then the impulse response $h(t)$ will contain singularity terms concentrated at $t = 0$. In particular, $h(t)$ will contain a term of the form

$$\sum_{r=0}^{M-N} \alpha_r u_r(t),$$

where the α_r are constants and the $u_r(t)$ are the singularity functions defined in Section 2.5.

(f) Find the impulse responses of the causal LTI systems described by the following differential equations:

(i) $\frac{dy(t)}{dt} + 2y(t) = 3\frac{dx(t)}{dt} + x(t)$

(ii) $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{d^3x(t)}{dt^3} + 2\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 3x(t)$

2.57. Consider a causal LTI system S whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] = -ay[n-1] + b_0x[n] + b_1x[n-1].$$

(a) Verify that S may be considered a cascade connection of two causal LTI systems S_1 and S_2 with the following input-output relationship:

$$S_1 : y_1[n] = b_0x_1[n] + b_1x_1[n-1],$$

$$S_2 : y_2[n] = -ay_2[n-1] + x_2[n].$$

(b) Draw a block diagram representation of S_1.

(c) Draw a block diagram representation of S_2.

(d) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_1 followed by the block diagram representation of S_2.

(e) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_2 followed by the block diagram representation of S_1.

(f) Show that the two unit-delay elements in the block diagram representation of S obtained in part (e) may be collapsed into one unit-delay element. The resulting block diagram is referred to as a *Direct Form II* realization of S, while the block diagrams obtained in parts (d) and (e) are referred to as *Direct Form I* realizations of S.

2.58. Consider a causal LTI system S whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$2y[n] - y[n-1] + y[n-3] = x[n] - 5x[n-4].$$

(a) Verify that S may be considered a cascade connection of two causal LTI systems S_1 and S_2 with the following input-output relationship:

$$S_1 : 2y_1[n] = x_1[n] - 5x_1[n-4],$$

$$S_2 : y_2[n] = \frac{1}{2}y_2[n-1] - \frac{1}{2}y_2[n-3] + x_2[n].$$

(b) Draw a block diagram representation of S_1.

(c) Draw a block diagram representation of S_2.

(d) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_1 followed by the block diagram representation of S_2.

(e) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_2 followed by the block diagram representation of S_1.

(f) Show that the four delay elements in the block diagram representation of S obtained in part (e) may be collapsed to three. The resulting block diagram is referred to as a *Direct Form II* realization of S, while the block diagrams obtained in parts (d) and (e) are referred to as *Direct Form I* realizations of S.

2.59. Consider a causal LTI system S whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) + b_1 \frac{dx(t)}{dt}.$$

(a) Show that

$$y(t) = A\int_{-\infty}^{t} y(\tau)\,d\tau + Bx(t) + C\int_{-\infty}^{t} x(\tau)\,d\tau,$$

and express the constants A, B, and C in terms of the constants a_0, a_1, b_0, and b_1.

(b) Show that S may be considered a cascade connection of the following two causal LTI systems:

$$S_1 : y_1(t) = Bx_1(t) + C\int_{-\infty}^{t} x(\tau)\,d\tau,$$

$$S_2 : y_2(t) = A\int_{-\infty}^{t} y_2(\tau)\,d\tau + x_2(t).$$

(c) Draw a block diagram representation of S_1.

(d) Draw a block diagram representation of S_2.

(e) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_1 followed by the block diagram representation of S_2.

(f) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_2 followed by the block diagram of representation S_1.

(g) Show that the two integrators in your answer to part (f) may be collapsed into one. The resulting block diagram is referred to as a *Direct Form II* realization of S, while the block diagrams obtained in parts (e) and (f) are referred to as *Direct Form I* realizations of S.

2.60. Consider a causal LTI system S whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$a_2 \frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) + b_1 \frac{dx(t)}{dt} + b_2 \frac{d^2x(t)}{dt^2}.$$

(a) Show that

$$y(t) = A\int_{-\infty}^{t} y(\tau)\,d\tau + B\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} y(\sigma)\,d\sigma\right)d\tau$$

$$+ Cx(t) + D\int_{-\infty}^{t} x(\tau)\,d\tau + E\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} x(\sigma)\,d\sigma\right)d\tau,$$

and express the constants A, B, C, D, and E in terms of the constants a_0, a_1, a_2, b_0, b_1, and b_2.

(b) Show that S may be considered a cascade connection of the following two causal LTI systems:

$$S_1 : y_1(t) = Cx_1(t) + D\int_{-\infty}^{t} x_1(\tau)\,d\tau + E\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} x_1(\sigma)\,d\sigma\right)d\tau,$$

$$S_2 : y_2(t) = A\int_{-\infty}^{t} y_2(\tau)\,d\tau + B\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} y_2(\sigma)\,d\sigma\right)d\tau + x_2(t).$$

(c) Draw a block diagram representation of S_1.

(d) Draw a block diagram representation of S_2.

(e) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_1 followed by the block diagram representation of S_2.

(f) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_2 followed by the block diagram representation of S_1.

(g) Show that the four integrators in your answer to part (f) may be collapsed into two. The resulting block diagram is referred to as a *Direct Form II* realization of S, while the block diagrams obtained in parts (e) and (f) are referred to as *Direct Form I* realizations of S.

EXTENSION PROBLEMS

2.61. **(a)** In the circuit shown in Figure P2.61(a), $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered to be the system output.

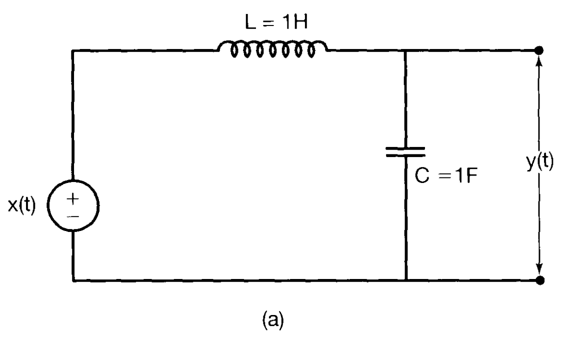


Figure P2.61a

(i) Determine the differential equation relating $x(t)$ and $y(t)$.

(ii) Show that the homogeneous solution of the differential equation from part (i) has the form $K_1 e^{j\omega_1 t} + K_2 e^{j\omega_2 t}$. Specify the values of ω_1 and ω_2.

(iii) Show that, since the voltage and current are restricted to be real, the natural response of the system is sinusoidal.

(b) In the circuit shown in Figure P2.61(b), $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered to be the system output.

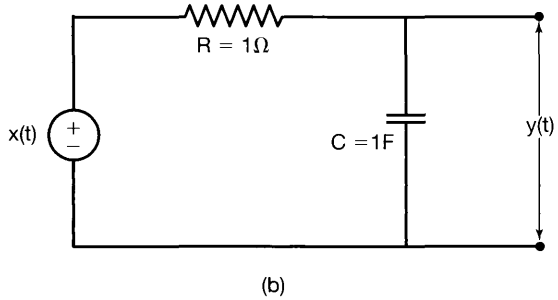


Figure P2.61b

(i) Determine the differential equation relating $x(t)$ and $y(t)$.
(ii) Show that the natural response of this system has the form Ke^{-at}, and specify the value of a.

(c) In the circuit shown in Figure P2.61(c), $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered to be the system output.

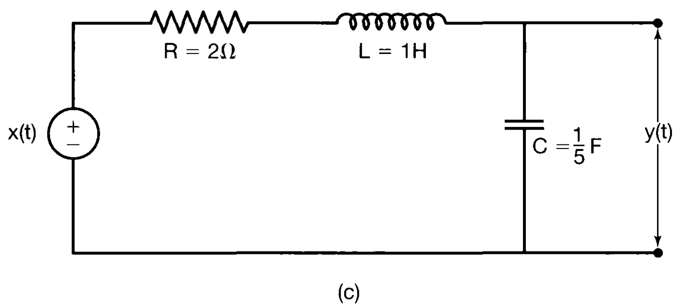


Figure P2.61c

(i) Determine the differential equation relating $x(t)$ and $y(t)$.
(ii) Show that the homogeneous solution of the differential equation from part (i) has the form $e^{-at}\{K_1e^{j2t} + K_2e^{-j2t}\}$, and specify the value of a.
(iii) Show that, since the voltage and current are restricted to be real, the natural response of the system is a decaying sinusoid.

2.62. (a) In the mechanical system shown in Figure P2.62(a), the force $x(t)$ applied to the mass represents the input, while the displacement $y(t)$ of the mass represents the output. Determine the differential equation relating $x(t)$ and $y(t)$. Show that the natural response of this system is periodic.

(b) Consider Figure P2.62(b), in which the force $x(t)$ is the input and the velocity $y(t)$ is the output. The mass of the car is m, while the coefficient of kinetic friction is ρ. Show that the natural response of this system decays with increasing time.

(c) In the mechanical system shown in Figure P2.62(c), the force $x(t)$ applied to the mass represents the input, while the displacement $y(t)$ of the mass represents the output.

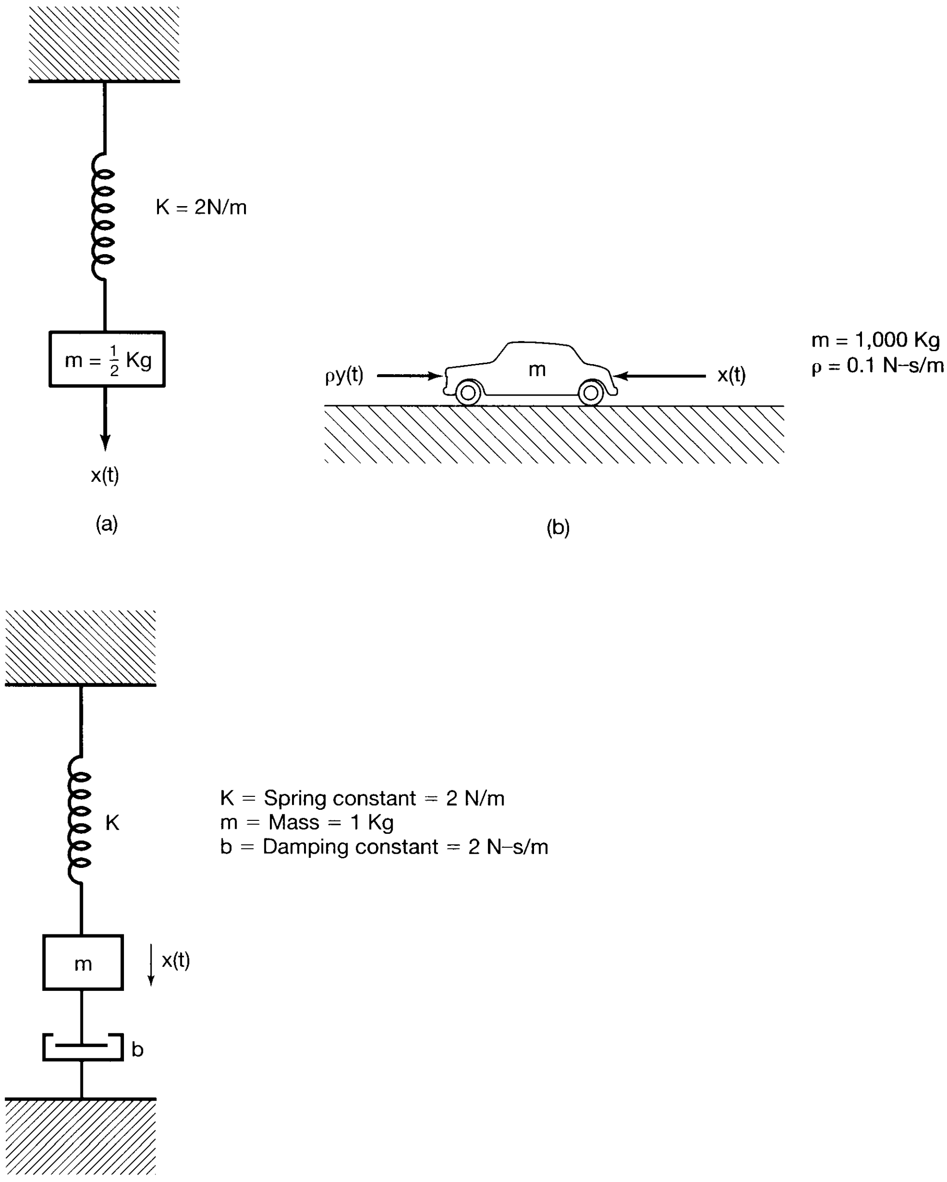


Figure P2.62

(i) Determine the differential equation relating $x(t)$ and $y(t)$.

(ii) Show that the homogeneous solution of the differential equation from part (i) has the form $e^{-at}\{K_1 e^{jt} + K_2 e^{-jt}\}$, and specify the value of a.

(iii) Show that, since the force and displacement are restricted to be real, the natural response of the system is a decaying sinusoid.

2.63. A \$100,000 mortgage is to be retired by *equal* monthly payments of D dollars. Interest, compounded monthly, is charged at the rate of 12% per annum on the unpaid balance; for example, after the first month, the total debt equals

$$\$100{,}000 + \left(\frac{0.12}{12}\right)\$100{,}000 = \$101{,}000.$$

The problem is to determine D such that after a specified time the mortgage is paid in full, leaving a net balance of zero.

(a) To set up the problem, let $y[n]$ denote the unpaid balance after the nth monthly payment. Assume that the principal is borrowed in month 0 and monthly payments begin in month 1. Show that $y[n]$ satisfies the difference equation

$$y[n] - \gamma y[n-1] = -D \quad n \geq 1 \qquad \text{(P2.63–1)}$$

with initial condition

$$y[0] = \$100{,}000,$$

where γ is a constant. Determine γ.

(b) Solve the difference equation of part (a) to determine

$$y[n] \quad \text{for } n \geq 0.$$

(*Hint:* The particular solution of eq. (P2.63–1) is a constant Y. Find the value of Y, and express $y[n]$ for $n \geq 1$ as the sum of particular and homogeneous solutions. Determine the unknown constant in the homogeneous solution by directly calculating $y[1]$ from eq. (P2.63–1) and comparing it to your solution.)

(c) If the mortgage is to be retired in 30 years after 360 monthly payments of D dollars, determine the appropriate value of D.

(d) What is the total payment to the bank over the 30-year period?

(e) Why do banks make loans?

2.64. One important use of inverse systems is in situations in which one wishes to remove distortions of some type. A good example of this is the problem of removing echoes from acoustic signals. For example, if an auditorium has a perceptible echo, then an initial acoustic impulse will be followed by attenuated versions of the sound at regularly spaced intervals. Consequently, an often-used model for this phenomenon is an LTI system with an impulse response consisting of a train of impulses, i.e.,

$$h(t) = \sum_{k=0}^{\infty} h_k \delta(t - kT). \qquad \text{(P2.64–1)}$$

Here the echoes occur T seconds apart, and h_k represents the gain factor on the kth echo resulting from an initial acoustic impulse.

(a) Suppose that $x(t)$ represents the original acoustic signal (the music produced by an orchestra, for example) and that $y(t) = x(t) * h(t)$ is the actual signal that is heard if no processing is done to remove the echoes. In order to remove the distortion introduced by the echoes, assume that a microphone is used to sense $y(t)$ and that the resulting signal is transduced into an electrical signal. We will also use $y(t)$ to denote this signal, as it represents the electrical equivalent of the acoustic signal, and we can go from one to the other via acoustic-electrical conversion systems.

The important point to note is that the system with impulse response given by eq. (P2.64–1) is invertible. Therefore, we can find an LTI system with impulse response $g(t)$ such that

$$y(t) * g(t) = x(t),$$

and thus, by processing the electrical signal $y(t)$ in this fashion and then converting back to an acoustic signal, we can remove the troublesome echoes.

The required impulse response $g(t)$ is also an impulse train:

$$g(t) = \sum_{k=0}^{\infty} g_k \delta(t - kT).$$

Determine the algebraic equations that the successive g_k must satisfy, and solve these equations for g_0, g_1, and g_2 in terms of h_k.

(b) Suppose that $h_0 = 1$, $h_1 = 1/2$, and $h_i = 0$ for all $i \geq 2$. What is $g(t)$ in this case?

(c) A good model for the generation of echoes is illustrated in Figure P2.64. Hence, each successive echo represents a fed-back version of $y(t)$, delayed by T seconds and scaled by α. Typically, $0 < \alpha < 1$, as successive echoes are attenuated.

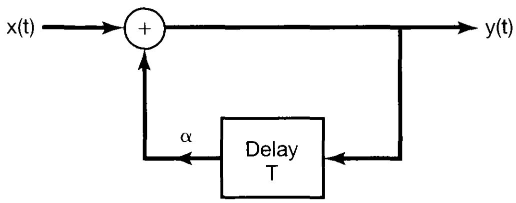


Figure P2.64

(i) What is the impulse response of this system? (Assume initial rest, i.e., $y(t) = 0$ for $t < 0$ if $x(t) = 0$ for $t < 0$.)

(ii) Show that the system is stable if $0 < \alpha < 1$ and unstable if $\alpha > 1$.

(iii) What is $g(t)$ in this case? Construct a realization of the inverse system using adders, coefficient multipliers, and T-second delay elements.

(d) Although we have phrased the preceding discussion in terms of continuous-time systems because of the application we have been considering, the same general ideas hold in discrete time. That is, the LTI system with impulse response

$$h[n] = \sum_{k=0}^{\infty} h_k \delta[n - kN]$$

is invertible and has as its inverse an LTI system with impulse response

$$g[n] = \sum_{k=0}^{\infty} g_k \delta[n - kN].$$

It is not difficult to check that the g_k satisfy the same algebraic equations as in part (a).

Consider now the discrete-time LTI system with impulse response

$$h[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN].$$

This system is *not* invertible. Find two inputs that produce the same output.

2.65. In Problem 1.45, we introduced and examined some of the basic properties of correlation functions for continuous-time signals. The discrete-time counterpart of the correlation function has essentially the same properties as those in continuous time, and both are extremely important in numerous applications (as is discussed in Problems 2.66 and 2.67). In this problem, we introduce the discrete-time correlation function and examine several more of its properties.

Let $x[n]$ and $y[n]$ be two real-valued discrete-time signals. The *autocorrelation functions* $\phi_{xx}[n]$ and $\phi_{yy}[n]$ of $x[n]$ and $y[n]$, respectively, are defined by the expressions

$$\phi_{xx}[n] = \sum_{m=-\infty}^{+\infty} x[m + n]x[m]$$

and

$$\phi_{yy}[n] = \sum_{m=-\infty}^{+\infty} y[m + n]y[m],$$

and the *cross-correlation functions* are given by

$$\phi_{xy}[n] = \sum_{m=-\infty}^{+\infty} x[m + n]y[m]$$

and

$$\phi_{yx}[n] = \sum_{m=-\infty}^{+\infty} y[m + n]x[m].$$

As in continuous time, these functions possess certain symmetry properties. Specifically, $\phi_{xx}[n]$ and $\phi_{yy}[n]$ are even functions, while $\phi_{xy}[n] = \phi_{yx}[-n]$.

(a) Compute the autocorrelation sequences for the signals $x_1[n]$, $x_2[n]$, $x_3[n]$, and $x_4[n]$ depicted in Figure P2.65.

(b) Compute the cross-correlation sequences

$$\phi_{x_i x_j}[n],\ i \neq j,\ \ i, j = 1, 2, 3, 4,$$

for $x_i[n]$, $i = 1, 2, 3, 4$, as shown in Figure P2.65.

(c) Let $x[n]$ be the input to an LTI system with unit sample response $h[n]$, and let the corresponding output be $y[n]$. Find expressions for $\phi_{xy}[n]$ and $\phi_{yy}[n]$ in terms of $\phi_{xx}[n]$ and $h[n]$. Show how $\phi_{xy}[n]$ and $\phi_{yy}[n]$ can be viewed as the output

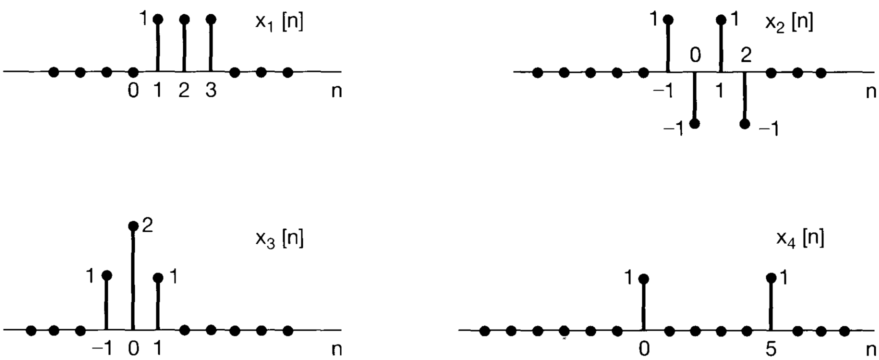


Figure P2.65

of LTI systems with $\phi_{xx}[n]$ as the input. (Do this by explicitly specifying the impulse response of each of the two systems.)

(d) Let $h[n] = x_1[n]$ in Figure P2.65, and let $y[n]$ be the output of the LTI system with impulse response $h[n]$ when the input $x[n]$ also equals $x_1[n]$. Calculate $\phi_{xy}[n]$ and $\phi_{yy}[n]$ using the results of part (c).

2.66. Let $h_1(t)$, $h_2(t)$, and $h_3(t)$, as sketched in Figure P2.66, be the impulse responses of three LTI systems. These three signals are known as *Walsh functions* and are of considerable practical importance because they can be easily generated by digital logic circuitry and because multiplication by each of them can be implemented in a simple fashion by a polarity-reversing switch.

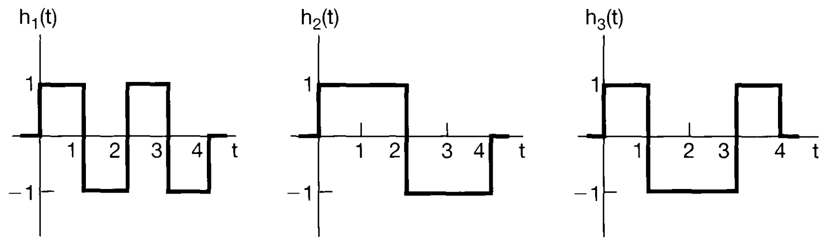


Figure P2.66

(a) Determine and sketch a choice for $x_1(t)$, a continuous-time signal with the following properties:

(i) $x_1(t)$ is real.

(ii) $x_1(t) = 0$ for $t < 0$.

(iii) $|x_1(t)| \leq 1$ for $t \geq 0$.

(iv) $y_1(t) = x_1(t) * h(t)$ is as large as possible at $t = 4$.

(b) Repeat part (a) for $x_2(t)$ and $x_3(t)$ by making $y_2(t) = x_2(t) * h_2(t)$ and $y_3(t) = x_3(t) * h_3(t)$ each as large as possible at $t = 4$.

(c) What is the value of

$$y_{ij}(t) = x_i(t) * h_j(t), \ i \neq j$$

at time $t = 4$ for $i, j = 1, 2, 3$?

The system with impulse response $h_i(t)$ is known as the *matched filter* for the signal $x_i(t)$ because the impulse response is tuned to $x_i(t)$ in order to produce the maximum output signal. In the next problem, we relate the concept of a matched filter to that of the correlation function for continuous-time signals.

2.67. The *cross-correlation function* between two continuous-time real signals $x(t)$ and $y(t)$ is

$$\phi_{xy}(t) = \int_{-\infty}^{+\infty} x(t+\tau)y(\tau)\,d\tau. \tag{P2.67–1}$$

The *autocorrelation function* of a signal $x(t)$ is obtained by setting $y(t) = x(t)$ in eq. (P2.67–1):

$$\phi_{xx}(t) = \int_{-\infty}^{+\infty} x(t+\tau)x(\tau)\,d\tau.$$

(a) Compute the autocorrelation function for each of the two signals $x_1(t)$ and $x_2(t)$ depicted in Figure P2.67(a).

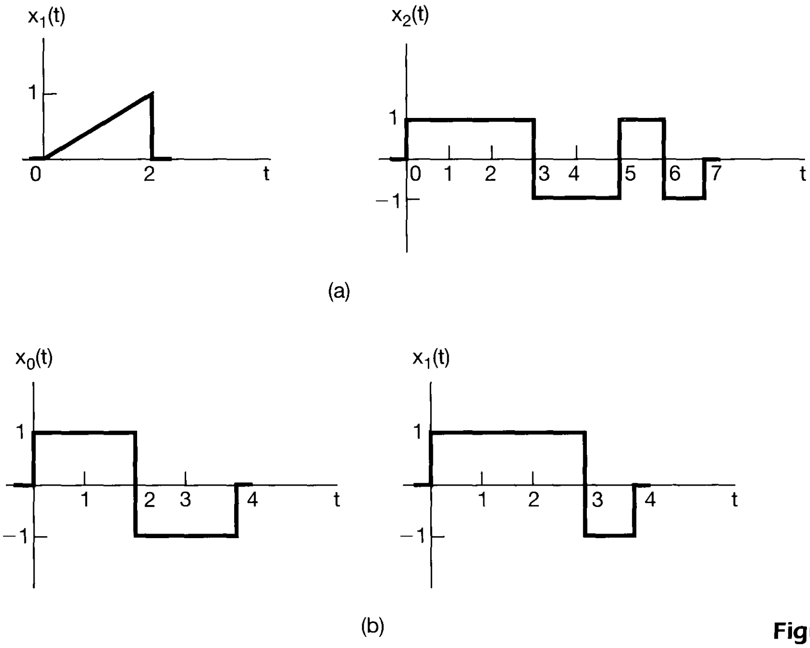


Figure P2.67

(b) Let $x(t)$ be a given signal, and assume that $x(t)$ is of finite duration—i.e., that $x(t) = 0$ for $t < 0$ and $t > T$. Find the impulse response of an LTI system so that $\phi_{xx}(t - T)$ is the output if $x(t)$ is the input.

(c) The system determined in part (b) is a *matched filter* for the signal $x(t)$. That this definition of a matched filter is identical to the one introduced in Problem 2.66 can be seen from the following:

Let $x(t)$ be as in part (b), and let $y(t)$ denote the response to $x(t)$ of an LTI system with real impulse response $h(t)$. Assume that $h(t) = 0$ for $t < 0$ and for $t > T$. Show that the choice for $h(t)$ that maximizes $y(T)$, subject to the constraint that

$$\int_0^T h^2(t)dt = M, \text{ a fixed positive number,} \qquad \text{(P2.67–2)}$$

is a scalar multiple of the impulse response determined in part (b). [*Hint:* Schwartz's inequality states that

$$\int_b^a u(t)v(t)dt \leq \left[\int_a^b u^2(t)dt\right]^{1/2} \left[\int_a^b v^2(t)dt\right]^{1/2}$$

for any two signals $u(t)$ and $v(t)$. Use this to obtain a bound on y(T).]

(d) The constraint given by eq. (P2.67–2) simply provides a scaling to the impulse response, as increasing M merely changes the scalar multiplier mentioned in part (c). Thus, we see that the particular choice for $h(t)$ in parts (b) and (c) is matched to the signal $x(t)$ to produce maximum output. This is an extremely important property in a number of applications, as we will now indicate.

In communication problems, one often wishes to transmit one of a small number of possible pieces of information. For example, if a complex message is encoded into a sequence of binary digits, we can imagine a system that transmits the information bit by bit. Each bit can then be transmitted by sending one signal, say, $x_0(t)$, if the bit is a 0, or a different signal $x_1(t)$ if a 1 is to be communicated. In this case, the receiving system for these signals must be capable of recognizing whether $x_0(t)$ or $x_1(t)$ has been received. Intuitively, what makes sense is to have two systems in the receiver, one tuned to $x_0(t)$ and one tuned to $x_1(t)$, where, by "tuned," we mean that the system gives a large output after the signal to which it is tuned is received. The property of producing a large output when a particular signal is received is exactly what the matched filter possesses.

In practice, there is always distortion and interference in the transmission and reception processes. Consequently, we want to maximize the difference between the response of a matched filter to the input to which it is matched and the response of the filter to one of the other signals that can be transmitted. To illustrate this point, consider the two signals $x_0(t)$ and $x_1(t)$ depicted in Figure P2.67(b). Let L_0 denote the matched filter for $x_0(t)$, and let L_1 denote the matched filter for $x_1(t)$.

(i) Sketch the responses of L_0 to $x_0(t)$ and $x_1(t)$. Do the same for L_1.

(ii) Compare the values of these responses at $t = 4$. How might you modify $x_0(t)$ so that the receiver would have an even easier job of distinguishing between $x_0(t)$ and $x_1(t)$ in that the response of L_0 to $x_1(t)$ and L_1 to $x_0(t)$ would both be zero at $t = 4$?

2.68. Another application in which matched filters and correlation functions play an important role is radar systems. The underlying principle of radar is that an electromagnetic pulse transmitted at a target will be reflected by the target and will subsequently return to the sender with a delay proportional to the distance to the target. Ideally, the received signal will simply be a shifted and possibly scaled version of the original transmitted signal.

Let $p(t)$ be the original pulse that is sent out. Show that

$$\phi_{pp}(0) = \max_t \phi_{pp}(t).$$

That is, $\phi_{pp}(0)$ is the largest value taken by $\phi_{pp}(t)$. Use this equation to deduce that, if the waveform that comes back to the sender is

$$x(t) = \alpha p(t - t_0),$$

where α is a positive constant, then

$$\phi_{xp}(t_0) = \max_t \phi_{xp}(t).$$

(*Hint:* Use Schwartz's inequality.)

Thus, the way in which simple radar ranging systems work is based on using a matched filter for the transmitted waveform $p(t)$ and noting the time at which the output of this system reaches its maximum value.

2.69. In Section 2.5, we characterized the unit doublet through the equation

$$x(t) * u_1(t) = \int_{-\infty}^{+\infty} x(t - \tau)u_1(\tau)\,d\tau = x'(t) \qquad \text{(P2.69–1)}$$

for any signal $x(t)$. From this equation, we derived the relationship

$$\int_{-\infty}^{+\infty} g(\tau)u_1(\tau)\,d\tau = -g'(0). \qquad \text{(P2.69–2)}$$

(a) Show that eq. (P2.69–2) is an equivalent characterization of $u_1(t)$ by showing that eq. (P2.69–2) implies eq. (P2.69–1). [*Hint:* Fix t, and define the signal $g(\tau) = x(t - \tau)$.]

Thus, we have seen that characterizing the unit impulse or unit doublet by how it behaves under convolution is equivalent to characterizing how it behaves under integration when multiplied by an arbitrary signal $g(t)$. In fact, as indicated in Section 2.5, the equivalence of these operational definitions holds for all signals and, in particular, for all singularity functions.

(b) Let $f(t)$ be a given signal. Show that

$$f(t)u_1(t) = f(0)u_1(t) - f'(0)\delta(t)$$

by showing that both functions have the same operational definitions.

(c) What is the value of

$$\int_{-\infty}^{\infty} x(\tau)u_2(\tau)\,d\tau?$$

Find an expression for $f(t)u_2(t)$ analogous to that in part (b) for $f(t)u_1(t)$.

2.70. In analogy with continuous-time singularity functions, we can define a set of discrete-time signals. Specifically, let

$$u_{-1}[n] = u[n],$$
$$u_0[n] = \delta[n],$$

and

$$u_1[n] = \delta[n] - \delta[n-1],$$

and define

$$u_k[n] = \underbrace{u_1[n] * u_1[n] * \cdots * u_1[n]}_{k \text{ times}}, \quad k > 0$$

and

$$u_k[n] = \underbrace{u_{-1}[n] * u_{-1}[n] * \cdots * u_{-1}[n]}_{|k| \text{ times}}, \quad k < 0.$$

Note that

$$x[n] * \delta[n] = x[n],$$
$$x[n] * u[n] = \sum_{m=-\infty}^{\infty} x[m],$$

and

$$x[n] * u_1[n] = x[n] - x[n-1],$$

(a) What is

$$\sum_{m=\infty}^{\infty} x[m]u_1[m]?$$

(b) Show that

$$x[n]u_1[n] = x[0]u_1[n] - [x[1] - x[0]]\delta[n-1]$$
$$= x[1]u_1[n] - [x[1] - x[0]]\delta[n].$$

(c) Sketch the signals $u_2[n]$ and $u_3[n]$.
(d) Sketch $u_{-2}[n]$ and $u_{-3}[n]$.
(e) Show that, in general, for $k > 0$,

$$u_k[n] = \frac{(-1)^n k!}{n!(k-n)!}[u[n] - u[n-k-1]]. \qquad \text{(P2.70–1)}$$

(*Hint:* Use induction. From part (c), it is evident that $u_k[n]$ satisfies eq. (P2.70–1) for $k = 2$ and 3. Then, assuming that eq. (P2.70–1) satisfies $u_k[n]$, write $u_{k+1}[n]$ in terms of $u_k[n]$, and show that the equation also satisfies $u_{k+1}[n]$.)

(f) Show that, in general, for $k > 0$,

$$u_{-k}[n] = \frac{(n+k-1)!}{n!(k-1)!}u[n]. \qquad \text{(P2.70–2)}$$

(*Hint:* Again, use induction. Note that

$$u_{-(k+1)}[n] - u_{-(k+1)}[n-1] = u_{-k}[n]. \qquad \text{(P2.70–3)}$$

Then, assuming that eq. (P2.70–2) is valid for $u_{-k}[n]$, use eq. (P2.70–3) to show that eq. (P2.70–2) is valid for $u_{-(k+1)}[n]$ as well.)

2.71. In this chapter, we have used several properties and ideas that greatly facilitate the analysis of LTI systems. Among these are two that we wish to examine a bit more closely. As we will see, in certain very special cases one must be careful in using these properties, which otherwise hold without qualification.

(a) One of the basic and most important properties of convolution (in both continuous and discrete time) is associativity. That is, if $x(t)$, $h(t)$, and $g(t)$ are three signals, then

$$x(t) * [g(t) * h(t)] = [x(t) * g(t)] * h(t) = [x(t) * h(t)] * g(t). \qquad \text{(P2.71–1)}$$

This relationship holds as long as all three expressions are well defined and finite. As that is usually the case in practice, we will in general use the associativity property without comments or assumptions. However, there are some cases in which it does *not* hold. For example, consider the system depicted in Figure P2.71, with $h(t) = u_1(t)$ and $g(t) = u(t)$. Compute the response of this system to the input

$$x(t) = 1 \text{ for all } t.$$

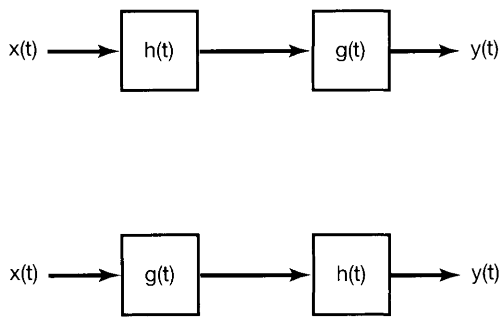


Figure P2.71

Do this in the three different ways suggested by eq. (P2.71–1) and by the figure:

(i) By first convolving the two impulse responses and then convolving the result with $x(t)$.

(ii) By first convolving $x(t)$ with $u_1(t)$ and then convolving the result with $u(t)$.

(iii) By first convolving $x(t)$ with $u(t)$ and then convolving the result with $u_1(t)$.

(b) Repeat part (a) for

$$x(t) = e^{-t}$$

and

$$h(t) = e^{-t}u(t),$$
$$g(t) = u_1(t) + \delta(t).$$

(c) Do the same for

$$x[n] = \left(\frac{1}{2}\right)^n,$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n],$$

$$g[n] = \delta[n] - \frac{1}{2}\delta[n-1].$$

Thus, in general, the associativity property of convolution holds if and only if the three expressions in eq. (P2.71–1) make sense (i.e., if and only if their interpretations in terms of LTI systems are meaningful). For example, in part (a) differentiating a constant and then integrating makes sense, but the process of integrating the constant from $t = -\infty$ and *then* differentiating does not, and it is only in such cases that associativity breaks down.

Closely related to the foregoing discussion is an issue involving inverse systems. Consider the LTI system with impulse response $h(t) = u(t)$. As we saw in part (a), there are inputs—specifically, $x(t) =$ nonzero constant—for which the output of this system is infinite, and thus, it is meaningless to consider the question of inverting such outputs to recover the input. However, if we limit ourselves to inputs that do yield finite outputs, that is, inputs which satisfy

$$\left| \int_{-\infty}^{t} x(\tau)\, d\tau \right| < \infty, \qquad \text{(P2.71–2)}$$

then the system *is* invertible, and the LTI system with impulse response $u_1(t)$ is its inverse.

(d) Show that the LTI system with impulse response $u_1(t)$ is *not* invertible. (*Hint:* Find two different inputs that both yield zero output for all time.) However, show that the system is invertible if we limit ourselves to inputs that satisfy eq. (P2.71–2). [*Hint:* In Problem 1.44, we showed that an LTI system is invertible if no input other than $x(t) = 0$ yields an output that is zero for all time; are there two inputs $x(t)$ that satisfy eq. (P2.71–2) and that yield identically zero responses when convolved with $u_1(t)$?]

What we have illustrated in this problem is the following:

(1) If $x(t)$, $h(t)$, and $g(t)$ are three signals, and if $x(t) * g(t)$, $x(t) * h(t)$, and $h(t) * g(t)$ are *all* well defined and finite, then the associativity property, eq. (P2.71–1), holds.

(2) Let $h(t)$ be the impulse response of an LTI system, and suppose that the impulse response $g(t)$ of a second system has the property

$$h(t) * g(t) = \delta(t). \qquad \text{(P2.71–3)}$$

Then, from (1), *for* all inputs $x(t)$ for which $x(t) * h(t)$ and $x(t) * g(t)$ are both well defined and finite, the two cascades of systems depicted in Figure P2.71 act as the identity system, and thus, the two LTI systems can be regarded as inverses of one another. For example, if $h(t) = u(t)$ and $g(t) = u_1(t)$, then, as long as we restrict ourselves to inputs satisfying eq. (P2.71–2), we can regard these two systems as inverses.

Therefore, we see that the associativity property of eq. (P2.71–1) and the definition of LTI inverses as given in eq. (P2.71–3) are valid, as long as all convolutions that are involved are finite. As this is certainly the case in any realistic problem, we will in general use these properties without comment or qualification. Note that, although we have phrased most of our discussion in terms of continuous-time signals and systems, the same points can also be made in discrete time [as should be evident from part (c)].

2.72. Let $\delta_\Delta(t)$ denote the rectangular pulse of height $\frac{1}{\Delta}$ for $0 < t \leq \Delta$. Verify that

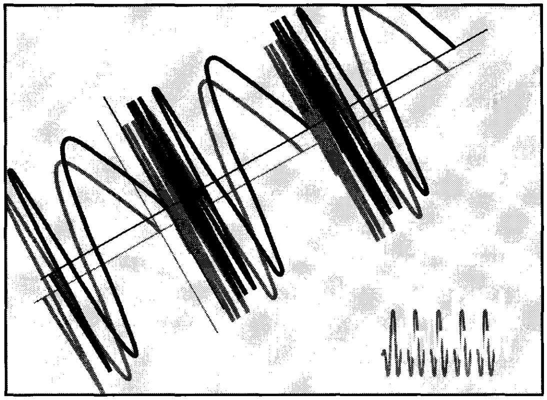
$$\frac{d}{dt}\delta_\Delta(t) = \frac{1}{\Delta}[\delta(t) - \delta(t - \Delta)].$$

2.73. Show by induction that

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!}u(t) \text{ for } k = 1, 2, 3 \ldots$$

3

FOURIER SERIES REPRESENTATION OF PERIODIC SIGNALS



3.0 INTRODUCTION

The representation and analysis of LTI systems through the convolution sum as developed in Chapter 2 is based on representing signals as linear combinations of shifted impulses. In this and the following two chapters, we explore an alternative representation for signals and LTI systems. As in Chapter 2, the starting point for our discussion is the development of a representation of signals as linear combinations of a set of basic signals. For this alternative representation we use complex exponentials. The resulting representations are known as the continuous-time and discrete-time Fourier series and transform. As we will see, these can be used to construct broad and useful classes of signals.

We then proceed as we did in Chapter 2. That is, because of the superposition property, the response of an LTI system to any input consisting of a linear combination of basic signals is the same linear combination of the individual responses to each of the basic signals. In Chapter 2, these responses were all shifted versions of the unit impulse response, leading to the convolution sum or integral. As we will find in the current chapter, the response of an LTI system to a complex exponential also has a particularly simple form, which then provides us with another convenient representation for LTI systems and with another way in which to analyze these systems and gain insight into their properties.

In this chapter, we focus on the representation of continuous-time and discrete-time periodic signals referred to as the Fourier series. In Chapters 4 and 5, we extend the analysis to the Fourier transform representation of broad classes of aperiodic, finite energy signals. Together, these representations provide one of the most powerful and important sets of tools and insights for analyzing, designing, and understanding signals and LTI systems, and we devote considerable attention in this and subsequent chapters to exploring the uses of Fourier methods.

We begin in the next section with a brief historical perspective in order to provide some insight into the concepts and issues that we develop in more detail in the sections and chapters that follow.

3.1 A HISTORICAL PERSPECTIVE

The development of Fourier analysis has a long history involving a great many individuals and the investigation of many different physical phenomena.[1] The concept of using "trigonometric sums"—that is, sums of harmonically related sines and cosines or periodic complex exponentials—to describe periodic phenomena goes back at least as far as the Babylonians, who used ideas of this type in order to predict astronomical events.[2] The modern history of the subject begins in 1748 with L. Euler, who examined the motion of a vibrating string. In Figure 3.1, we have indicated the first few of what are known as the "normal modes" of such a string. If we consider the vertical deflection $f(t, x)$ of the string at time t and at a distance x along the string, then for any fixed instant of time, the normal modes are harmonically related sinusoidal functions of x. What Euler noted was that if the configuration of a vibrating string at some point in time is a linear combination of these normal modes, so is the configuration at any subsequent time. Furthermore, Euler showed that one could calculate the coefficients for the linear combination at the later time in a very straightforward manner from the coefficients at the earlier time. In doing this, Euler performed the same type of calculation as we will in the next section in deriving one of the properties of trigonometric sums that make them so useful for the analysis of LTI systems. Specifically, we will see that if the input to an LTI system is expressed as a linear combination of periodic complex exponentials or sinusoids, the output can also be expressed in this form, with coefficients that are related in a straightforward way to those of the input.

The property described in the preceding paragraph would not be particularly useful, unless it were true that a large class of interesting functions could be represented by linear combinations of complex exponentials. In the middle of the 18th century, this point was the subject of heated debate. In 1753, D. Bernoulli argued on physical grounds that all physical motions of a string could be represented by linear combinations of normal modes, but he did not pursue this mathematically, and his ideas were not widely accepted. In fact, Euler himself discarded trigonometric series, and in 1759 J. L. Lagrange strongly criticized the use of trigonometric series in the examination of vibrating strings. His criticism was based on his own belief that it was impossible to represent signals with corners (i.e., with discontinuous slopes) using trigonometric series. Since such a configuration arises from

[1] The historical material in this chapter was taken from the following references: I. Grattan-Guiness, *Joseph Fourier, 1768–1830* (Cambridge, MA: The MIT Press, 1972); G. F. Simmons, *Differential Equations: With Applications and Historical Notes* (New York: McGraw-Hill Book Company, 1972); C. Lanczos, *Discourse on Fourier Series* (London: Oliver and Boyd, 1966); R. E. Edwards, *Fourier Series: A Modern Introduction* (New York: Springer-Verlag, 2nd ed., 1970); and A. D. Aleksandrov, A. N. Kolmogorov, and M. A. Lavrent'ev, *Mathematics: Its Content, Methods, and Meaning*, trans. S. H. Gould, Vol. II; trans. K. Hirsch, Vol. III (Cambridge, MA: The MIT Press, 1969). Of these, Grattan-Guiness' work offers the most complete account of Fourier's life and contributions. Other references are cited in several places in the chapter.

[2] H. Dym and H. P. McKean, *Fourier Series and Integrals* (New York: Academic Press, 1972). This text and the book of Simmons cited in footnote 1 also contain discussions of the vibrating-string problem and its role in the development of Fourier analysis.

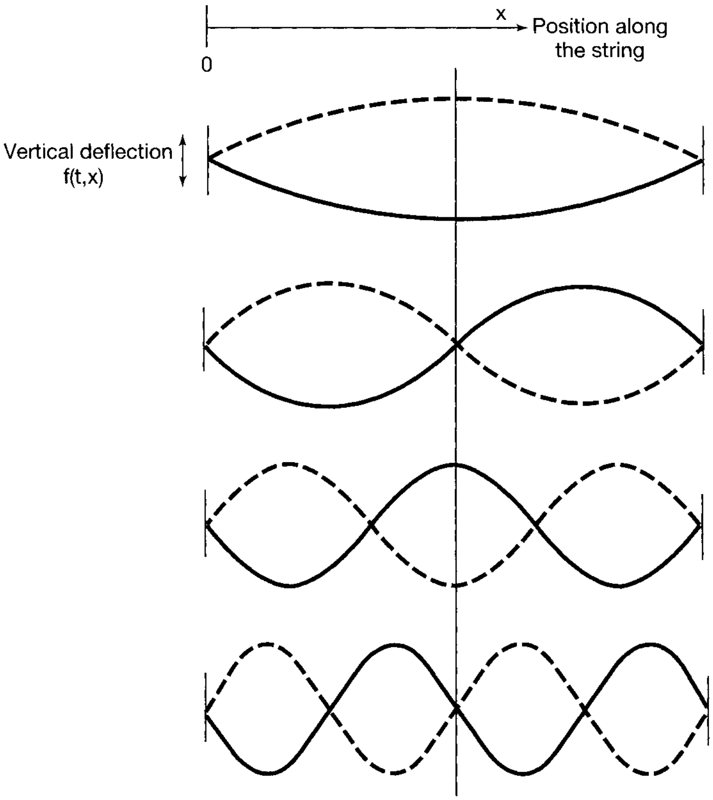


Figure 3.1 Normal modes of a vibrating string. (Solid lines indicate the configuration of each of these modes at some fixed instant of time, *t*.)

the plucking of a string (i.e., pulling it taut and then releasing it), Lagrange argued that trigonometric series were of very limited use.

It was in this somewhat hostile and skeptical environment that Jean Baptiste Joseph Fourier (Figure 3.2) presented his ideas half a century later. Fourier was born on March

Figure 3.2 Jean Baptiste Joseph Fourier [picture from J. B. J. Fourier, *Oeuvres de Fourier,* Vol. II (Paris: Gauthier-Villars et Fils, 1980)].

21, 1768, in Auxerre, France, and by the time of his entrance into the controversy concerning trigonometric series, he had already had a lifetime of experiences. His many contributions—in particular, those concerned with the series and transform that carry his name—are made even more impressive by the circumstances under which he worked. His revolutionary discoveries, although not completely appreciated during his own lifetime, have had a major impact on the development of mathematics and have been and still are of great importance in an extremely wide range of scientific and engineering disciplines.

In addition to his studies in mathematics, Fourier led an active political life. In fact, during the years that followed the French Revolution, his activities almost led to his downfall, as he narrowly avoided the guillotine on two separate occasions. Subsequently, Fourier became an associate of Napoleon Bonaparte, accompanied him on his expeditions to Egypt (during which time Fourier collected the information he would use later as the basis for his treatises on Egyptology), and in 1802 was appointed by Bonaparte to the position of prefect of a region of France centered in Grenoble. It was there, while serving as prefect, that Fourier developed his ideas on trigonometric series.

The physical motivation for Fourier's work was the phenomenon of heat propagation and diffusion. This in itself was a significant step in that most previous research in mathematical physics had dealt with rational and celestial mechanics. By 1807, Fourier had completed a work, Fourier had found series of harmonically related sinusoids to be useful in representing the temperature distribution through a body. In addition, he claimed that "any" periodic signal could be represented by such a series. While his treatment of this topic was significant, many of the basic ideas behind it had been discovered by others. Also, Fourier's mathematical arguments were still imprecise, and it remained for P. L. Dirichlet in 1829 to provide precise conditions under which a periodic signal could be represented by a Fourier series.[3] Thus, Fourier did not actually contribute to the mathematical theory of Fourier series. However, he did have the clear insight to see the potential for this series representation, and it was to a great extent his work and his claims that spurred much of the subsequent work on Fourier series. In addition, Fourier took this type of representation one very large step farther than any of his predecessors: He obtained a representation for *aperiodic* signals—not as weighted *sums* of harmonically related sinusoids—but as weighted *integrals* of sinusoids that are *not* all harmonically related. It is this extension from Fourier series to the Fourier integral or transform that is the focus of Chapters 4 and 5. Like the Fourier series, the Fourier transform remains one of the most powerful tools for the analysis of LTI systems.

Four distinguished mathematicians and scientists were appointed to examine the 1807 paper of Fourier. Three of the four—S. F. Lacroix, G. Monge, and P. S. de Laplace—were in favor of publication of the paper, but the fourth, J. L. Lagrange, remained adamant in rejecting trigonometric series, as he had done 50 years earlier. Because of Lagrange's vehement objections, Fourier's paper never appeared. After several other attempts to have his work accepted and published by the Institut de France, Fourier undertook the writing of another version of his work, which appeared as the text *Théorie analytique de la chaleur*.[4]

[3]Both S. D. Poisson and A. L. Cauchy had obtained results about the convergence of Fourier series before 1829, but Dirichlet's work represented such a significant extension of their results that he is usually credited with being the first to consider Fourier series convergence in a rigorous fashion.

[4]See J. B. J. Fourier, *The Analytical Theory of Heat*, trans. A. Freeman (New York: Dover, 1955).

This book was published in 1822, 15 years after Fourier had first presented his results to the Institut.

Toward the end of his life Fourier received some of the recognition he deserved, but the most significant tribute to him has been the enormous impact of his work on so many disciplines within the fields of mathematics, science, and engineering. The theory of integration, point-set topology, and eigenfunction expansions are just a few examples of topics in mathematics that have their roots in the analysis of Fourier series and integrals.[5] Furthermore, in addition to the original studies of vibration and heat diffusion, there are numerous other problems in science and engineering in which sinusoidal signals, and therefore Fourier series and transforms, play an important role. For example, sinusoidal signals arise naturally in describing the motion of the planets and the periodic behavior of the earth's climate. Alternating-current sources generate sinusoidal voltages and currents, and, as we will see, the tools of Fourier analysis enable us to analyze the response of an LTI system, such as a circuit, to such sinusoidal inputs. Also, as illustrated in Figure 3.3, waves in the ocean consist of the linear combination of sinusoidal waves with different spatial periods or *wavelengths*. Signals transmitted by radio and television stations are sinusoidal in nature as well, and as a quick perusal of any text on Fourier analysis will show, the range of applications in which sinusoidal signals arise and in which the tools of Fourier analysis are useful extends far beyond these few examples.

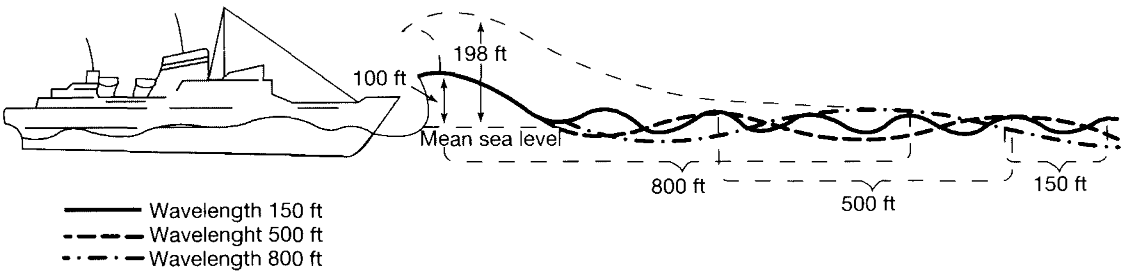


Figure 3.3 Ship encountering the superposition of three wave trains, each with a different spatial period. When these waves reinforce one another, a very large wave can result. In more severe seas, a giant wave indicated by the dotted line could result. Whether such a reinforcement occurs at any location depends upon the relative phases of the components that are superposed. [Adapted from an illustration by P. Mion in "Nightmare Waves Are All Too Real to Deepwater Sailors," by P. Britton, *Smithsonian* 8 (February 1978), pp. 64–65].

While many of the applications in the preceding paragraph, as well as the original work of Fourier and his contemporaries on problems of mathematical physics, focus on phenomena in continuous time, the tools of Fourier analysis for discrete-time signals and systems have their own distinct historical roots and equally rich set of applications. In particular, discrete-time concepts and methods are fundamental to the discipline of numerical analysis. Formulas for the processing of discrete sets of data points to produce numerical approximations for interpolation, integration, and differentiation were being investigated as early as the time of Newton in the 1600s. In addition, the problem of predicting the motion of a heavenly body, given a sequence of observations of the body, spurred the

[5]For more on the impact of Fourier's work on mathematics, see W. A. Coppel, "J. B. Fourier—on the occasion of His Two Hundredth Birthday," *American Mathematical Monthly*, 76 (1969), 468–83.

investigation of harmonic time series in the 18th and 19th centuries by eminent scientists and mathematicians, including Gauss, and thus provided a second setting in which much of the initial work was done on discrete-time signals and systems.

In the mid-1960s an algorithm, now known as the fast Fourier transform, or FFT, was introduced. This algorithm, which was independently discovered by Cooley and Tukey in 1965, also has a considerable history and can, in fact, be found in Gauss' notebooks.[6] What made its modern discovery so important was the fact that the FFT proved to be perfectly suited for efficient digital implementation, and it reduced the time required to compute transforms by orders of magnitude. With this tool, many interesting but previously impractical ideas utilizing the discrete-time Fourier series and transform suddenly became practical, and the development of discrete-time signal and system analysis techniques moved forward at an accelerated pace.

What has emerged out of this long history is a powerful and cohesive framework for the analysis of continuous-time and discrete-time signals and systems and an extraordinarily broad array of existing and potential applications. In this and the following chapters, we will develop the basic tools of that framework and examine some of its important implications.

3.2 THE RESPONSE OF LTI SYSTEMS TO COMPLEX EXPONENTIALS

As we indicated in Section 3.0, it is advantageous in the study of LTI systems to represent signals as linear combinations of basic signals that possess the following two properties:

1. The set of basic signals can be used to construct a broad and useful class of signals.
2. The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals.

Much of the importance of Fourier analysis results from the fact that both of these properties are provided by the set of complex exponential signals in continuous and discrete time—i.e., signals of the form e^{st} in continuous time and z^n in discrete time, where s and z are complex numbers. In subsequent sections of this and the following two chapters, we will examine the first property in some detail. In this section, we focus on the second property and, in this way, provide motivation for the use of Fourier series and transforms in the analysis of LTI systems.

The importance of complex exponentials in the study of LTI systems stems from the fact that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude; that is,

$$\text{continuous time: } e^{st} \longrightarrow H(s)e^{st}, \tag{3.1}$$

$$\text{discrete time: } z^n \longrightarrow H(z)z^n, \tag{3.2}$$

where the complex amplitude factor $H(s)$ or $H(z)$ will in general be a function of the complex variable s or z. A signal for which the system output is a (possibly complex)

[6] M. T. Heideman, D. H. Johnson, and C. S. Burrus, "Gauss and the History of the Fast Fourier Transform," *The IEEE ASSP Magazine 1* (1984), pp. 14–21.

constant times the input is referred to as an *eigenfunction* of the system, and the amplitude factor is referred to as the system's *eigenvalue*.

To show that complex exponentials are indeed eigenfunctions of LTI systems, let us consider a continuous-time LTI system with impulse response $h(t)$. For an input $x(t)$, we can determine the output through the use of the convolution integral, so that with $x(t) = e^{st}$

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}\,d\tau. \end{aligned} \tag{3.3}$$

Expressing $e^{s(t-\tau)}$ as $e^{st}e^{-s\tau}$, and noting that e^{st} can be moved outside the integral, we see that eq. (3.3) becomes

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau. \tag{3.4}$$

Assuming that the integral on the right-hand side of eq. (3.4) converges, the response to e^{st} is of the form

$$y(t) = H(s)e^{st}, \tag{3.5}$$

where $H(s)$ is a complex constant whose value depends on s and which is related to the system impulse response by

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau. \tag{3.6}$$

Hence, we have shown that complex exponentials are eigenfunctions of LTI systems. The constant $H(s)$ for a specific value of s is then the eigenvalue associated with the eigenfunction e^{st}.

In an exactly parallel manner, we can show that complex exponential sequences are eigenfunctions of discrete-time LTI systems. That is, suppose that an LTI system with impulse response $h[n]$ has as its input the sequence

$$x[n] = z^n, \tag{3.7}$$

where z is a complex number. Then the output of the system can be determined from the convolution sum as

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}. \end{aligned} \tag{3.8}$$

From this expression, we see that if the input $x[n]$ is the complex exponential given by eq. (3.7), then, assuming that the summation on the right-hand side of eq. (3.8) converges, the output is the same complex exponential multiplied by a constant that depends on the value of z. That is,

$$y[n] = H(z)z^n, \tag{3.9}$$

where

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}. \tag{3.10}$$

Consequently, as in the continuous-time case, complex exponentials are eigenfunctions of discrete-time LTI systems. The constant $H(z)$ for a specified value of z is the eigenvalue associated with the eigenfunction z^n.

For the analysis of LTI systems, the usefulness of decomposing more general signals in terms of eigenfunctions can be seen from an example. Let $x(t)$ correspond to a linear combination of three complex exponentials; that is,

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}. \tag{3.11}$$

From the eigenfunction property, the response to each separately is

$$a_1 e^{s_1 t} \longrightarrow a_1 H(s_1) e^{s_1 t},$$
$$a_2 e^{s_2 t} \longrightarrow a_2 H(s_2) e^{s_2 t},$$
$$a_3 e^{s_3 t} \longrightarrow a_3 H(s_3) e^{s_3 t},$$

and from the superposition property the response to the sum is the sum of the responses, so that

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}. \tag{3.12}$$

More generally, in continuous time, eq. (3.5), together with the superposition property, implies that the representation of signals as a linear combination of complex exponentials leads to a convenient expression for the response of an LTI system. Specifically, if the input to a continuous-time LTI system is represented as a linear combination of complex exponentials, that is, if

$$x(t) = \sum_k a_k e^{s_k t}, \tag{3.13}$$

then the output will be

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}. \tag{3.14}$$

In an exactly analogous manner, if the input to a discrete-time LTI system is represented as a linear combination of complex exponentials, that is, if

$$x[n] = \sum_k a_k z_k^n, \tag{3.15}$$

then the output will be

$$y[n] = \sum_k a_k H(z_k) z_k^n. \tag{3.16}$$

In other words, for both continuous time and discrete time, if the input to an LTI system is represented as a linear combination of complex exponentials, then the output can also be represented as a linear combination of the same complex exponential signals. Each coefficient in this representation of the output is obtained as the product of the corresponding coefficient a_k of the input and the system's eigenvalue $H(s_k)$ or $H(z_k)$ associated with the eigenfunction $e^{s_k t}$ or z_k^n, respectively. It was precisely this fact that Euler discovered for the problem of the vibrating string, that Gauss and others used in the analysis of time series, and that motivated Fourier and others after him to consider the question of how broad a class of signals could be represented as a linear combination of complex exponentials. In the next few sections we examine this question for periodic signals, first in continuous time and then in discrete time, and in Chapters 4 and 5 we consider the extension of these representations to aperiodic signals. Although in general, the variables s and z in eqs. (3.1)–(3.16) may be arbitrary complex numbers, Fourier analysis involves restricting our attention to particular forms for these variables. In particular, in continuous time we focus on purely imaginary values of s—i.e., $s = j\omega$—and thus, we consider only complex exponentials of the form $e^{j\omega t}$. Similarly, in discrete time we restrict the range of values of z to those of unit magnitude—i.e., $z = e^{j\omega}$—so that we focus on complex exponentials of the form $e^{j\omega n}$.

Example 3.1

As an illustration of eqs. (3.5) and (3.6), consider an LTI system for which the input $x(t)$ and output $y(t)$ are related by a time shift of 3, i.e.,

$$y(t) = x(t-3). \tag{3.17}$$

If the input to this system is the complex exponential signal $x(t) = e^{j2t}$, then, from eq. (3.17),

$$y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}. \tag{3.18}$$

Equation (3.18) is in the form of eq. (3.5), as we would expect, since e^{j2t} is an eigenfunction. The associated eigenvalue is $H(j2) = e^{-j6}$. It is straightforward to confirm eq. (3.6) for this example. Specifically, from eq. (3.17), the impulse response of the system is $h(t) = \delta(t-3)$. Substituting into eq. (3.6), we obtain

$$H(s) = \int_{-\infty}^{+\infty} \delta(\tau - 3)e^{-s\tau}\,d\tau = e^{-3s},$$

so that $H(j2) = e^{-j6}$.

As a second example, in this case illustrating eqs. (3.11) and (3.12), consider the input signal $x(t) = \cos(4t) + \cos(7t)$. From eq. (3.17), $y(t)$ will of course be

$$y(t) = \cos(4(t-3)) + \cos(7(t-3)). \tag{3.19}$$

To see that this will also result from eq. (3.12), we first expand $x(t)$ using Euler's relation:

$$x(t) = \tfrac{1}{2}e^{j4t} + \tfrac{1}{2}e^{-j4t} + \tfrac{1}{2}e^{j7t} + \tfrac{1}{2}\,e^{-j7t}. \tag{3.20}$$

From eqs. (3.11) and (3.12),

$$y(t) = \tfrac{1}{2}e^{-j12}e^{j4t} + \tfrac{1}{2}e^{j12}e^{-j4t} + \tfrac{1}{2}e^{-j21}e^{j7t} + \tfrac{1}{2}e^{j21}e^{-j7t},$$

or

$$\begin{aligned} y(t) &= \tfrac{1}{2}e^{j4(t-3)} + \tfrac{1}{2}e^{-j4(t-3)} + \tfrac{1}{2}e^{j7(t-3)} + \tfrac{1}{2}e^{-j7(t-3)} \\ &= \cos(4(t-3)) + \cos(7(t-3)). \end{aligned}$$

For this simple example, multiplication of each periodic exponential component of $x(t)$—for example, $\frac{1}{2}e^{j4t}$—by the corresponding eigenvalue—e.g., $H(j4) = e^{-j12}$—effectively causes the input component to shift in time by 3. Obviously, in this case we can determine $y(t)$ in eq. (3.19) by inspection rather than by employing eqs. (3.11) and (3.12). However, as we will see, the general property embodied in eqs. (3.11) and (3.12) not only allows us to calculate the responses of more complex LTI systems, but also provides the basis for the frequency domain representation and analysis of LTI systems.

3.3 FOURIER SERIES REPRESENTATION OF CONTINUOUS-TIME PERIODIC SIGNALS

3.3.1 Linear Combinations of Harmonically Related Complex Exponentials

As defined in Chapter 1, a signal is periodic if, for some positive value of T,

$$x(t) = x(t+T) \quad \text{for all } t. \tag{3.21}$$

The fundamental period of $x(t)$ is the minimum positive, nonzero value of T for which eq. (3.21) is satisfied, and the value $\omega_0 = 2\pi/T$ is referred to as the fundamental frequency.

In Chapter 1 we also introduced two basic periodic signals, the sinusoidal signal

$$x(t) = \cos \omega_0 t \tag{3.22}$$

and the periodic complex exponential

$$x(t) = e^{j\omega_0 t}. \tag{3.23}$$

Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $T = 2\pi/\omega_0$. Associated with the signal in eq. (3.23) is the set of *harmonically related* complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{3.24}$$

Each of these signals has a fundamental frequency that is a multiple of ω_0, and therefore, each is periodic with period T (although for $|k| \geq 2$, the fundamental period of $\phi_k(t)$ is a fraction of T). Thus, a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \tag{3.25}$$

is also periodic with period T. In eq. (3.25), the term for $k = 0$ is a constant. The terms for $k = +1$ and $k = -1$ both have fundamental frequency equal to ω_0 and are collectively referred to as the *fundamental components* or the *first harmonic components*. The two terms for $k = +2$ and $k = -2$ are periodic with half the period (or, equivalently, twice the frequency) of the fundamental components and are referred to as the *second harmonic components*. More generally, the components for $k = +N$ and $k = -N$ are referred to as the Nth harmonic components.

The representation of a periodic signal in the form of eq. (3.25) is referred to as the *Fourier series* representation. Before developing the properties of this representation, let us consider an example.

Example 3.2

Consider a periodic signal $x(t)$, with fundamental frequency 2π, that is expressed in the form of eq. (3.25) as

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}, \tag{3.26}$$

where

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

Rewriting eq. (3.26) and collecting each of the harmonic components which have the same fundamental frequency, we obtain

$$\begin{aligned} x(t) = 1 &+ \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) \\ &+ \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t}). \end{aligned} \tag{3.27}$$

Equivalently, using Euler's relation, we can write $x(t)$ in the form

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t. \tag{3.28}$$

In Figure 3.4, we illustrate graphically how the signal $x(t)$ is built up from its harmonic components.

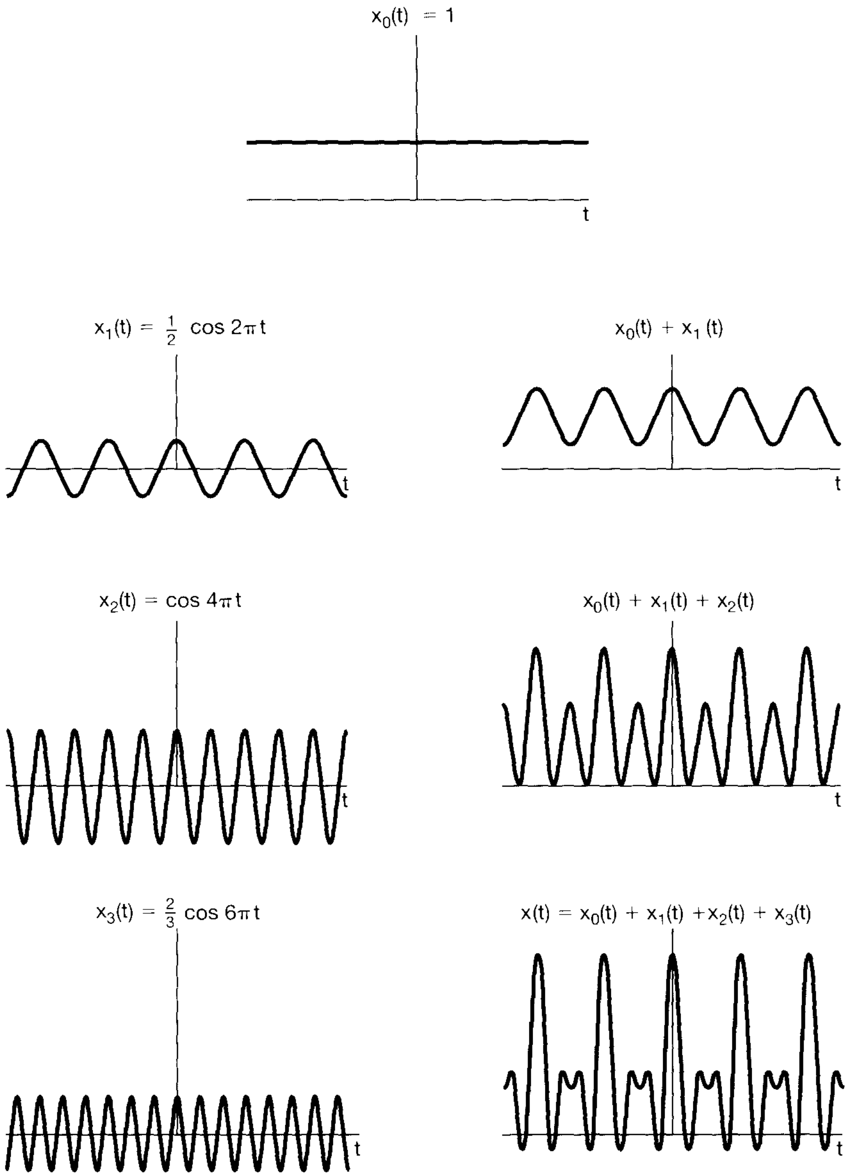


Figure 3.4 Construction of the signal $x(t)$ in Example 3.2 as a linear combination of harmonically related sinusoidal signals.

Equation (3.28) is an example of an alternative form for the Fourier series of real periodic signals. Specifically, suppose that $x(t)$ is real and can be represented in the form of eq. (3.25). Then, since $x^*(t) = x(t)$, we obtain

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}.$$

Replacing k by $-k$ in the summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^{*} e^{jk\omega_0 t},$$

which, by comparison with eq. (3.25), requires that $a_k = a_{-k}^{*}$, or equivalently, that

$$a_k^{*} = a_{-k}. \tag{3.29}$$

Note that this is the case in Example 3.2, where the a_k's are in fact real and $a_k = a_{-k}$.

To derive the alternative forms of the Fourier series, we first rearrange the summation in eq. (3.25) as

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}].$$

Substituting a_k^{*} for a_{-k} from eq. (3.29), we obtain

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k^{*} e^{-jk\omega_0 t}].$$

Since the two terms inside the summation are complex conjugates of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\{a_k e^{jk\omega_0 t}\}. \tag{3.30}$$

If a_k is expressed in polar form as

$$a_k = A_k e^{j\theta_k},$$

then eq. (3.30) becomes

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\{A_k e^{j(k\omega_0 t + \theta_k)}\}.$$

That is,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k). \tag{3.31}$$

Equation (3.31) is one commonly encountered form for the Fourier series of real periodic signals in continuous time. Another form is obtained by writing a_k in rectangular form as

$$a_k = B_k + jC_k,$$

where B_k and C_k are both real. With this expression for a_k, eq. (3.30) takes the form

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]. \tag{3.32}$$

In Example 3.2 the a_k's are all real, so that $a_k = A_k = B_k$, and therefore, both representations, eqs. (3.31) and (3.32), reduce to the same form, eq. (3.28).

Thus, for real periodic functions, the Fourier series in terms of complex exponentials, as given in eq. (3.25), is mathematically equivalent to either of the two forms in eqs. (3.31) and (3.32) that use trigonometric functions. Although the latter two are common forms for Fourier series,[7] the complex exponential form of eq. (3.25) is particularly convenient for our purposes, so we will use that form almost exclusively.

Equation (3.29) illustrates one of many properties associated with Fourier series. These properties are often quite useful in gaining insight and for computational purposes, and in Section 3.5 we collect together the most important of them. The derivation of several of them is considered in problems at the end of the chapter. In Section 4.3, we also will develop the majority of the properties within the broader context of the Fourier transform.

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

Assuming that a given periodic signal can be represented with the series of eq. (3.25), we need a procedure for determining the coefficients a_k. Multiplying both sides of eq. (3.25) by $e^{-jn\omega_0 t}$, we obtain

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}. \tag{3.33}$$

Integrating both sides from 0 to $T = 2\pi/\omega_0$, we have

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}\,dt.$$

Here, T is the fundamental period of $x(t)$, and consequently, we are integrating over one period. Interchanging the order of integration and summation yields

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}\,dt\right]. \tag{3.34}$$

The evaluation of the bracketed integral is straightforward. Rewriting this integral using Euler's formula, we obtain

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos(k-n)\omega_0 t\,dt + j\int_0^T \sin(k-n)\omega_0 t\,dt. \tag{3.35}$$

For $k \neq n$, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are periodic sinusoids with fundamental period $(T/|k-n|)$. Therefore, in eq. (3.35), we are integrating over an interval (of length T) that is an integral number of periods of these signals. Since the integral may be viewed as measuring the total area under the functions over the interval, we see that for $k \neq n$, both of the integrals on the right-hand side of eq. (3.35) are zero. For $k = n$, the integrand on the left-hand side of eq. (3.35) equals 1, and thus, the integral equals T. In sum, we then have

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases},$$

[7]In fact, in his original work, Fourier used the sine-cosine form of the Fourier series given in eq. (3.32).

and consequently, the right-hand side of eq. (3.34) reduces to Ta_n. Therefore,

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt, \tag{3.36}$$

which provides the equation for determining the coefficients. Furthermore, note that in evaluating eq. (3.35), the only fact that we used concerning the interval of integration was that we were integrating over an interval of length T, which is an integral number of periods of $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$. Therefore, we will obtain the same result if we integrate over any interval of length T. That is, if we denote integration over *any* interval of length T by $\int_T$, we have

$$\int_T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases},$$

and consequently,

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\,dt. \tag{3.37}$$

To summarize, if $x(t)$ has a Fourier series representation [i.e., if it can be expressed as a linear combination of harmonically related complex exponentials in the form of eq. (3.25)], then the coefficients are given by eq. (3.37). This pair of equations, then, defines the Fourier series of a periodic continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \tag{3.38}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt. \tag{3.39}$$

Here, we have written equivalent expressions for the Fourier series in terms of the fundamnetal frequency ω_0 and the fundamental period T. Equation (3.38) is referred to as the *synthesis* equation and eq. (3.39) as the *analysis* equation. The set of coefficients $\{a_k\}$ are often called the *Fourier series coefficients* or the *spectral coefficients* of $x(t)$.[8] These complex coefficients measure the portion of the signal $x(t)$ that is at each harmonic of the fundamental component. The coefficient a_0 is the dc or constant component of $x(t)$ and is given by eq. (3.39) with $k = 0$. That is,

$$a_0 = \frac{1}{T}\int_T x(t)\,dt, \tag{3.40}$$

which is simply the average value of $x(t)$ over one period.

Equations (3.38) and (3.39) were known to both Euler and Lagrange in the middle of the 18th century. However, they discarded this line of analysis without having

[8]The term "spectral coefficient" is derived from problems such as the spectroscopic decomposition of light into spectral lines (i.e., into its elementary components at different frequencies). The intensity of any line in such a decomposition is a direct measure of the fraction of the total light energy at the frequency corresponding to the line.

examined the question of how large a class of periodic signals could, in fact, be represented in such a fashion. Before we turn to this question in the next section, let us illustrate the continuous-time Fourier series by means of a few examples.

Example 3.3

Consider the signal

$$x(t) = \sin \omega_0 t,$$

whose fundamental frequency is ω_0. One approach to determining the Fourier series coefficients for this signal is to apply eq. (3.39). For this simple case, however, it is easier to expand the sinusoidal signal as a linear combination of complex exponentials and identify the Fourier series coefficients by inspection. Specifically, we can express $\sin \omega_0 t$ as

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}.$$

Comparing the right-hand sides of this equation and eq. (3.38), we obtain

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$
$$a_k = 0, \qquad k \neq +1 \text{ or } -1.$$

Example 3.4

Let

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has fundamental frequency ω_0. As with Example 3.3, we can again expand $x(t)$ directly in terms of complex exponentials, so that

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

Collecting terms, we obtain

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Thus, the Fourier series coefficients for this example are

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1-j),$$
$$a_k = 0, \ |k| > 2.$$

In Figure 3.5, we show a bar graph of the magnitude and phase of a_k.

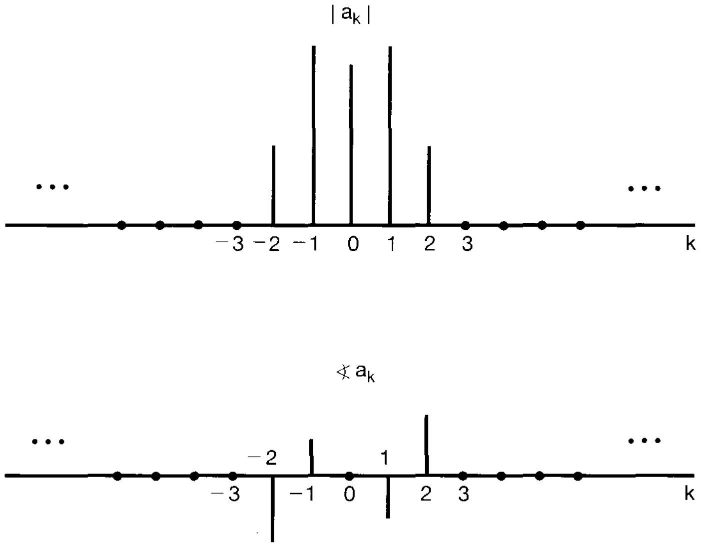


Figure 3.5 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 3.4.

Example 3.5

The periodic square wave, sketched in Figure 3.6 and defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \tag{3.41}$$

is a signal that we will encounter a number of times throughout this book. This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

To determine the Fourier series coefficients for $x(t)$, we use eq. (3.39). Because of the symmetry of $x(t)$ about $t = 0$, it is convenient to choose $-T/2 \le t < T/2$ as the

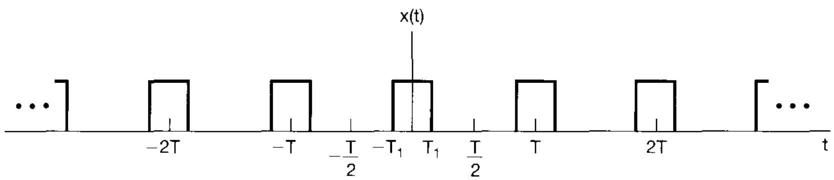


Figure 3.6 Periodic square wave.

interval over which the integration is performed, although any interval of length T is equally valid and thus will lead to the same result. Using these limits of integration and substituting from eq. (3.41), we have first, for $k = 0$,

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}. \tag{3.42}$$

As mentioned previously, a_0 is interpreted to be the average value of $x(t)$, which in this case equals the fraction of each period during which $x(t) = 1$. For $k \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1},$$

which we may rewrite as

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]. \tag{3.43}$$

Noting that the term in brackets is $\sin k\omega_0 T_1$, we can express the coefficients a_k as

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0, \tag{3.44}$$

where we have used the fact that $\omega_0 T = 2\pi$.

Figure 3.7 is a bar graph of the Fourier series coefficients for this example. In particular, the coefficients are plotted for a fixed value of T_1 and several values of T. For this specific example, the Fourier coefficients are real, and consequently, they can be depicted graphically with only a single graph. More generally, of course, the Fourier coefficients are complex, so that two graphs, corresponding to the real and imaginary parts, or magnitude and phase, of each coefficient, would be required. For $T = 4T_1$, $x(t)$ is a square wave that is unity for half the period and zero for half the period. In this case, $\omega_0 T_1 = \pi/2$, and from eq. (3.44),

$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \quad k \neq 0, \tag{3.45}$$

while

$$a_0 = \frac{1}{2}. \tag{3.46}$$

From eq. (3.45), $a_k = 0$ for k even and nonzero. Also, $\sin(\pi k/2)$ alternates between ± 1 for successive odd values of k. Therefore,

$$a_1 = a_{-1} = \frac{1}{\pi},$$

$$a_3 = a_{-3} = -\frac{1}{3\pi},$$

$$a_5 = a_{-5} = \frac{1}{5\pi},$$

$$\vdots$$

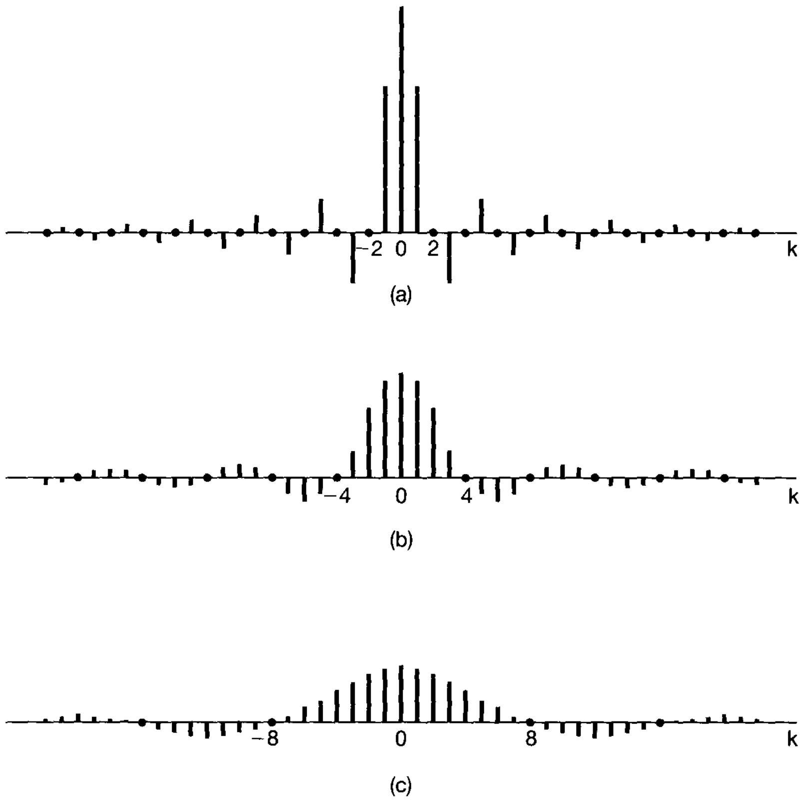


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

3.4 CONVERGENCE OF THE FOURIER SERIES

Although Euler and Lagrange would have been happy with the results of Examples 3.3 and 3.4, they would have objected to Example 3.5, since $x(t)$ is discontinuous while each of its harmonic components is continuous. Fourier, on the other hand, considered the same example and maintained that the Fourier series representation of the square wave is valid. In fact, Fourier maintained that *any* periodic signal could be represented by a Fourier series. Although this is not quite true, it *is* true that Fourier series can be used to represent an extremely large class of periodic signals, including the square wave and all other periodic signals with which we will be concerned in this book and which are of interest in practice.

To gain an understanding of the square-wave example and, more generally, of the question of the validity of Fourier series representations, let us examine the problem of approximating a given periodic signal $x(t)$ by a linear combination of a finite number of harmonically related complex exponentials—that is, by a finite series of the form

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}. \tag{3.47}$$

Let $e_N(t)$ denote the approximation error; that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}. \tag{3.48}$$

In order to determine how good any particular approximation is, we need to specify a quantitative measure of the size of the approximation error. The criterion that we will use is the energy in the error over one period:

$$E_N = \int_T |e_N(t)|^2 \, dt. \tag{3.49}$$

As shown in Problem 3.66, the particular choice for the coefficients in eq. (3.47) that minimize the energy in the error is

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \, dt. \tag{3.50}$$

Comparing eqs. (3.50) and (3.39), we see that eq. (3.50) is identical to the expression used to determine the Fourier series coefficients. Thus, if $x(t)$ has a Fourier series representation, the best approximation using only a finite number of harmonically related complex exponentials is obtained by truncating the Fourier series to the desired number of terms. As N increases, new terms are added and E_N decreases. If, in fact, $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.

Let us turn now to the question of when a periodic signal $x(t)$ does in fact have a Fourier series representation. Of course, for any signal, we can attempt to obtain a set of Fourier coefficients through the use of eq. (3.39). However, in some cases, the integral in eq. (3.39) may diverge; that is, the value obtained for some of the a_k may be infinite. Moreover, even if all of the coefficients obtained from eq. (3.39) are finite, when these coefficients are substituted into the synthesis equation (3.38), the resulting infinite series may not converge to the original signal $x(t)$.

Fortunately, there are no convergence difficulties for large classes of periodic signals. For example, every continuous periodic signal has a Fourier series representation for which the energy E_N in the approximation error approaches 0 as N goes to ∞. This is also true for many discontinuous signals. Since we will find it very useful to include discontinuous signals such as square waves in our discussions, it is worthwhile to investigate the issue of convergence in a bit more detail. Specifically, there are two somewhat different classes of conditions that a periodic signal can satisfy to guarantee that it can be represented by a Fourier series. In discussing these, we will not attempt to provide a complete mathematical justification; more rigorous treatments can be found in many texts on Fourier analysis.[9]

[9]See, for example, R. V. Churchill, *Fourier Series and Boundary Value Problems*, 3rd ed. (New York: McGraw-Hill Book Company, 1978); W. Kaplan, *Operational Methods for Linear Systems* (Reading, MA: Addison-Wesley Publishing Company, 1962); and the book by Dym and McKean referenced in footnote 2 of this chapter.

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period, i.e., signals for which

$$\int_T |x(t)|^2\, dt < \infty. \tag{3.51}$$

When this condition is satisfied, we are guaranteed that the coefficients a_k obtained from eq. (3.39) are finite. Furthermore, let $x_N(t)$ be the approximation to $x(t)$ obtained by using these coefficients for $|k| \leq N$:

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}. \tag{3.52}$$

Then we are guaranteed that the energy E_N in the approximation error, as defined in eq. (3.49), converges to 0 as we add more and more terms, i.e., as $N \rightarrow \infty$. That is, if we define

$$e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \tag{3.53}$$

then

$$\int_T |e(t)|^2\, dt = 0. \tag{3.54}$$

As we will see in an example at the end of this section, eq. (3.54) does *not* imply that the signal $x(t)$ and its Fourier series representation

$$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \tag{3.55}$$

are equal at every value of t. What it does say is that there is no energy in their difference.

The type of convergence guaranteed when $x(t)$ has finite energy over a single period is quite useful. In this case eq. (3.54) states that the difference between $x(t)$ and its Fourier series representation has zero energy. Since physical systems respond to signal energy, from this perspective $x(t)$ and its Fourier series representation are indistinguishable. Because most of the periodic signals that we consider do have finite energy over a single period, they have Fourier series representations. Moreover, an alternative set of conditions, developed by P. L. Dirichlet and also satisfied by essentially all of the signals with which we will be concerned, guarantees that $x(t)$ *equals* its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. At these values, the infinite series of eq. (3.55) converges to the average of the values on either side of the discontinuity.

The Dirichlet conditions are as follows:

Condition 1. Over any period, $x(t)$ must be *absolutely integrable*; that is,

$$\int_T |x(t)|\, dt < \infty. \tag{3.56}$$

As with square integrability, this guarantees that each coefficient a_k will be finite, since

$$|a_k| \leq \frac{1}{T}\int_T |x(t)e^{-jk\omega_0 t}|\,dt = \frac{1}{T}\int_T |x(t)|\,dt.$$

So if

$$\int_T |x(t)|\,dt < \infty,$$

then

$$|a_k| < \infty.$$

A periodic signal that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1;$$

that is, $x(t)$ is periodic with period 1. This signal is illustrated in Figure 3.8(a).

Condition 2. In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

An example of a function that meets Condition 1 but not Condition 2 is

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1, \tag{3.57}$$

as illustrated in Figure 3.8(b). For this function, which is periodic with $T = 1$,

$$\int_0^1 |x(t)|\,dt < 1.$$

The function has, however, an infinite number of maxima and minima in the interval.

Condition 3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

An example of a function that violates Condition 3 is illustrated in Figure 3.8(c). The signal, of period $T = 8$, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.

As can be seen from the examples given in Figure 3.8, signals that do not satisfy the Dirichlet conditions are generally pathological in nature and consequently do not typically arise in practical contexts. For this reason, the question of the convergence of Fourier series will not play a particularly significant role in the remainder of the book. For a periodic signal that has no discontinuities, the Fourier series representation converges and equals the original signal at every value of t. For a periodic signal with a finite number of discontinuities in each period, the Fourier series representation equals the signal every-

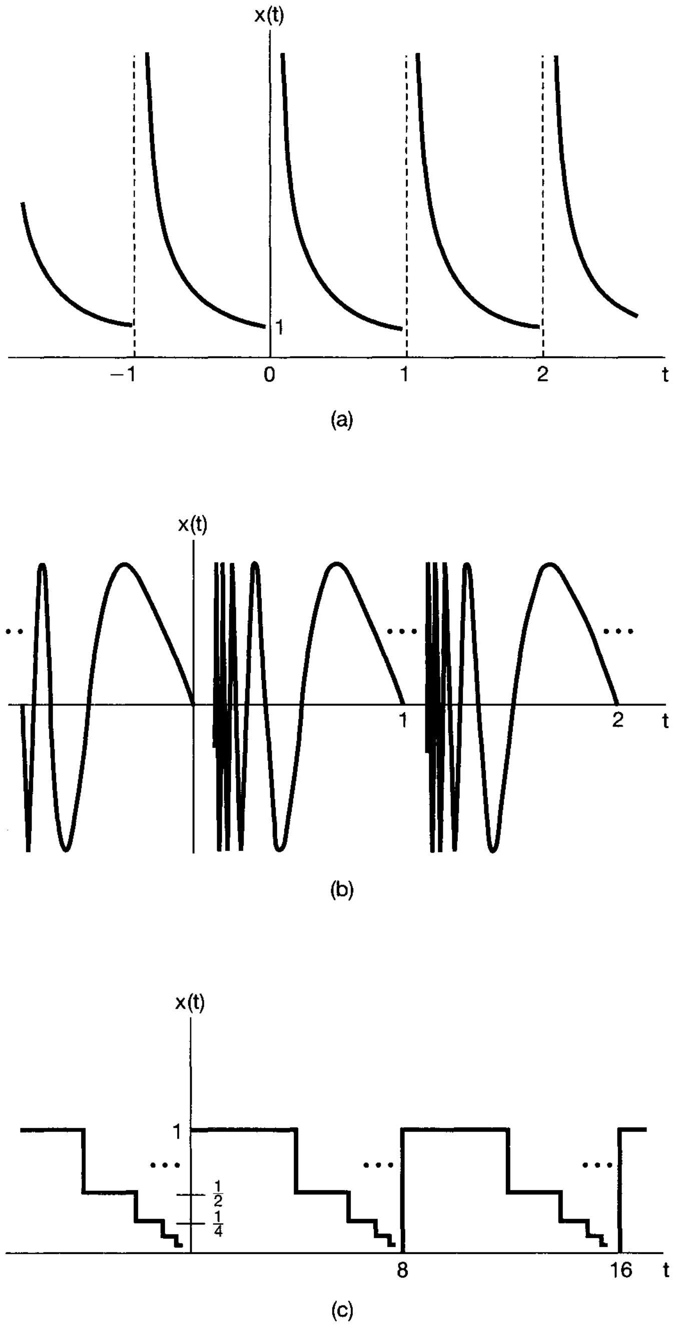


Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

where except at the isolated points of discontinuity, at which the series converges to the average value of the signal on either side of the discontinuity. In this case the difference between the original signal and its Fourier series representation contains no energy, and consequently, the two signals can be thought of as being the same for all practical purposes. Specifically, since the signals differ only at isolated points, the integrals of both signals over any interval *are* identical. For this reason, the two signals behave identically under convolution and consequently are identical from the standpoint of the analysis of LTI systems.

To gain some additional understanding of *how* the Fourier series converges for a periodic signal with discontinuities, let us return to the example of a square wave. In particular, in 1898,[10] an American physicist, Albert Michelson, constructed a harmonic analyzer, a device that, for any periodic signal $x(t)$, would compute the truncated Fourier series approximation of eq. (3.52) for values of N up to 80. Michelson tested his device on many functions, with the expected result that $x_N(t)$ looked very much like $x(t)$. However, when he tried the square wave, he obtained an important and, to him, very surprising result. Michelson was concerned about the behavior he observed and thought that his device might have had a defect. He wrote about the problem to the famous mathematical physicist Josiah Gibbs, who investigated it and reported his explanation in 1899.

What Michelson had observed is illustrated in Figure 3.9, where we have shown $x_N(t)$ for several values of N for $x(t)$, a symmetric square wave ($T = 4T_1$). In each case, the partial sum is superimposed on the original square wave. Since the square wave satisfies the Dirichlet conditions, the limit as $N \to \infty$ of $x_N(t)$ at the discontinuities should be the average value of the discontinuity. We see from the figure that this is in fact the case, since for any N, $x_N(t)$ has exactly that value at the discontinuities. Furthermore, for any other value of t, say, $t = t_1$, we are guaranteed that

$$\lim_{N \to \infty} x_N(t_1) = x(t_1).$$

Therefore, the squared error in the Fourier series representation of the square wave has zero area, as in eqs. (3.53) and (3.54).

For this example, the interesting effect that Michelson observed is that the behavior of the partial sum in the vicinity of the discontinuity exhibits ripples and that the peak amplitude of these ripples does not seem to decrease with increasing N. Gibbs showed that these are in fact the case. Specifically, for a discontinuity of unity height, the partial sum exhibits a maximum value of 1.09 (i.e., an overshoot of 9% of the height of the discontinuity), no matter how large N becomes. One must be careful to interpret this correctly, however. As stated before, for any *fixed* value of t, say, $t = t_1$, the partial sums will converge to the correct value, and at the discontinuity they will converge to one-half the sum of the values of the signal on either side of the discontinuity. However, the closer t_1 is chosen to the point of discontinuity, the larger N must be in order to reduce the error below a specified amount. Thus, as N increases, the ripples in the partial sums become compressed toward the discontinuity, but for *any* finite value of N, the peak amplitude of the ripples remains constant. This behavior has come to be known as the *Gibbs phenomenon*. The implication is that the truncated Fourier series approximation $x_N(t)$ of a discontinuous signal $x(t)$ will in general exhibit high-frequency ripples and overshoot $x(t)$ near the discontinuities. If such an approximation is used in practice, a large enough value of N should be chosen so as to guarantee that the total energy in these ripples is insignificant. In the limit, of course, we know that the energy in the approximation error vanishes and that the Fourier series representation of a discontinuous signal such as the square wave converges.

[10]The historical information used in this example is taken from the book by Lanczos referenced in footnote 1 of this chapter.

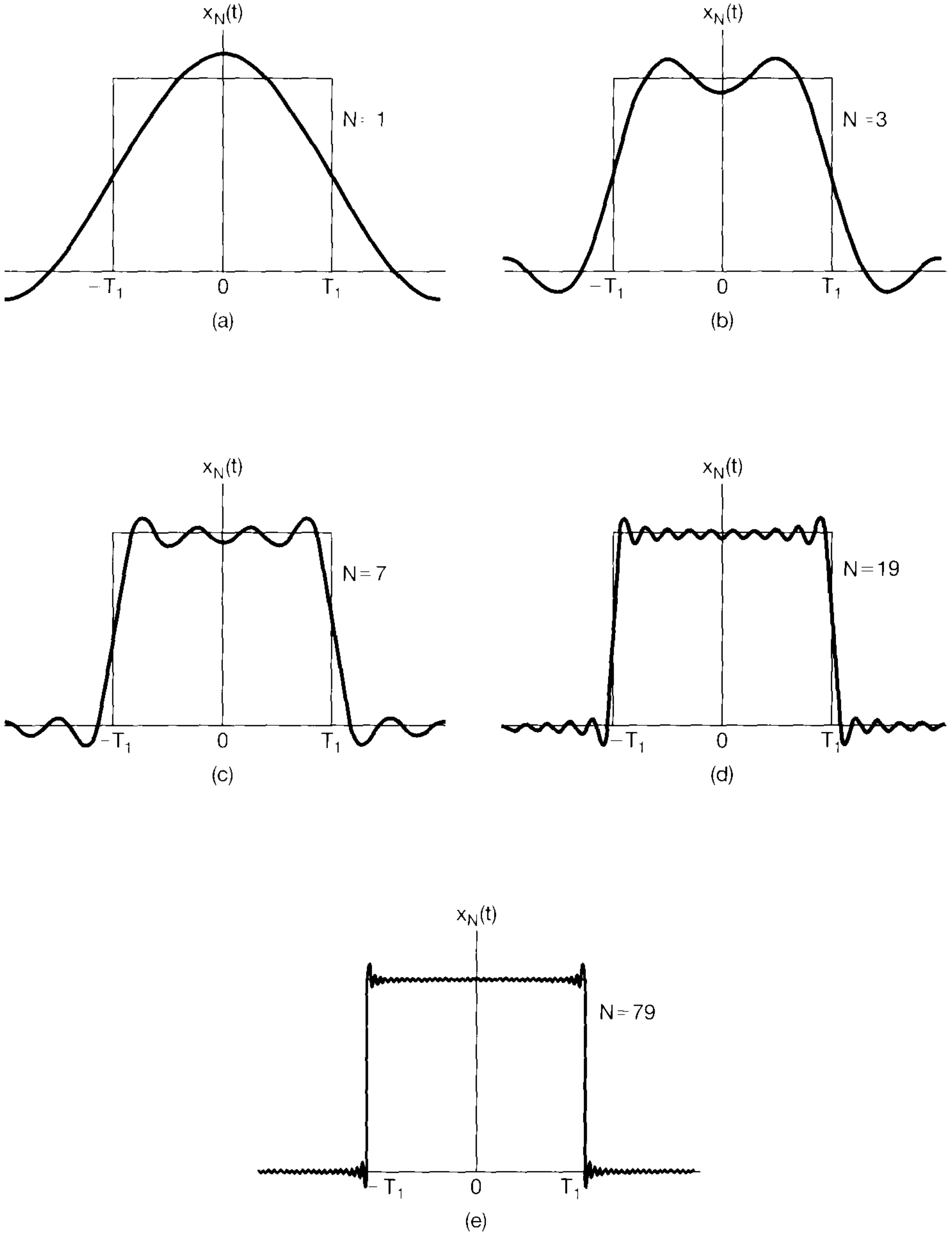


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

3.5 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

As mentioned earlier, Fourier series representations possess a number of important properties that are useful for developing conceptual insights into such representations, and they can also help to reduce the complexity of the evaluation of the Fourier series of many signals. In Table 3.1 we have summarized these properties, several of which are considered in the problems at the end of this chapter. In Chapter 4, in which we develop the Fourier transform, we will see that most of these properties can be deduced from corresponding properties of the continuous-time Fourier transform. Consequently we limit ourselves here to the discussion of several of these properties to illustrate how they may be derived, interpreted, and used.

Throughout the following discussion of selected properties from Table 3.1, we will find it convenient to use a shorthand notation to indicate the relationship between a periodic signal and its Fourier series coefficients. Specifically, suppose that $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. Then if the Fourier series coefficients of $x(t)$ are denoted by a_k, we will use the notation

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

to signify the pairing of a periodic signal with its Fourier series coefficients.

3.5.1 Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and which have Fourier series coefficients denoted by a_k and b_k, respectively. That is,

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k.$$

Since $x(t)$ and $y(t)$ have the same period T, it easily follows that any linear combination of the two signals will also be periodic with period T. Furthermore, the Fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination of the Fourier series coefficients for $x(t)$ and $y(t)$. That is,

$$z(t) = Ax(t) + By(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k. \tag{3.58}$$

The proof of this follows directly from the application of eq. (3.39). We also note that the linearity property is easily extended to a linear combination of an arbitrary number of signals with period T.

3.5.2 Time Shifting

When a time shift is applied to a periodic signal $x(t)$, the period T of the signal is preserved. The Fourier series coefficients b_k of the resulting signal $y(t) = x(t - t_0)$ may be expressed as

$$b_k = \frac{1}{T}\int_T x(t - t_0)e^{-jk\omega_0 t}dt. \tag{3.59}$$

Letting $\tau = t - t_0$ in the integral, and noting that the new variable τ will also range over an interval of duration T, we obtain

$$\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau$$
$$= e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k, \tag{3.60}$$

where a_k is the kth Fourier series coefficient of $x(t)$. That is, if

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x(t - t_0) \stackrel{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k.$$

One consequence of this property is that, when a periodic signal is shifted in time, the *magnitudes* of its Fourier series coefficients remain unaltered. That is, $|b_k| = |a_k|$.

3.5.3 Time Reversal

The period T of a periodic signal $x(t)$ also remains unchanged when the signal undergoes time reversal. To determine the Fourier series coefficients of $y(t) = x(-t)$, let us consider the effect of time reversal on the synthesis equation (3.38):

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}. \tag{3.61}$$

Making the substitution $k = -m$, we obtain

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m}e^{jm2\pi t/T}. \tag{3.62}$$

We observe that the right-hand side of this equation has the form of a Fourier series synthesis equation for $x(-t)$, where the Fourier series coefficients b_k are

$$b_k = a_{-k}. \tag{3.63}$$

That is, if

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x(-t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{-k}.$$

In other words time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of Fourier series coefficients. An interesting consequence of the time-reversal property is that if $x(t)$ is even—that is, if $x(-t) = x(t)$—then its Fourier series coefficients are also even—i.e., $a_{-k} = a_k$. Similarly, if $x(t)$ is odd, so that $x(-t) = -x(t)$, then so are its Fourier series coefficients—i.e., $a_{-k} = -a_k$.

3.5.4 Time Scaling

Time scaling is an operation that in general changes the period of the underlying signal. Specifically, if $x(t)$ is periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$. Since the time-scaling operation applies directly to each of the harmonic components of $x(t)$, we may easily conclude that the Fourier coefficients for each of those components remain the same. That is, if $x(t)$ has the Fourier series representation in eq. (3.38), then

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

is the Fourier series representation of $x(\alpha t)$. We emphasize that, while the Fourier coefficients have not changed, the Fourier series representation *has* changed because of the change in the fundamental frequency.

3.5.5 Multiplication

Suppose that $x(t)$ and $y(t)$ are both periodic with period T and that

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, we can expand it in a Fourier series with Fourier series coefficients h_k expressed in terms of those for $x(t)$ and $y(t)$. The result is

$$x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \tag{3.64}$$

One way to derive this relationship (see Problem 3.46) is to multiply the Fourier series representations of $x(t)$ and $y(t)$ and to note that the kth harmonic component in the product will have a coefficient which is the sum of terms of the form $a_l b_{k-l}$. Observe that the sum on the right-hand side of eq. (3.64) may be interpreted as the discrete-time convolution of the sequence representing the Fourier coefficients of $x(t)$ and the sequence representing the Fourier coefficients of $y(t)$.

3.5.6 Conjugation and Conjugate Symmetry

Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation *and* time reversal on the corresponding Fourier series coefficients. That is, if

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}. \tag{3.65}$$

This property is easily proved by applying complex conjugation to both sides of eq. (3.38) and replacing the summation variable k by its negative.

Some interesting consequences of this property may be derived for $x(t)$ real—that is, when $x(t) = x^*(t)$. In particular, in this case, we see from eq. (3.65) that the Fourier series coefficients will be *conjugate symmetric*, i.e.,

$$a_{-k} = a_k^*, \tag{3.66}$$

as we previously saw in eq. (3.29). This in turn implies various symmetry properties (listed in Table 3.1) for the magnitudes, phases, real parts, and imaginary parts of the Fourier series coefficients of real signals. For example, from eq. (3.66), we see that if $x(t)$ is real, then a_0 is real and

$$|a_k| = |a_{-k}|.$$

Also, if $x(t)$ is real and even, then, from Section 3.5.3, $a_k = a_{-k}$. However, from eq. (3.66) we see that $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is, if $x(t)$ is real and even, then so are its Fourier series coefficients. Similarly, it can be shown that if $x(t)$ is real and odd, then its Fourier series coefficients are purely imaginary and odd. Thus, for example, $a_0 = 0$ if $x(t)$ is real and odd. This and the other symmetry properties of the Fourier series are examined further in Problem 3.42.

3.5.7 Parseval's Relation for Continuous-Time Periodic Signals

As shown in Problem 3.46, Parseval's relation for continuous-time periodic signals is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2, \tag{3.67}$$

where the a_k are the Fourier series coefficients of $x(t)$ and T is the period of the signal.

Note that the left-hand side of eq. (3.67) is the average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$. Also,

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2, \tag{3.68}$$

so that $|a_k|^2$ is the average power in the kth harmonic component of $x(t)$. Thus, what Parseval's relation states is that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

3.5.8 Summary of Properties of the Continuous-Time Fourier Series

In Table 3.1, we summarize these and other important properties of continuous-time Fourier series.

3.5.9 Examples

Fourier series properties, such as those listed in Table 3.1, may be used to circumvent some of the algebra involved in determining the Fourier coefficients of a given signal. In the next

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t - \tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2. \tag{3.69}$$

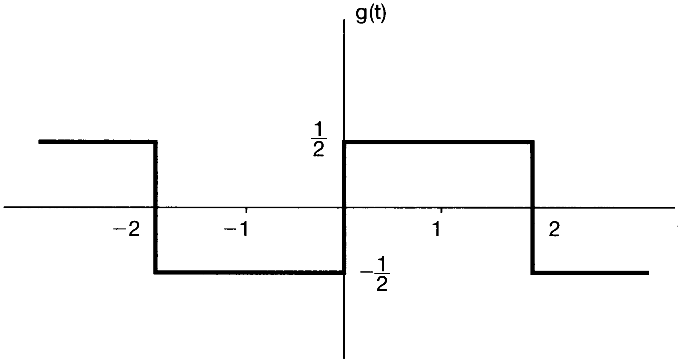


Figure 3.10 Periodic signal for Example 3.6.

The time-shift property in Table 3.1 indicates that, if the Fourier Series coefficients of $x(t)$ are denoted by a_k, the Fourier coefficients of $x(t-1)$ may be expressed as

$$b_k = a_k e^{-jk\pi/2}. \tag{3.70}$$

The Fourier coefficients of the *dc offset* in $g(t)$—i.e., the term $-1/2$ on the right-hand side of eq. (3.69)—are given by

$$c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}. \tag{3.71}$$

Applying the linearity property in Table 3.1, we conclude that the coefficients for $g(t)$ may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases},$$

where each a_k may now be replaced by the corresponding expression from eqs. (3.45) and (3.46), yielding

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}. \tag{3.72}$$

Example 3.7

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$ shown in Figure 3.11. The derivative of this signal is the signal $g(t)$ in Exam-

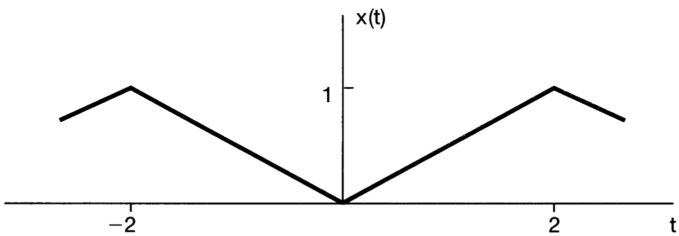


Figure 3.11 Triangular wave signal in Example 3.7.

ple 3.6. Denoting the Fourier coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k, we see that the differentiation property in Table 3.1 indicates that

$$d_k = jk(\pi/2)e_k. \tag{3.73}$$

This equation can be used to express e_k in terms of d_k, except when $k = 0$. Specifically, from eq. (3.72),

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2}, \quad k \neq 0. \tag{3.74}$$

For $k = 0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

$$e_0 = \frac{1}{2}.$$

Example 3.8

Let us examine some properties of the Fourier series representation of a periodic train of impulses, or *impulse train.* This signal and its representation in terms of complex exponentials will play an important role when we discuss the topic of sampling in Chapter 7. The impulse train with period T may be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT); \tag{3.75}$$

it is illustrated in Figure 3.12(a). To determine the Fourier series coefficients a_k, we use eq. (3.39) and select the interval of integration to be $-T/2 \leq t \leq T/2$, avoiding the placement of impulses at the integration limits. Within this interval, $x(t)$ is the same as $\delta(t)$, and it follows that

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk2\pi t/T}dt = \frac{1}{T}. \tag{3.76}$$

In other words, all the Fourier series coefficients of the impulse train are identical. These coefficients are also real valued and even (with respect to the index k). This is to be expected, since, according to Table 3.1, any real and even signal (such as our impulse train) should have real and even Fourier coefficients.

The impulse train also has a straightforward relationship to square-wave signals such as $g(t)$ in Figure 3.6, which we repeat in Figure 3.12(b). The derivative of $g(t)$ is the signal $q(t)$ illustrated in Figure 3.12(c). We may interpret $q(t)$ as the difference of two shifted versions of the impulse train $x(t)$. That is,

$$q(t) = x(t + T_1) - x(t - T_1). \tag{3.77}$$

Using the properties of Fourier series, we can now compute the Fourier series coefficients of $q(t)$ and $g(t)$ without any further direct evaluation of the Fourier series analysis equation. First, from the time-shifting and linearity properties, we see from eq. (3.77) that the Fourier series coefficients b_k of $q(t)$ may be expressed in terms of the Fourier series coefficients a_k of $x(t)$; that is,

$$b_k = e^{jk\omega_0 T_1}a_k - e^{-jk\omega_0 T_1}a_k,$$

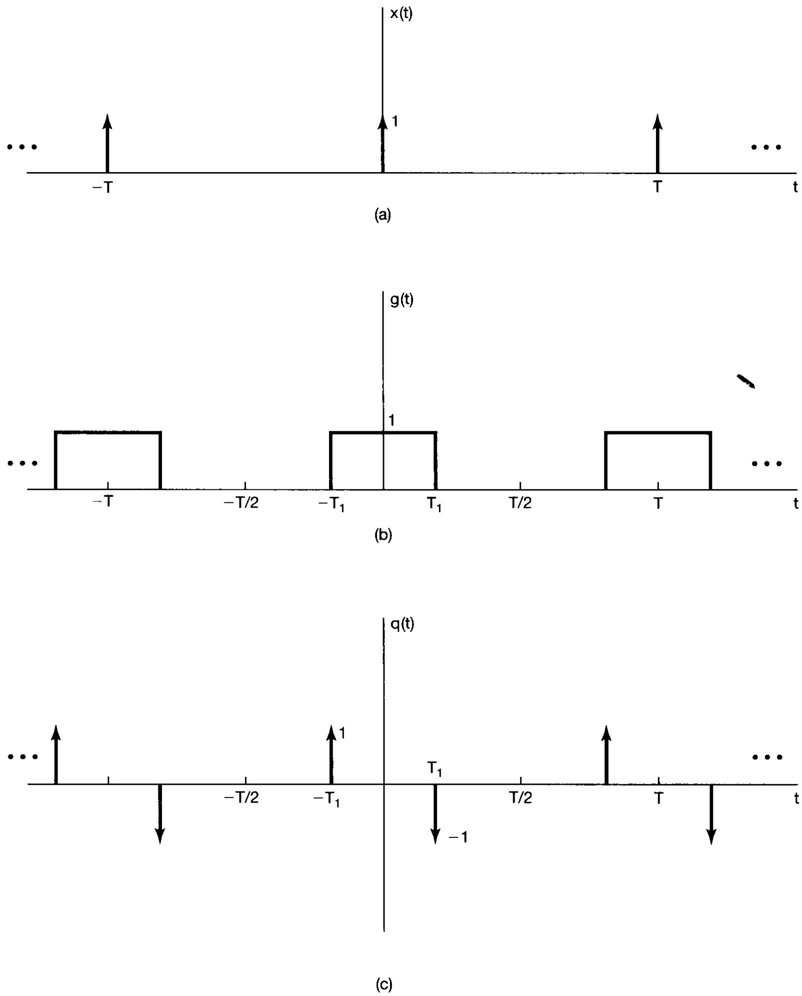


Figure 3.12 (a) Periodic train of impulses; (b) periodic square wave; (c) derivative of the periodic square wave in (b).

where $\omega_0 = 2\pi/T$. Using eq. (3.76), we then have

$$b_k = \frac{1}{T}[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}] = \frac{2j\sin(k\omega_0 T_1)}{T}.$$

Finally, since $q(t)$ is the derivative of $g(t)$, we can use the differentiation property in Table 3.1 to write

$$b_k = jk\omega_0 c_k, \tag{3.78}$$

where the c_k are the Fourier series coefficients of $g(t)$. Thus,

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0, \tag{3.79}$$

where we have used the fact that $\omega_0 T = 2\pi$. Note that eq. (3.79) is valid for $k \neq 0$, since we cannot solve for c_0 from eq. (3.78) with $k = 0$. However, since c_0 is just the average value of $g(t)$ over one period, we can determine it by inspection from Figure 3.12(b):

$$c_0 = \frac{2T_1}{T}. \tag{3.80}$$

Eqs. (3.80) and (3.79) are identical to eqs. (3.42) and (3.44), respectively, for the Fourier series coefficients of the square wave derived in Example 3.5.

The next example is chosen to illustrate the use of many of the properties in Table 3.1.

Example 3.9

Suppose we are given the following facts about a signal $x(t)$:

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k.
3. $a_k = 0$ for $|k| > 1$.
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2}a_{-k}$ is odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$.

Let us show that this information is sufficient to determine the signal $x(t)$ to within a sign factor. According to Fact 3, $x(t)$ has at most three nonzero Fourier series coefficients a_k: a_0, a_1, and a_{-1}. Then, since $x(t)$ has fundamental frequency $\omega_0 = 2\pi/4 = \pi/2$, it follows that

$$x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1}e^{-j\pi t/2}.$$

Since $x(t)$ is real (Fact 1), we can use the symmetry properties in Table 3.1 to conclude that a_0 is real and $a_1 = a*_{-1}$. Consequently,

$$x(t) = a_0 + a_1 e^{j\pi t/2} + (a_1 e^{j\pi t/2})^* = a_0 + 2\Re e\{a_1 e^{j\pi t/2}\}. \tag{3.81}$$

Let us now determine the signal corresponding to the Fourier coefficients b_k given in Fact 4. Using the time-reversal property from Table 3.1, we note that a_{-k} corresponds to the signal $x(-t)$. Also, the time-shift property in the table indicates that multiplication of the kth Fourier coefficient by $e^{-jk\pi/2} = e^{-jk\omega_0}$ corresponds to the underlying signal being shifted by 1 to the right (i.e., having t replaced by $t - 1$). We conclude that the coefficients b_k correspond to the signal $x(-(t-1)) = x(-t+1)$, which, according to Fact 4, must be odd. Since $x(t)$ is real, $x(-t+1)$ must also be real. From Table 3.1, it then follows that the Fourier coefficients of $x(-t+1)$ must be purely imaginary and odd. Thus, $b_0 = 0$ and $b_{-1} = -b_1$. Since time-reversal and time-shift operations cannot change the average power per period, Fact 5 holds even if $x(t)$ is replaced by $x(-t+1)$. That is,

$$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = 1/2. \tag{3.82}$$

We can now use Parseval's relation to conclude that

$$|b_1|^2 + |b_{-1}|^2 = 1/2. \tag{3.83}$$

Substituting $b_1 = -b_{-1}$ in this equation, we obtain $|b_1| = 1/2$. Since b_1 is also known to be purely imaginary, it must be either $j/2$ or $-j/2$.

Now we can translate these conditions on b_0 and b_1 into equivalent statements on a_0 and a_1. First, since $b_0 = 0$, Fact 4 implies that $a_0 = 0$. With $k = 1$, this condition implies that $a_1 = e^{-j\pi/2}b_{-1} = -jb_{-1} = jb_1$. Thus, if we take $b_1 = j/2$, then $a_1 = -1/2$, and therefore, from eq. (3.81), $x(t) = -\cos(\pi t/2)$. Alternatively, if we take $b_1 = -j/2$, then $a_1 = 1/2$, and therefore, $x(t) = \cos(\pi t/2)$.

3.6 FOURIER SERIES REPRESENTATION OF DISCRETE-TIME PERIODIC SIGNALS

In this section, we consider the Fourier series representation of discrete-time periodic signals. While the discussion closely parallels that of Section 3.3, there are some important differences. In particular, the Fourier series representation of a discrete-time periodic signal is a *finite* series, as opposed to the infinite series representation required for continuous-time periodic signals. As a consequence, there are no mathematical issues of convergence such as those discussed in Section 3.4.

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

As defined in Chapter 1, a discrete-time signal $x[n]$ is periodic with period N if

$$x[n] = x[n + N]. \tag{3.84}$$

The fundamental period is the smallest positive integer N for which eq. (3.84) holds, and $\omega_0 = 2\pi/N$ is the fundamental frequency. For example, the complex exponential $e^{j(2\pi/N)n}$ is periodic with period N. Furthermore, the set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{3.85}$$

All of these signals have fundamental frequencies that are multiples of $2\pi/N$ and thus are harmonically related.

As mentioned in Section 1.3.3, there are only N distinct signals in the set given by eq. (3.85). This is a consequence of the fact that discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical. Specifically, $\phi_0[n] = \phi_N[n]$, $\phi_1[n] = \phi_{N+1}[n]$, and, in general,

$$\phi_k[n] = \phi_{k+rN}[n]. \tag{3.86}$$

That is, when k is changed by any integer multiple of N, the identical sequence is generated. This differs from the situation in continuous time in which the signals $\phi_k(t)$ defined in eq. (3.24) are all different from one another.

We now wish to consider the representation of more general periodic sequences in terms of linear combinations of the sequences $\phi_k[n]$ in eq. (3.85). Such a linear combination has the form

$$x[n] = \sum_{k} a_k \phi_k[n] = \sum_{k} a_k e^{jk\omega_0 n} = \sum_{k} a_k e^{jk(2\pi/N)n}. \tag{3.87}$$

Since the sequences $\phi_k[n]$ are distinct only over a range of N successive values of k, the summation in eq. (3.87) need only include terms over this range. Thus, the summation is on k, as k varies over a range of N successive integers, beginning with any value of k. We indicate this by expressing the limits of the summation as $k = \langle N \rangle$. That is,

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}. \tag{3.88}$$

For example, k could take on the values $k = 0, 1, \ldots, N-1$, or $k = 3, 4, \ldots, N+2$. In either case, by virtue of eq. (3.86), exactly the same set of complex exponential sequences appears in the summation on the right-hand side of eq. (3.88). Equation (3.88) is referred to as the *discrete-time Fourier series* and the coefficients a_k as the *Fourier series coefficients*.

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

Suppose now that we are given a sequence $x[n]$ that is periodic with fundamental period N. We would like to determine whether a representation of $x[n]$ in the form of eq. (3.88) exists and, if so, what the values of the coefficients a_k are. This question can be phrased in terms of finding a solution to a set of linear equations. Specifically, if we evaluate eq. (3.88) for N successive values of n corresponding to one period of $x[n]$, we obtain

$$\begin{aligned} x[0] &= \sum_{k=\langle N \rangle} a_k, \\ x[1] &= \sum_{k=\langle N \rangle} a_k e^{j2\pi k/N}, \\ &\vdots \\ x[N-1] &= \sum_{k=\langle N \rangle} a_k e^{j2\pi k(N-1)/N}. \end{aligned} \tag{3.89}$$

Thus, eq. (3.89) represents a set of N linear equations for the N unknown coefficients a_k as k ranges over a set of N successive integers. It can be shown that this set of equations is linearly independent and consequently can be solved to obtain the coefficients a_k in terms of the given values of $x[n]$. In Problem 3.32, we consider an example in which the Fourier series coefficients are obtained by explicitly solving the set of N equations given in eq. (3.89). However, by following steps parallel to those used in continuous time, it is possible to obtain a closed-form expression for the coefficients a_k in terms of the values of the sequence $x[n]$.

The basis for this result is the fact, shown in Problem 3.54, that

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}. \tag{3.90}$$

Equation (3.90) states that the sum over one period of the values of a periodic complex exponential is zero, unless that complex exponential is a constant.

Now consider the Fourier series representation of eq. (3.88). Multiplying both sides by $e^{-jr(2\pi/N)n}$ and summing over N terms, we obtain

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n}. \tag{3.91}$$

Interchanging the order of summation on the right-hand side, we have

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}. \tag{3.92}$$

From the identity in eq. (3.90), the innermost sum on n on the right-hand side of eq. (3.92) is zero, unless $k - r$ is zero or an integer multiple of N. Therefore, if we choose values for r over the same range as that over which k varies in the outer summation, the innermost sum on the right-hand side of eq. (3.92) equals N if $k = r$ and 0 if $k \neq r$. The right-hand side of eq. (3.92) then reduces to Na_r, and we have

$$a_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n}. \tag{3.93}$$

This provides a closed-form expression for obtaining the Fourier series coefficients, and we have the *discrete-time Fourier series pair*:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}, \tag{3.94}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}. \tag{3.95}$$

These equations play the same role for discrete-time periodic signals that eqs. (3.38) and (3.39) play for continuous-time periodic signals, with eq. (3.94) the *synthesis* equation and eq. (3.95) the *analysis* equation. As in continuous time, the discrete-time Fourier series coefficients a_k are often referred to as the *spectral coefficients* of $x[n]$. These coefficients specify a decomposition of $x[n]$ into a sum of N harmonically related complex exponentials.

Referring to eq. (3.88), we see that if we take k in the range from 0 to $N - 1$, we have

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \ldots + a_{N-1}\phi_{N-1}[n]. \tag{3.96}$$

Similarly, if k ranges from 1 to N, we obtain

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \ldots + a_N\phi_N[n]. \tag{3.97}$$

From eq. (3.86), $\phi_0[n] = \phi_N[n]$, and therefore, upon comparing eqs. (3.96) and (3.97), we conclude that $a_0 = a_N$. Similarly, by letting k range over any set of N consecutive integers and using eq. (3.86), we can conclude that

$$a_k = a_{k+N}. \tag{3.98}$$

That is, if we consider more than N sequential values of k, the values a_k repeat periodically with period N. It is important that this fact be interpreted carefully. In particular, since there are only N distinct complex exponentials that are periodic with period N, the discrete-time Fourier series representation is a finite series with N terms. Therefore, if we fix the N consecutive values of k over which we define the Fourier series in eq. (3.94), we will obtain a set of exactly N Fourier coefficients from eq. (3.95). On the other hand, at times it will be convenient to use different sets of N values of k, and consequently, it is useful to regard eq. (3.94) as a sum over any *arbitrary* set of N successive values of k. For this reason, it is sometimes convenient to think of a_k as a sequence defined for all values of k, but where only N successive elements in the sequence will be used in the Fourier series representation. Furthermore, since the $\phi_k[n]$ repeat periodically with period N as we vary k [eq. (3.86)], so must the a_k [eq. (3.98)]. This viewpoint is illustrated in the next example.

Example 3.10

Consider the signal

$$x[n] = \sin \omega_0 n, \tag{3.99}$$

which is the discrete-time counterpart of the signal $x(t) = \sin \omega_0 t$ of Example 3.3. $x[n]$ is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers. For the case when $2\pi/\omega_0$ is an integer N, that is, when

$$\omega_0 = \frac{2\pi}{N},$$

$x[n]$ is periodic with fundamental period N, and we obtain a result that is exactly analogous to the continuous-time case. Expanding the signal as a sum of two complex exponentials, we get

$$x[n] = \frac{1}{2j}e^{j(2\pi/N)n} - \frac{1}{2j}e^{-j(2\pi/N)n}. \tag{3.100}$$

Comparing eq. (3.100) with eq. (3.94), we see by inspection that

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \tag{3.101}$$

and the remaining coefficients over the interval of summation are zero. As described previously, these coefficients repeat with period N; thus, a_{N+1} is also equal to $(1/2j)$ and a_{N-1} equals $(-1/2j)$. The Fourier series coefficients for this example with $N = 5$ are illustrated in Figure 3.13. The fact that they repeat periodically is indicated. However, only one period is utilized in the synthesis equation (3.94).

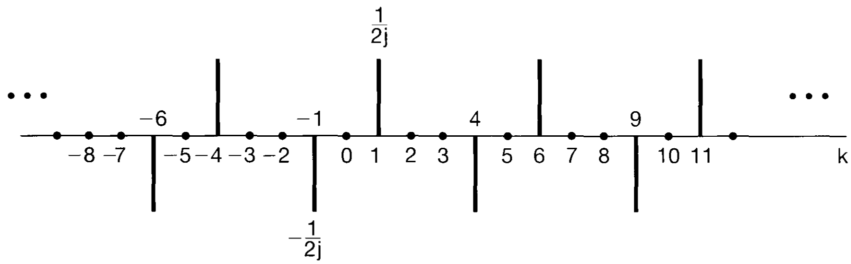


Figure 3.13 Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Consider now the case when $2\pi/\omega_0$ is a ratio of integers—that is, when

$$\omega_0 = \frac{2\pi M}{N}.$$

Assuming that M and N do not have any common factors, $x[n]$ has a fundamental period of N. Again expanding $x[n]$ as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j}e^{jM(2\pi/N)n} - \frac{1}{2j}e^{-jM(2\pi/N)n},$$

from which we can determine by inspection that $a_M = (1/2j)$, $a_{-M} = (-1/2j)$, and the remaining coefficients over one period of length N are zero. The Fourier coefficients for this example with $M = 3$ and $N = 5$ are depicted in Figure 3.14. Again, we have indicated the periodicity of the coefficients. For example, for $N = 5$, $a_2 = a_{-3}$, which in our example equals $(-1/2j)$. Note, however, that over any period of length 5 there are only two nonzero Fourier coefficients, and therefore there are only two nonzero terms in the synthesis equation.

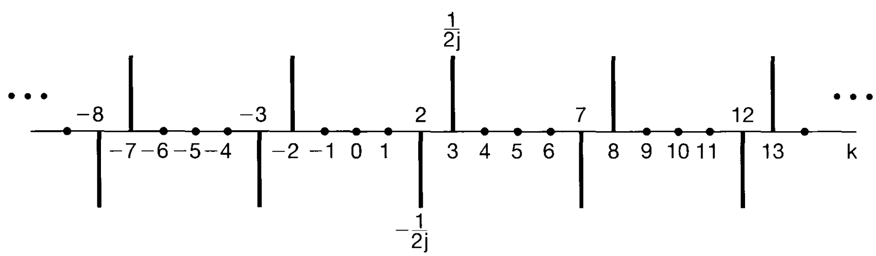


Figure 3.14 Fourier coefficients for $x[n] = \sin 3(2\pi/5)n$.

Example 3.11

Consider the signal

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$$

This signal is periodic with period N, and, as in Example 3.10, we can expand $x[n]$ directly in terms of complex exponentials to obtain

$$x[n] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}] + \frac{1}{2}[e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}].$$

Collecting terms, we find that

$$x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j(2\pi/N)n} + \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}.$$

Thus the Fourier series coefficients for this example are

$$a_0 = 1,$$
$$a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j,$$
$$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}j,$$
$$a_{-2} = -\frac{1}{2}j,$$

with $a_k = 0$ for other values of k in the interval of summation in the synthesis equation (3.94). Again, the Fourier coefficients are periodic with period N, so, for example, $a_N = 1$, $a_{3N-1} = \frac{3}{2} + \frac{1}{2}j$, and $a_{2-N} = \frac{1}{2}j$. In Figure 3.15(a) we have plotted the real and imaginary parts of these coefficients for $N = 10$, while the magnitude and phase of the coefficients are depicted in Figure 3.15(b).

Note that in Example 3.11, $a_{-k} = a_k^*$ for all values of k. In fact, this equality holds whenever $x[n]$ is real. The property is identical to one that we discussed in Section 3.3 for continuous-time periodic signals, and as in continuous time, one implication is that there are two alternative forms for the discrete-time Fourier series of real periodic sequences. These forms are analogous to the continuous-time Fourier series representations given in eqs. (3.31) and (3.32) and are examined in Problem 3.52. For our purposes, the exponential form of the Fourier series, as given in eqs. (3.94) and (3.95), is particularly convenient, and we will use it exclusively.

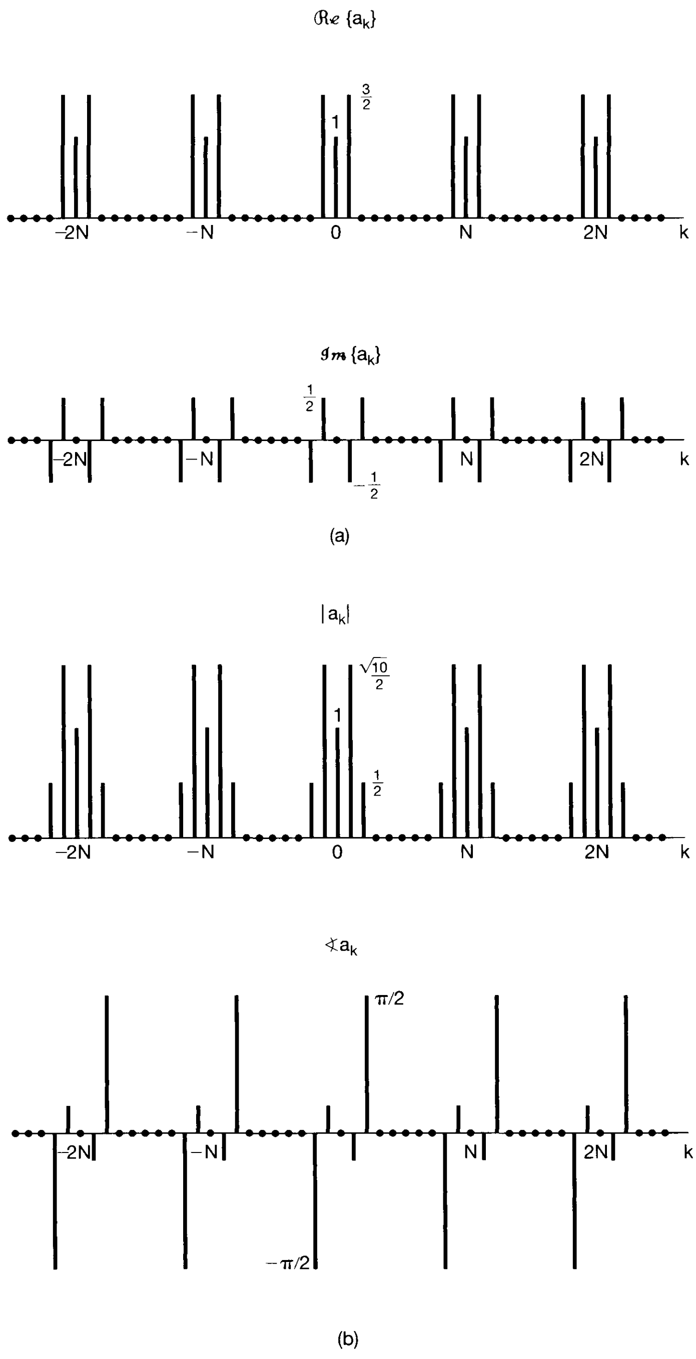


Figure 3.15 (a) Real and imaginary parts of the Fourier series coefficients in Example 3.11; (b) magnitude and phase of the same coefficients.

Example 3.12

In this example, we consider the discrete-time periodic square wave shown in Figure 3.16. We can evaluate the Fourier series for this signal using eq. (3.95). Because $x[n] = 1$ for $-N_1 \leq n \leq N_1$, it is particularly convenient to choose the length-N interval of summation in eq. (3.95) so that it includes the range $-N_1 \leq n \leq N_1$. In this case, we can express eq. (3.95) as

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}. \tag{3.102}$$

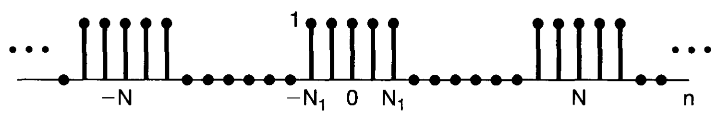


Figure 3.16 Discrete-time periodic square wave.

Letting $m = n + N_1$, we observe that eq. (3.102) becomes

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} \\ &= \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}. \end{aligned} \tag{3.103}$$

The summation in eq. (3.103) consists of the sum of the first $2N_1 + 1$ terms in a geometric series, which can be evaluated using the result of Problem 1.54. This yields

$$\begin{aligned} a_k &= \frac{1}{N} e^{jk(2\pi/N)N_1} \left(\frac{1 - e^{-jk2\pi(2N_1+1)/N}}{1 - e^{-jk(2\pi/N)}} \right) \\ &= \frac{1}{N} \frac{e^{-jk(2\pi/2N)}[e^{jk2\pi(N_1+1/2)/N} - e^{-jk2\pi(N_1+1/2)/N}]}{e^{-jk(2\pi/2N)}[e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)}]} \\ &= \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, \quad k \neq 0, \pm N, \pm 2N, \ldots \end{aligned} \tag{3.104}$$

and

$$a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots. \tag{3.105}$$

The coefficients a_k for $2N_1 + 1 = 5$ are sketched for $N = 10$, 20, and 40 in Figures 3.17(a), (b), and (c), respectively.

In discussing the convergence of the continuous-time Fourier series in Section 3.4, we considered the example of a symmetric square wave and observed how the finite sum in eq. (3.52) converged to the square wave as the number of terms approached infinity. In par-

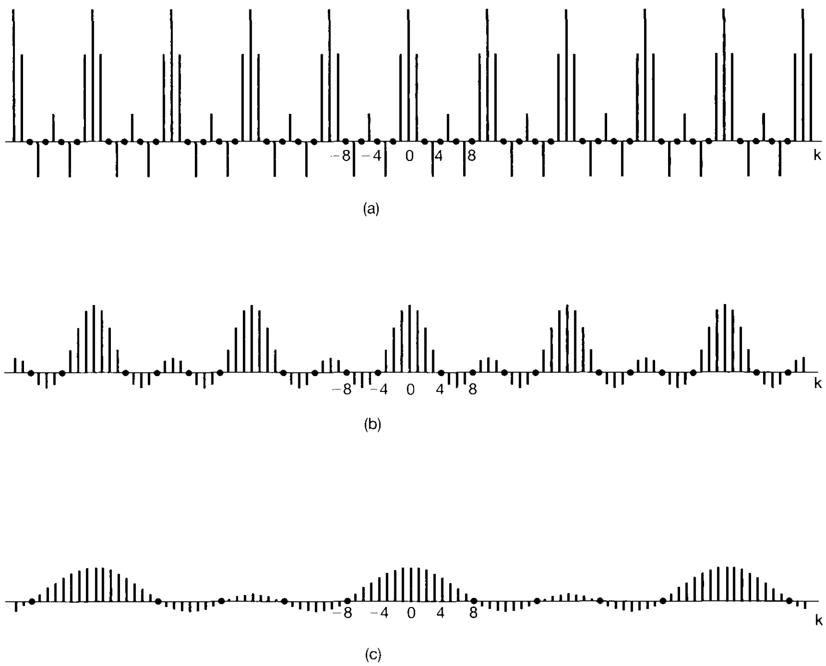


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

ticular, we observed the Gibbs phenomenon at the discontinuity, whereby, as the number of terms increased, the ripples in the partial sum (Figure 3.9) became compressed toward the discontinuity, with the peak amplitude of the ripples remaining constant independently of the number of terms in the partial sum. Let us consider the analogous sequence of partial sums for the discrete-time square wave, where, for convenience, we will assume that the period N is odd. In Figure 3.18, we have depicted the signals

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n} \tag{3.106}$$

for the example of Figure 3.16 with $N = 9$, $2N_1 + 1 = 5$, and for several values of M. For $M = 4$, the partial sum exactly equals $x[n]$. We see in particular that in contrast to the continuous-time case, there are no convergence issues and there is no Gibbs phenomenon. In fact, there are no convergence issues with the discrete-time Fourier series in general. The reason for this stems from the fact that any discrete-time periodic sequence $x[n]$ is completely specified by a *finite* number N of parameters, namely, the values of the sequence over one period. The Fourier series analysis equation (3.95) simply transforms this set of N parameters into an equivalent set—the values of the N Fourier coefficients—and

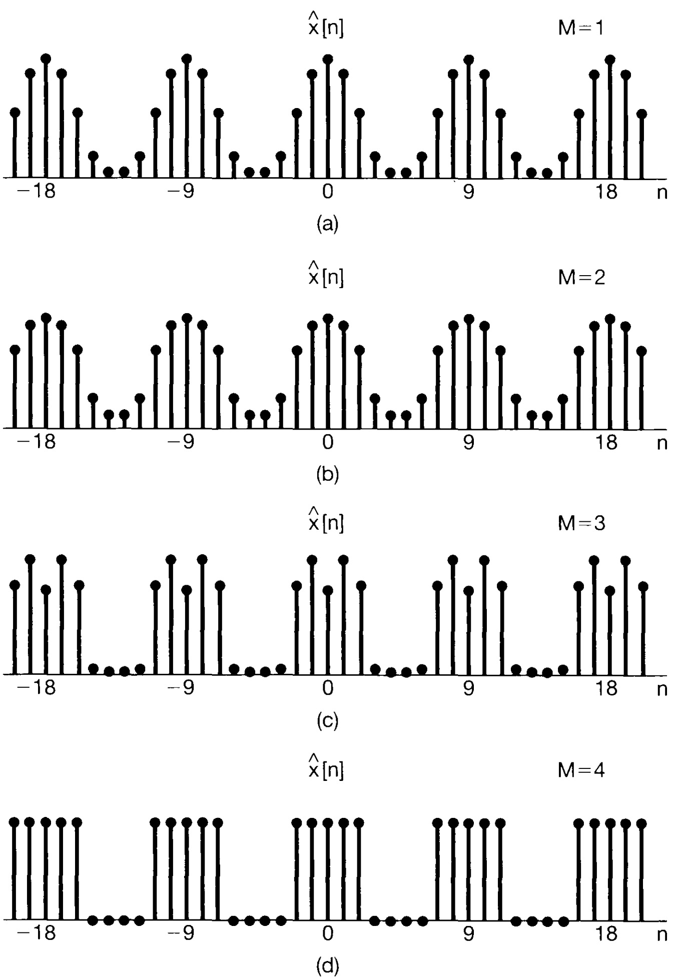


Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

the synthesis equation (3.94) tells us how to recover the values of the original sequence in terms of a *finite* series. Thus, if N is odd and we take $M = (N - 1)/2$ in eq. (3.106), the sum includes exactly N terms, and consequently, from the synthesis equations, we have $\hat{x}[n] = x[n]$. Similarly, if N is even and we let

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk(2\pi/N)n}, \tag{3.107}$$

then with $M = N/2$, this sum consists of N terms, and again, we can conclude from eq. (3.94) that $\hat{x}[n] = x[n]$.

In contrast, a continuous-time periodic signal takes on a continuum of values over a single period, and an infinite number of Fourier coefficients are required to represent it.

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	Na_kb_k
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$$

The derivations of many of these properties are very similar to those of the corresponding properties for continuous-time Fourier series, and several such derivations are considered in the problems at the end of the chapter. In addition, in Chapter 5 we will see that most of the properties can be inferred from corresponding properties of the discrete-time Fourier transform. Consequently, we limit the discussion in the following subsections to only a few of these properties, including several that have important differences relative to those for continuous time. We also provide examples illustrating the usefulness of various discrete-time Fourier series properties for developing conceptual insights and helping to reduce the complexity of the evaluation of the Fourier series of many periodic sequences.

As with continuous time, it is often convenient to use a shorthand notation to indicate the relationship between a periodic signal and its Fourier series coefficients. Specifically, if $x[n]$ is a periodic signal with period N and with Fourier series coefficients denoted by a_k, then we will write

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k.$$

3.7.1 Multiplication

The multiplication property of the Fourier series representation is one example of a property that reflects the difference between continuous time and discrete time. From Table 3.1, the product of two continuous-time signals of period T results in a periodic signal with period T whose sequence of Fourier series coefficients is the *convolution* of the sequences of Fourier series coefficients of the two signals being multiplied. In discrete time, suppose that

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

and

$$y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k$$

are both periodic with period N. Then the product $x[n]y[n]$ is also periodic with period N, and, as shown in Problem 3.57, its Fourier coefficients, d_k, are given by

$$x[n]y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=\langle N\rangle} a_l b_{k-l}. \tag{3.108}$$

Equation (3.108) is analogous to the definition of convolution, except that the summation variable is now restricted to an interval of N consecutive samples. As shown in Problem 3.57, the summation can be taken over *any* set of N consecutive values of l. We refer to this type of operation as a *periodic convolution* between the two periodic sequences of Fourier coefficients. The usual form of the convolution sum (where the summation variable ranges from $-\infty$ to ∞) is sometimes referred to as *aperiodic convolution*, to distinguish it from periodic convolution.

3.7.2 First Difference

The discrete-time parallel to the differentiation property of the continuous-time Fourier series involves the use of the first-difference operation, which is defined as $x[n]-x[n-1]$.

If $x[n]$ is periodic with period N, then so is $y[n]$, since shifting $x[n]$ or linearly combining $x[n]$ with another periodic signal whose period is N always results in a periodic signal with period N. Also, if

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then the Fourier coefficients corresponding to the first difference of $x[n]$ may be expressed as

$$x[n] - x[n-1] \stackrel{\mathcal{FS}}{\longleftrightarrow} (1 - e^{-jk(2\pi/N)})a_k, \tag{3.109}$$

which is easily obtained by applying the time-shifting and linearity properties in Table 3.2. A common use of this property is in situations where evaluation of the Fourier series coefficients is easier for the first difference than for the original sequence. (See Problem 3.31.)

3.7.3 Parseval's Relation for Discrete-Time Periodic Signals

As shown in Problem 3.57, Parseval's relation for discrete-time periodic signals is given by

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2, \tag{3.110}$$

where the a_k are the Fourier series coefficients of $x[n]$ and N is the period. As in the continuous-time case, the left-hand side of Parseval's relation is the average power in one period for the periodic signal $x[n]$. Similarly, $|a_k|^2$ is the average power in the kth harmonic component of $x[n]$. Thus, once again, Parseval's relation states that the average power in a periodic signal equals the sum of the average powers in all of its harmonic components. In discrete time, of course, there are only N distinct harmonic components, and since the a_k are periodic with period N, the sum on the right-hand side of eq. (3.110) can be taken over any N consecutive values of k.

3.7.4 Examples

In this subsection, we present several examples illustrating how properties of the discrete-time Fourier series can be used to characterize discrete-time periodic signals and to compute their Fourier series representations. Specifically, Fourier series properties, such as those listed in Table 3.2, may be used to simplify the process of determining the Fourier series coefficients of a given signal. This involves first expressing the given signal in terms of other signals whose Fourier series coefficients are already known or are simpler to compute. Then, using Table 3.2, we can express the Fourier series coefficients of the given signal in terms of the Fourier series coefficients of the other signals. This is illustrated in Example 3.13. Example 3.14 then illustrates the determination of a sequence from some partial information. In Example 3.15 we illustrate the use of the periodic convolution property in Table 3.2.

Example 3.13

Let us consider the problem of finding the Fourier series coefficients a_k of the sequence $x[n]$ shown in Figure 3.19(a). This sequence has a fundamental period of 5. We observe that $x[n]$ may be viewed as the sum of the square wave $x_1[n]$ in Figure 3.19(b) and the *dc sequence* $x_2[n]$ in Figure 3.19(c). Denoting the Fourier series coefficients of $x_1[n]$ by b_k and those of $x_2[n]$ by c_k, we use the linearity property of Table 3.2 to conclude that

$$a_k = b_k + c_k. \tag{3.111}$$

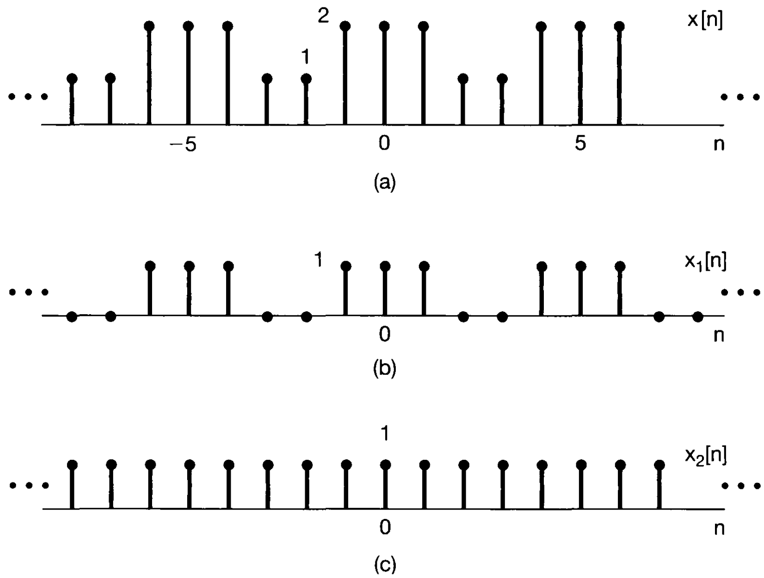


Figure 3.19 (a) Periodic sequence $x[n]$ for Example 3.13 and its representation as a sum of (b) the square wave $x_1[n]$ and (c) the dc sequence $x_2[n]$.

From Example 3.12 (with $N_1 = 1$ and $N = 5$), the Fourier series coefficients b_k corresponding to $x_1[n]$ can be expressed as

$$b_k = \begin{cases} \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \dots \\ \dfrac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \dots \end{cases}. \tag{3.112}$$

The sequence $x_2[n]$ has only a dc value, which is captured by its zeroth Fourier series coefficient:

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1. \tag{3.113}$$

Since the discrete-time Fourier series coefficients are periodic, it follows that $c_k = 1$ whenever k is an integer multiple of 5. The remaining coefficients of $x_2[n]$ must be zero, because $x_2[n]$ contains only a dc component. We can now substitute the expressions for b_k and c_k into eq. (3.111) to obtain

$$a_k = \begin{cases} b_k = \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \dots \\ \dfrac{8}{5}, & \text{for } k = 0, \pm 5, \pm 10, \dots \end{cases}. \tag{3.114}$$

Example 3.14

Suppose we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

Let us determine the sequence $x[n]$. We denote the Fourier series coefficients of $x[n]$ by a_k. From Fact 2, we conclude that $a_0 = 1/3$. Noting that $(-1)^n = e^{-j\pi n} = e^{-j(2\pi/6)3n}$, we see from Fact 3 that $a_3 = 1/6$. From Parseval's relation (see Table 3.2), the average power in $x[n]$ is

$$P = \sum_{k=0}^{5} |a_k|^2. \tag{3.115}$$

Since each nonzero coefficient contributes a positive amount to P, and since the values of a_0 and a_3 are prespecified, the value of P is minimized by choosing $a_1 = a_2 = a_4 = a_5 = 0$. It then follows that

$$x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n, \tag{3.116}$$

which is sketched in Figure 3.20.

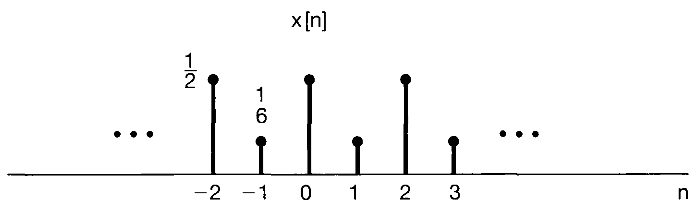


Figure 3.20 Sequence $x[n]$ that is consistent with the properties specified in Example 3.14.

Example 3.15

In this example we determine and sketch a periodic sequence, given an algebraic expression for its Fourier series coefficients. In the process, we will also exploit the periodic convolution property (see Table 3.2) of the discrete-time Fourier series. Specifically, as stated in the table and as shown in Problem 3.58, if $x[n]$ and $y[n]$ are periodic with period N, then the signal

$$w[n] = \sum_{r=\langle N\rangle} x[r]y[n-r]$$

is also periodic with period N. Here, the summation may be taken over any set of N consecutive values of r. Furthermore, the Fourier series coefficients of $w[n]$ are equal to Na_kb_k, where a_k and b_k are the Fourier coefficients of $x[n]$ and $y[n]$, respectively.

Suppose now that we are told that a signal $w[n]$ is periodic with a fundamental period of $N = 7$ and with Fourier series coefficients

$$c_k = \frac{\sin^2(3\pi k/7)}{7\sin^2(\pi k/7)}. \tag{3.117}$$

We observe that $c_k = 7d_k^2$, where d_k denotes the sequence of Fourier series coefficients of a square wave $x[n]$, as in Example 3.12, with $N_1 = 1$ and $N = 7$. Using the periodic convolution property, we see that

$$w[n] = \sum_{r=\langle 7\rangle} x[r]x[n-r] = \sum_{r=-3}^{3} x[r]x[n-r], \tag{3.118}$$

where, in the last equality, we have chosen to sum over the interval $-3 \leq r \leq 3$. Except for the fact that the sum is limited to a finite interval, the product-and-sum method for evaluating convolution is applicable here. In fact, we can convert eq. (3.118) to an ordinary convolution by defining a signal $\hat{x}[n]$ that equals $x[n]$ for $-3 \leq n \leq 3$ and is zero otherwise. Then, from eq. (3.118),

$$w[n] = \sum_{r=-3}^{3} \hat{x}[r]x[n-r] = \sum_{r=-\infty}^{+\infty} \hat{x}[r]x[n-r].$$

That is, $w[n]$ is the aperiodic convolution of the sequences $\hat{x}[n]$ and $x[n]$.

The sequences $x[r]$, $\hat{x}[r]$, and $x[n-r]$ are sketched in Figure 3.21 (a)–(c). From the figure we can immediately calculate $w[n]$. In particular we see that $w[0] = 3$; $w[-1] = w[1] = 2$; $w[-2] = w[2] = 1$; and $w[-3] = w[3] = 0$. Since $w[n]$ is periodic with period 7, we can then sketch $w[n]$ as shown in Figure 3.21(d).

3.8 FOURIER SERIES AND LTI SYSTEMS

In the preceding few sections, we have seen that the Fourier series representation can be used to construct any periodic signal in discrete time and essentially all periodic continuous-time signals of practical importance. In addition, in Section 3.2 we saw that the response of an LTI system to a linear combination of complex exponentials takes a particularly simple form. Specifically, in continuous time, if $x(t) = e^{st}$ is the input to a continuous-time LTI system, then the output is given by $y(t) = H(s)e^{st}$, where, from eq. (3.6),

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau, \tag{3.119}$$

in which $h(t)$ is the impulse response of the LTI system.

Similarly, if $x[n] = z^n$ is the input to a discrete-time LTI system, then the output is given by $y[n] = H(z)z^n$, where, from eq. (3.10),

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}, \tag{3.120}$$

in which $h[n]$ is the impulse response of the LTI system.

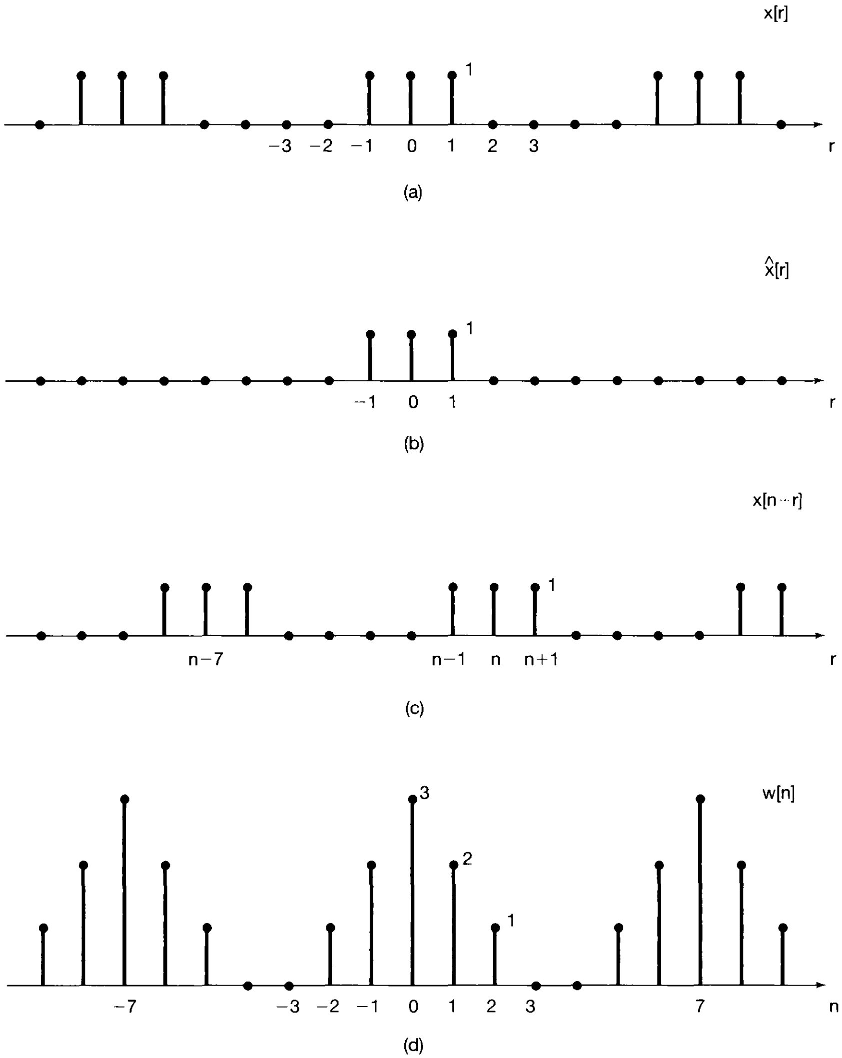


Figure 3.21 (a) The square-wave sequence $x[r]$ in Example 3.15; (b) the sequence $\hat{x}[r]$ equal to $x[r]$ for $-3 \leq r \leq 3$ and zero otherwise; (c) the sequence $x[n-r]$; (d) the sequence $w[n]$ equal to the periodic convolution of $x[n]$ with itself and to the aperiodic convolution of $\hat{x}[n]$ with $x[n]$.

When s or z are general complex numbers, $H(s)$ and $H(z)$ are referred to as the *system functions* of the corresponding systems. For continuous-time signals and systems in this and the following chapter, we focus on the specific case in which $\mathcal{R}e\{s\} = 0$, so that $s = j\omega$, and consequently, e^{st} is of the form $e^{j\omega t}$. This input is a complex exponential at frequency ω. The system function of the form $s = j\omega$—i.e., $H(j\omega)$ viewed as a function of ω—is referred to as the *frequency response* of the system and is given by

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt. \tag{3.121}$$

Similarly, for discrete-time signals and systems, we focus in this chapter and in Chapter 5 on values of z for which $|z| = 1$, so that $z = e^{j\omega}$ and z^n is of the form $e^{j\omega n}$. Then the system function $H(z)$ for z restricted to the form $z = e^{j\omega}$ is referred to as the frequency response of the system and is given by

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}. \tag{3.122}$$

The response of an LTI system to a complex exponential signal of the form $e^{j\omega t}$ (in continuous time) or $e^{j\omega n}$ (in discrete time) is particularly simple to express in terms of the frequency response of the system. Furthermore, as a result of the superposition property for LTI systems, we can express the response of an LTI system to a linear combination of complex exponentials with equal ease. In Chapters 4 and 5, we will see how we can use these ideas together with continuous-time and discrete-time Fourier transforms to analyze the response of LTI systems to aperiodic signals. In the remainder of this chapter, as a first look at this important set of concepts and results, we focus on interpreting and understanding this notion in the context of periodic signals.

Consider first the continuous-time case, and let $x(t)$ be a periodic signal with a Fourier series representation given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}. \tag{3.123}$$

Suppose that we apply this signal as the input to an LTI system with impulse response $h(t)$. Then, since each of the complex exponentials in eq. (3.123) is an eigenfunction of the system, as in eq. (3.13) with $s_k = jk\omega_0$, it follows that the output is

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}. \tag{3.124}$$

Thus, $y(t)$ is also periodic with the same fundamental frequency as $x(t)$. Furthermore, if $\{a_k\}$ is the set of Fourier series coefficients for the input $x(t)$, then $\{a_k H(jk\omega_0)\}$ is the set of coefficients for the output $y(t)$. That is, the effect of the LTI system is to modify individually each of the Fourier coefficients of the input through multiplication by the value of the frequency response at the corresponding frequency.

Example 3.16

Suppose that the periodic signal $x(t)$ discussed in Example 3.2 is the input signal to an LTI system with impulse response

$$h(t) = e^{-t}u(t).$$

To calculate the Fourier series coefficients of the output $y(t)$, we first compute the frequency response:

$$\begin{aligned} H(j\omega) &= \int_0^{\infty} e^{-\tau} e^{-j\omega\tau}\, d\tau \\ &= -\frac{1}{1+j\omega} e^{-\tau} e^{-j\omega\tau}\Big|_0^{\infty} \\ &= \frac{1}{1+j\omega}. \end{aligned} \tag{3.125}$$

Therefore, using eqs. (3.124) and (3.125), together with the fact that $\omega_0 = 2\pi$ in this example, we obtain

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}, \tag{3.126}$$

with $b_k = a_k H(jk2\pi)$, so that

$$\begin{aligned} & b_0 = 1, \\ b_1 &= \frac{1}{4}\left(\frac{1}{1+j2\pi}\right), & b_{-1} &= \frac{1}{4}\left(\frac{1}{1-j2\pi}\right), \\ b_2 &= \frac{1}{2}\left(\frac{1}{1+j4\pi}\right), & b_{-2} &= \frac{1}{2}\left(\frac{1}{1-j4\pi}\right), \\ b_3 &= \frac{1}{3}\left(\frac{1}{1+j6\pi}\right), & b_{-3} &= \frac{1}{3}\left(\frac{1}{1-j6\pi}\right). \end{aligned} \tag{3.127}$$

Note that $y(t)$ must be a real-valued signal, since it is the convolution of $x(t)$ and $h(t)$, which are both real. This can be verified by examining eq. (3.127) and observing that $b_k^* = b_{-k}$. Therefore, $y(t)$ can also be expressed in either of the forms given in eqs. (3.31) and (3.32); that is,

$$y(t) = 1 + 2\sum_{k=1}^{3} D_k \cos(2\pi kt + \theta_k), \tag{3.128}$$

or

$$y(t) = 1 + 2\sum_{k=1}^{3} [E_k \cos 2\pi kt - F_k \sin 2\pi kt], \tag{3.129}$$

where

$$b_k = D_k e^{j\theta_k} = E_k + jF_k, \quad k = 1, 2, 3. \tag{3.130}$$

These coefficients can be evaluated directly from eq. (3.127). For example,

$$\begin{aligned} D_1 &= |b_1| = \frac{1}{4\sqrt{1+4\pi^2}}, & \theta_1 &= \sphericalangle b_1 = -\tan^{-1}(2\pi), \\ E_1 &= \mathcal{R}e\{b_1\} = \frac{1}{4(1+4\pi^2)}, & F_1 &= \mathcal{I}m\{b_1\} = -\frac{\pi}{2(1+4\pi^2)}. \end{aligned}$$

In discrete time, the relationship between the Fourier series coefficients of the input and output of an LTI system exactly parallels eqs. (3.123) and (3.124). Specifically, let $x[n]$ be a periodic signal with Fourier series representation given by

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

If we apply this signal as the input to an LTI system with impulse response $h[n]$, then, as in eq. (3.16) with $z_k = e^{jk(2\pi/N)}$, the output is

$$y[n] = \sum_{k=\langle N\rangle} a_k H(e^{j2\pi k/N}) e^{jk(2\pi/N)n}. \tag{3.131}$$

Thus, $y[n]$ is also periodic with the same period as $x[n]$, and the kth Fourier coefficient of $y[n]$ is the product of the kth Fourier coefficient of the input and the value of the frequency response of the LTI system, $H(e^{j2\pi k/N})$, at the corresponding frequency.

Example 3.17

Consider an LTI system with impulse response $h[n] = \alpha^n u[n]$, $-1 < \alpha < 1$, and with the input

$$x[n] = \cos\left(\frac{2\pi n}{N}\right). \tag{3.132}$$

As in Example 3.10, $x[n]$ can be written in Fourier series form as

$$x[n] = \frac{1}{2} e^{j(2\pi/N)n} + \frac{1}{2} e^{-j(2\pi/N)n}.$$

Also, from eq. (3.122),

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n. \tag{3.133}$$

This geometric series can be evaluated using the result of Problem 1.54, yielding

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}. \tag{3.134}$$

Using eq. (3.131), we then obtain the Fourier series for the output:

$$\begin{aligned} y[n] &= \frac{1}{2} H\left(e^{j2\pi/N}\right) e^{j(2\pi/N)n} + \frac{1}{2} H\left(e^{-j2\pi/N}\right) e^{-j(2\pi/N)n} \\ &= \frac{1}{2}\left(\frac{1}{1 - \alpha e^{-j2\pi/N}}\right) e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1 - \alpha e^{j2\pi/N}}\right) e^{-j(2\pi/N)n}. \end{aligned} \tag{3.135}$$

If we write

$$\frac{1}{1 - \alpha e^{-j2\pi/N}} = re^{j\theta},$$

then eq. (3.135) reduces to

$$y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right). \tag{3.136}$$

For example, if $N = 4$,

$$\frac{1}{1 - \alpha e^{-j2\pi/4}} = \frac{1}{1 + \alpha j} = \frac{1}{\sqrt{1 + \alpha^2}} e^{j(-\tan^{-1}(\alpha))},$$

and thus,

$$y[n] = \frac{1}{\sqrt{1 + \alpha^2}} \cos\left(\frac{\pi n}{2} - \tan^{-1}(\alpha)\right).$$

We note that for expressions such as eqs. (3.124) and (3.131) to make sense, the frequency responses $H(j\omega)$ and $H(e^{j\omega})$ in eqs. (3.121) and (3.122) must be well defined and finite. As we will see in Chapters 4 and 5, this will be the case if the LTI systems under consideration are stable. For example, the LTI system in Example 3.16, with impulse response $h(t) = e^{-t}u(t)$, is stable and has a well-defined frequency response given by eq. (3.125). On the other hand, an LTI system with impulse response $h(t) = e^{t}u(t)$ is unstable, and it is easy to check that the integral in eq. (3.121) for $H(j\omega)$ diverges for any value of ω. Similarly, the LTI system in Example 3.17, with impulse response $h[n] = \alpha^n u[n]$, is stable for $|\alpha| < 1$ and has frequency response given by eq. (3.134). However, if $|\alpha| > 1$, the system is unstable, and then the summation in eq. (3.133) diverges.

3.9 FILTERING

In a variety of applications, it is of interest to change the relative amplitudes of the frequency components in a signal or perhaps eliminate some frequency components entirely, a process referred to as *filtering*. Linear time-invariant systems that change the shape of the spectrum are often referred to as *frequency-shaping filters.* Systems that are designed to pass some frequencies essentially undistorted and significantly attenuate or eliminate others are referred to as *frequency-selective filters.* As indicated by eqs. (3.124) and (3.131), the Fourier series coefficients of the output of an LTI system are those of the input multiplied by the frequency response of the system. Consequently, filtering can be conveniently accomplished through the use of LTI systems with an appropriately chosen frequency response, and frequency-domain methods provide us with the ideal tools to examine this very important class of applications. In this and the following two sections, we take a first look at filtering through a few examples.

3.9.1 Frequency-Shaping Filters

One application in which frequency-shaping filters are often encountered is audio systems. For example, LTI filters are typically included in such systems to permit the listener to modify the relative amounts of low-frequency energy (bass) and high-frequency energy (treble). These filters correspond to LTI systems whose frequency responses can be changed by manipulating the tone controls. Also, in high-fidelity audio systems, a so-called equalizing filter is often included in the preamplifier to compensate for the frequency-response characteristics of the speakers. Overall, these cascaded filtering stages are frequently referred to as the equalizing or equalizer circuits for the audio system. Figure 3.22 illustrates the three stages of the equalizer circuits for one particular series of audio speakers. In this figure, the magnitude of the frequency response for each of these stages is shown on a log-log plot. Specifically, the magnitude is in units of $20\log_{10}|H(j\omega)|$, referred to as decibels or dB. The frequency axis is labeled in Hz (i.e., $\omega/2\pi$) along a logarithmic scale. As will be discussed in more detail in Section 6.2.3, a logarithmic display of the magnitude of the frequency response in this form is common and useful.

Taken together, the equalizing circuits in Figure 3.22 are designed to compensate for the frequency response of the speakers and the room in which they are located and to allow the listener to control the overall frequency response. In particular, since the three systems are connected in cascade, and since each system modifies a complex exponential input $Ke^{j\omega t}$ by multiplying it by the system frequency response at that frequency, it follows that the overall frequency response of the cascade of the three systems is the product of the three frequency responses. The first two filters, indicated in Figures 3.22(a) and (b), together make up the control stage of the system, as the frequency behavior of these filters can be adjusted by the listener. The third filter, illustrated in Figure 3.22(c), is the equalizer stage, which has the fixed frequency response indicated. The filter in Figure 3.22(a) is a low-frequency filter controlled by a two-position switch, to provide one of the two frequency responses indicated. The second filter in the control stage has two continuously adjustable slider switches to vary the frequency response within the limits indicated in Figure 3.22(b).

Another class of frequency-shaping filters often encountered is that for which the filter output is the derivative of the filter input, i.e., $y(t) = dx(t)/dt$. With $x(t)$ of the form $x(t) = e^{j\omega t}$, $y(t)$ will be $y(t) = j\omega e^{j\omega t}$, from which it follows that the frequency response is

$$H(j\omega) = j\omega. \tag{3.137}$$

The frequency response characteristics of a differentiating filter are shown in Figure 3.23. Since $H(j\omega)$ is complex in general, and in this example in particular, $H(j\omega)$ is frequently displayed (as in the figure) as separate plots of $|H(j\omega)|$ and $\measuredangle H(j\omega)$. The shape of this frequency response implies that a complex exponential input $e^{j\omega t}$ will receive greater amplification for larger values of ω. Consequently, differentiating filters are useful in enhancing rapid variations or transitions in a signal.

One purpose for which differentiating filters are often used is to enhance edges in picture processing. A black-and-white picture can be thought of as a two-dimensional "continuous-time" signal $x(t_1, t_2)$, where t_1 and t_2 are the horizontal and vertical coordinates, respectively, and $x(t_1, t_2)$ is the brightness of the image. If the image is repeated periodically in the horizontal and vertical directions, then it can be represented by a two-dimensional Fourier series (see Problem 3.70) consisting of sums of products of complex

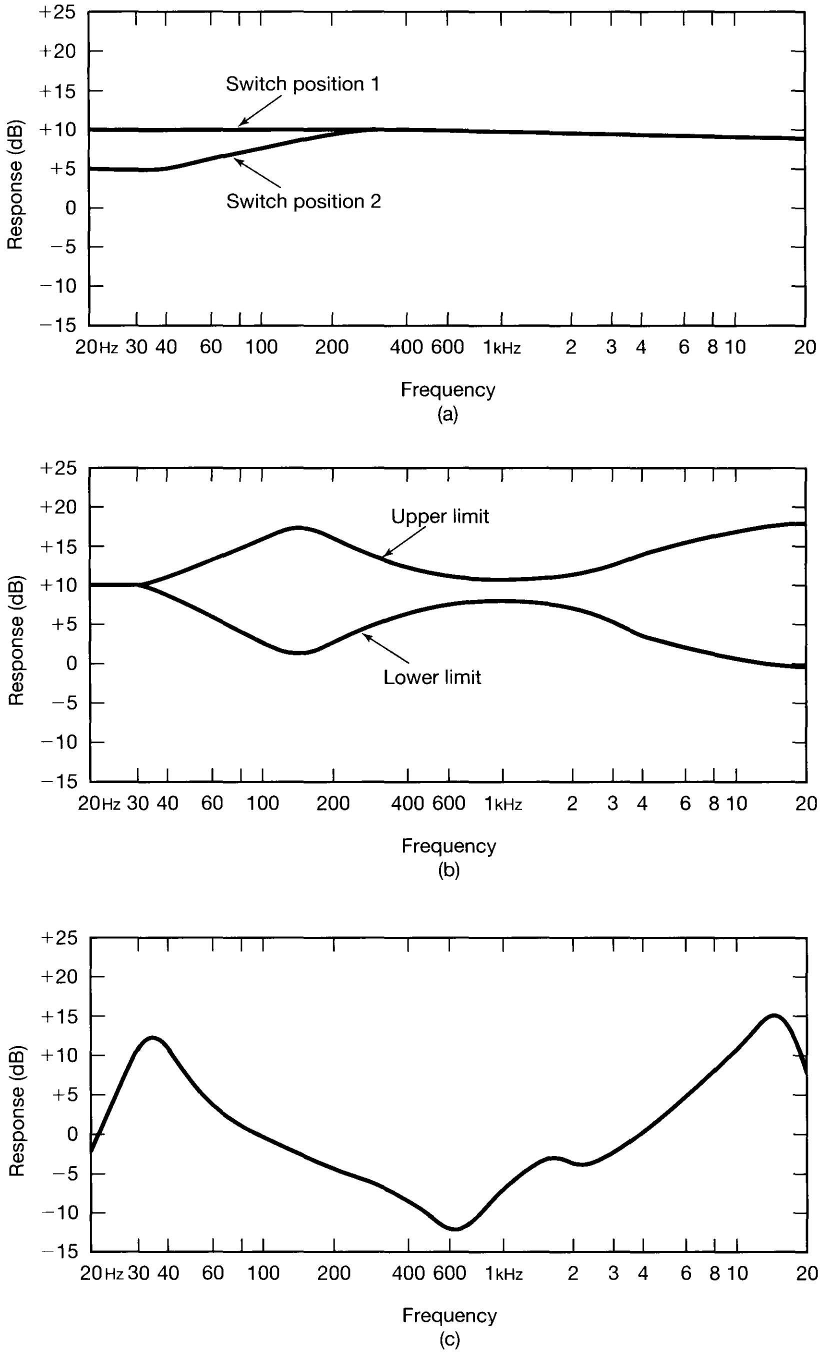


Figure 3.22 Magnitudes of the frequency responses of the equalizer circuits for one particular series of audio speakers, shown on a scale of $20 \log_{10} |H(j\omega)|$, which is referred to as a decibel (or dB) scale. (a) Low-frequency filter controlled by a two-position switch; (b) upper and lower frequency limits on a continuously adjustable shaping filter; (c) fixed frequency response of the equalizer stage.

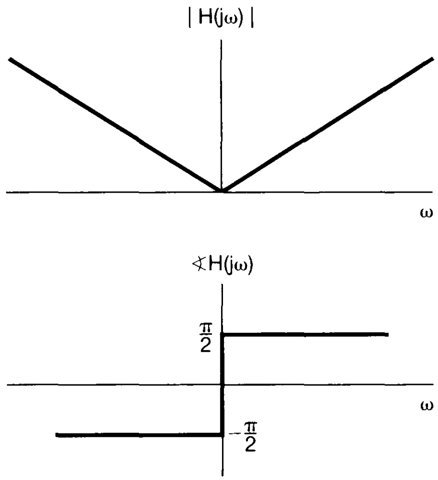


Figure 3.23 Characteristics of the frequency response of a filter for which the output is the derivative of the input.

exponentials, $e^{j\omega_1 t_1}$ and $e^{j\omega_2 t_2}$, that oscillate at possibly different frequencies in each of the two coordinate directions. Slow variations in brightness in a particular direction are represented by the lower harmonics in that direction. For example, consider an edge corresponding to a sharp transition in brightness that runs vertically in an image. Since the brightness is constant or slowly varying along the edge, the frequency content of the edge in the vertical direction is concentrated at low frequencies. In contrast, since there is an abrupt variation in brightness across the edge, the frequency content of the edge in the horizontal direction is concentrated at higher frequencies. Figure 3.24 illustrates the effect on an image of the two-dimensional equivalent of a differentiating filter.[11] Figure 3.24(a) shows two original images and Figure 3.24(b) the result of processing those images with the filter. Since the derivative at the edges of a picture is greater than in regions where the brightness varies slowly with distance, the effect of the filter is to enhance the edges.

Discrete-time LTI filters also find a broad array of applications. Many of these involve the use of discrete-time systems, implemented using general- or special-purpose digital processors, to process continuous-time signals, a topic we discuss at some length in Chapter 7. In addition, the analysis of time series information, including demographic data and economic data sequences such as the stock market average, commonly involves the use of discrete-time filters. Often the long-term variations (which correspond to low frequencies) have a different significance than the short-term variations (which correspond to high frequencies), and it is useful to analyze these components separately. Reshaping the relative weighting of the components is typically accomplished using discrete-time filters.

As one example of a simple discrete-time filter, consider an LTI system that successively takes a two-point average of the input values:

$$y[n] = \frac{1}{2}(x[n] + x[n-1]). \tag{3.138}$$

[11]Specifically each image in Figure 3.24(b) is the magnitude of the two-dimensional gradient of its counterpart image in Figure 3.24(a) where the magnitude of the gradient of $f(x, y)$ is

$$\left[\left(\frac{\partial f(x,y)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y)}{\partial y}\right)^2\right]^{1/2}.$$

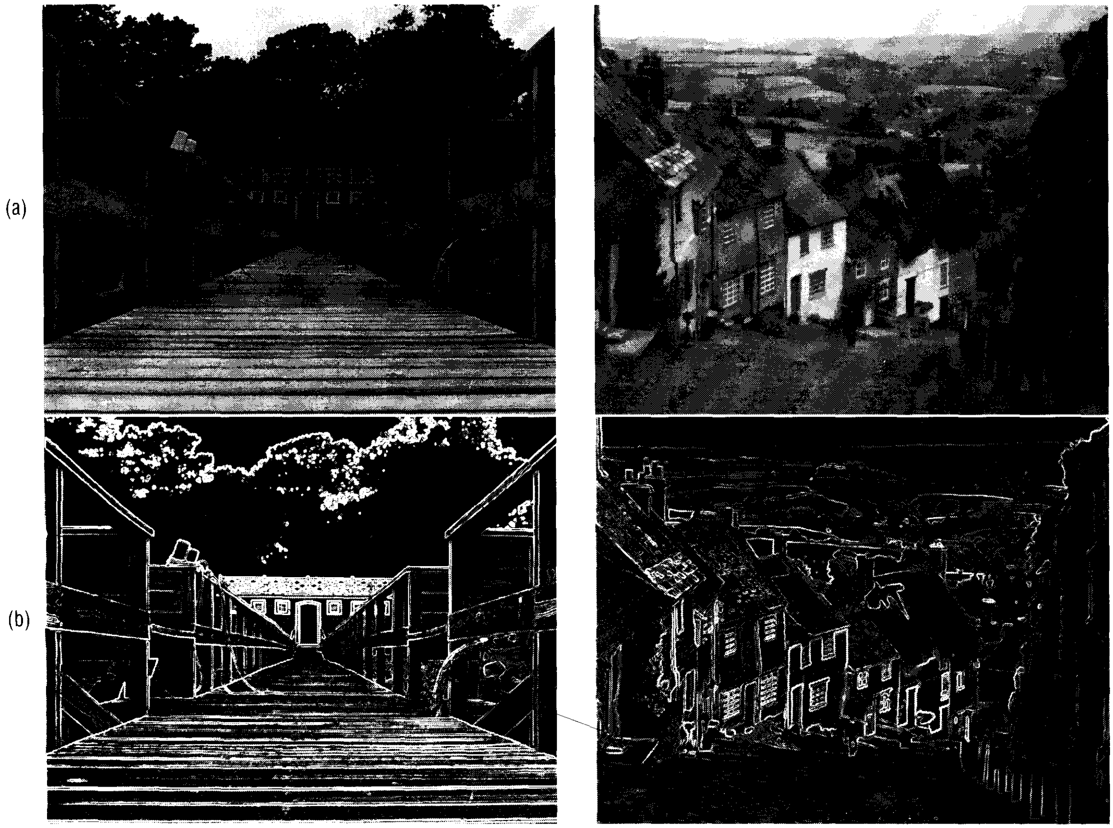


Figure 3.24 Effect of a differentiating filter on an image: (a) two original images; (b) the result of processing the original images with a differentiating filter.

In this case $h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$, and from eq. (3.122), we see that the frequency response of the system is

$$H(e^{j\omega}) = \frac{1}{2}[1 + e^{-j\omega}] = e^{-j\omega/2}\cos(\omega/2). \tag{3.139}$$

The magnitude of $H(e^{j\omega})$ is plotted in Figure 3.25(a), and $\sphericalangle H(e^{j\omega})$ is shown in Figure 3.25(b). As discussed in Section 1.3.3, low frequencies for discrete-time complex exponentials occur near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$, and high frequencies near $\omega = \pm\pi, \pm 3\pi, \ldots$. This is a result of the fact that $e^{j(\omega+2\pi)n} = e^{j\omega n}$, so that in discrete time we need only consider a 2π interval of values of ω in order to cover a complete range of distinct discrete-time frequencies. As a consequence, any discrete-time frequency responses $H(e^{j\omega})$ must be periodic with period 2π, a fact that can also be deduced directly from eq. (3.122).

For the specific filter defined in eqs. (3.138) and (3.139), we see from Figure 3.25(a) that $|H(e^{j\omega})|$ is large for frequencies near $\omega = 0$ and decreases as we increase $|\omega|$ toward π, indicating that higher frequencies are attentuated more than lower ones. For example, if the input to this system is constant—i.e., a zero-frequency complex exponential

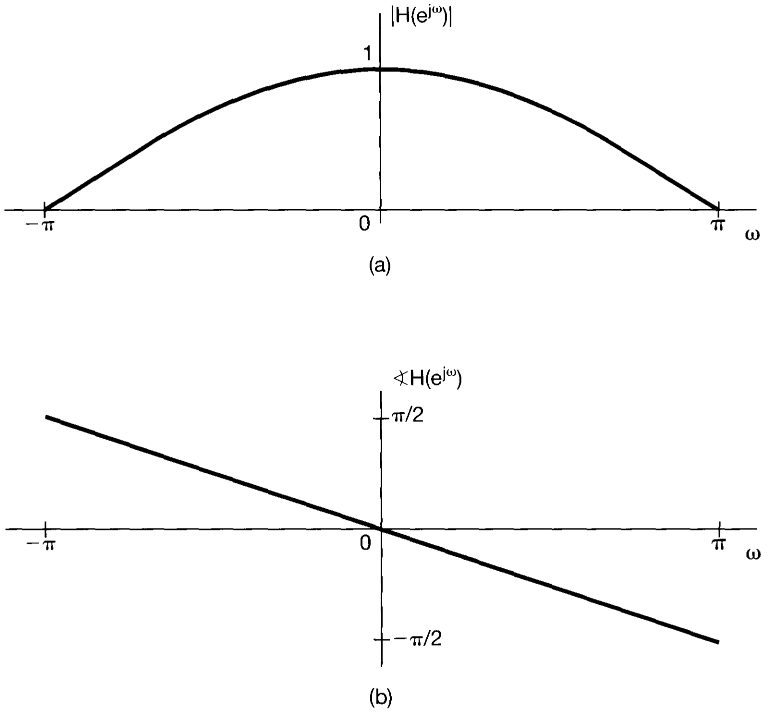


Figure 3.25 (a) Magnitude and (b) phase for the frequency response of the discrete-time LTI system $y[n] = 1/2(x[n] + x[n-1])$.

$x[n] = Ke^{j0 \cdot n} = K$—then the output will be

$$y[n] = H(e^{j \cdot 0})Ke^{j\omega 0 \cdot n} = K = x[n].$$

On the other hand, if the input is the high-frequency signal $x[n] = Ke^{j\pi n} = K(-1)^n$, then the output will be

$$y[n] = H(e^{j\pi})Ke^{j\pi \cdot n} = 0.$$

Thus, this system separates out the long-term constant value of a signal from its high-frequency fluctuations and, consequently, represents a first example of frequency-selective filtering, a topic we look at more carefully in the next subsection.

3.9.2 Frequency-Selective Filters

Frequency-selective filters are a class of filters specifically intended to accurately or approximately select some bands of frequencies and reject others. The use of frequency-selective filters arises in a variety of situations. For example, if noise in an audio recording is in a higher frequency band than the music or voice on the recording is, it can be removed by frequency-selective filtering. Another important application of frequency-selective filters is in communication systems. As we discuss in detail in Chapter 8, the basis for amplitude modulation (AM) systems is the transmission of information from many different sources simultaneously by putting the information from each channel into a separate frequency band and extracting the individual channels or bands at the receiver using frequency-selective filters. Frequency-selective filters for separating the individual channels and frequency-shaping filters (such as the equalizer illustrated in Figure 3.22) for adjusting the quality of the tone form a major part of any home radio and television receiver.

While frequency selectivity is not the only issue of concern in applications, its broad importance has led to a widely accepted set of terms describing the characteristics of frequency-selective filters. In particular, while the nature of the frequencies to be passed by a frequency-selective filter varies considerably from application to application, several basic types of filter are widely used and have been given names indicative of their function. For example, a *lowpass filter* is a filter that passes low frequencies—i.e., frequencies around $\omega = 0$—and attenuates or rejects higher frequencies. A *highpass filter* is a filter that passes high frequencies and attentuates or rejects low ones, and a *bandpass filter* is a filter that passes a band of frequencies and attenuates frequencies both higher and lower than those in the band that is passed. In each case, the *cutoff frequencies* are the frequencies defining the boundaries between frequencies that are passed and frequencies that are rejected—i.e., the frequencies in the *passband* and *stopband.*

Numerous questions arise in defining and assessing the quality of a frequency-selective filter. How effective is the filter at passing frequencies in the passband? How effective is it at attentuating frequencies in the stopband? How sharp is the transition near the cutoff frequency—i.e., from nearly free of distortion in the passband to highly attenuated in the stopband? Each of these questions involves a comparison of the characteristics of an actual frequency-selective filter with those of a filter with idealized behavior. Specifically, an *ideal frequency-selective filter* is a filter that exactly passes complex exponentials at one set of frequencies without any distortion and completely rejects signals at all other frequencies. For example, a continuous-time *ideal lowpass filter* with cutoff frequency ω_c is an LTI system that passes complex exponentials $e^{j\omega t}$ for values of ω in the range $-\omega_c \leq \omega \leq \omega_c$ and rejects signals at all other frequencies. That is, the frequency response of a continuous-time ideal lowpass filter is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \tag{3.140}$$

as shown in Figure 3.26.

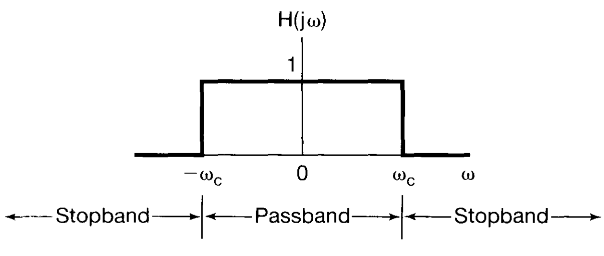


Figure 3.26 Frequency response of an ideal lowpass filter.

Figure 3.27(a) depicts the frequency response of an ideal continuous-time highpass filter with cutoff frequency ω_c, and Figure 3.27(b) illustrates an ideal continuous-time bandpass filter with lower cutoff frequency ω_{c1} and upper cutoff frequency ω_{c2}. Note that each of these filters is symmetric about $\omega = 0$, and thus, there appear to be two passbands for the highpass and bandpass filters. This is a consequence of our having adopted the

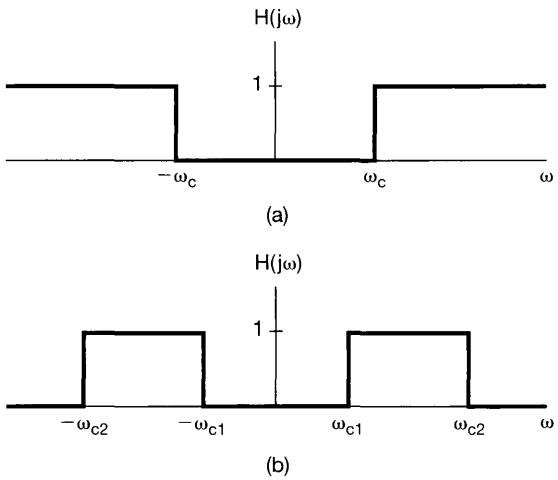


Figure 3.27 (a) Frequency response of an ideal highpass filter; (b) frequency response of an ideal bandpass filter.

use of the complex exponential signal $e^{j\omega t}$, rather than the sinusoidal signals $\sin \omega t$ and $\cos \omega t$, at frequency ω. Since $e^{j\omega t} = \cos \omega t + j \sin \omega t$ and $e^{-j\omega t} = \cos \omega t - j \sin \omega t$, both of these complex exponentials are composed of sinusoidal signals at the same frequency ω. For this reason, we usually define ideal filters so that they have the symmetric frequency response behavior seen in Figures 3.26 and 3.27.

In a similar fashion, we can define the corresponding set of ideal discrete-time frequency-selective filters, the frequency responses for which are depicted in Figure 3.28.

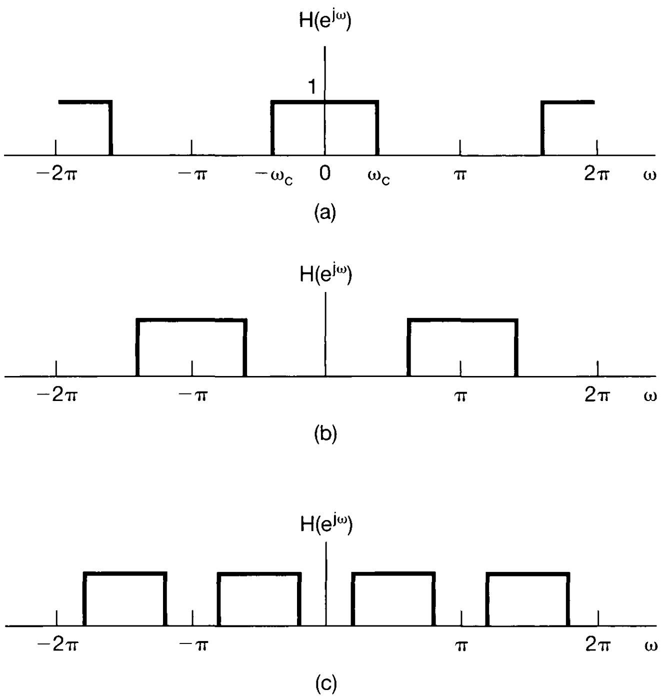


Figure 3.28 Discrete-time ideal frequency-selective filters: (a) lowpass; (b) highpass; (c) bandpass.

In particular, Figure 3.28(a) depicts an ideal discrete-time lowpass filter, Figure 3.28(b) is an ideal highpass filter, and Figure 3.28(c) is an ideal bandpass filter. Note that, as discussed in the preceding section, the characteristics of the continuous-time and discrete-time ideal filters differ by virtue of the fact that, for discrete-time filters, the frequency response $H(e^{j\omega})$ must be periodic with period 2π, with low frequencies near even multiples of π and high frequencies near odd multiples of π.

As we will see on numerous occasions, ideal filters are quite useful in describing idealized system configurations for a variety of applications. However, they are not realizable in practice and must be approximated. Furthermore, even if they could be realized, some of the characteristics of ideal filters might make them undesirable for particular applications, and a nonideal filter might in fact be preferable.

In detail, the topic of filtering encompasses many issues, including design and implementation. While we will not delve deeply into the details of filter design methodologies, in the remainder of this chapter and the following chapters we will see a number of other examples of both continuous-time and discrete-time filters and will develop the concepts and techniques that form the basis of this very important engineering discipline.

3.10 EXAMPLES OF CONTINUOUS-TIME FILTERS DESCRIBED BY DIFFERENTIAL EQUATIONS

In many applications, frequency-selective filtering is accomplished through the use of LTI systems described by linear constant-coefficient differential or difference equations. The reasons for this are numerous. For example, many physical systems that can be interpreted as performing filtering operations are characterized by differential or difference equations. A good example of this that we will examine in Chapter 6 is an automobile suspension system, which in part is designed to filter out high-frequency bumps and irregularities in road surfaces. A second reason for the use of filters described by differential or difference equations is that they are conveniently implemented using either analog or digital hardware. Furthermore, systems described by differential or difference equations offer an extremely broad and flexible range of designs, allowing one, for example, to produce filters that are close to ideal or that possess other desirable characteristics. In this and the next section, we consider several examples that illustrate the implementation of continuous-time and discrete-time frequency-selective filters through the use of differential and difference equations. In Chapters 4–6, we will see other examples of these classes of filters and will gain additional insights into the properties that make them so useful.

3.10.1 A Simple RC Lowpass Filter

Electrical circuits are widely used to implement continuous-time filtering operations. One of the simplest examples of such a circuit is the first-order *RC* circuit depicted in Figure 3.29, where the source voltage $v_s(t)$ is the system input. This circuit can be used to perform either a lowpass or highpass filtering operation, depending upon what we take as the output signal. In particular, suppose that we take the capacitor voltage $v_c(t)$ as the output. In this case, the output voltage is related to the input voltage through the linear

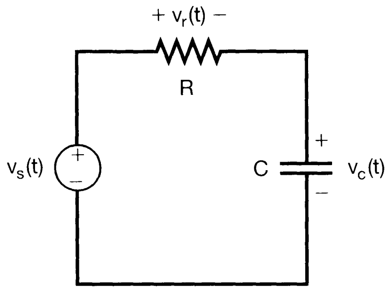


Figure 3.29 First-order *RC* filter.

constant-coefficient differential equation

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t). \tag{3.141}$$

Assuming initial rest, the system described by eq. (3.141) is LTI. In order to determine its frequency response $H(j\omega)$, we note that, by definition, with input voltage $v_s(t) = e^{j\omega t}$, we must have the output voltage $v_c(t) = H(j\omega)e^{j\omega t}$. If we substitute these expressions into eq. (3.141), we obtain

$$RC\frac{d}{dt}[H(j\omega)e^{j\omega t}] + H(j\omega)e^{j\omega t} = e^{j\omega t}, \tag{3.142}$$

or

$$RC\,j\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}, \tag{3.143}$$

from which it follows directly that

$$H(j\omega)e^{j\omega t} = \frac{1}{1 + RCj\omega}e^{j\omega t}, \tag{3.144}$$

or

$$H(j\omega) = \frac{1}{1 + RCj\omega}. \tag{3.145}$$

The magnitude and phase of the frequency response $H(j\omega)$ for this example are shown in Figure 3.30. Note that for frequencies near $\omega = 0$, $|H(j\omega)| \approx 1$, while for larger values of ω (positive or negative), $|H(j\omega)|$ is considerably smaller and in fact steadily decreases as $|\omega|$ increases. Thus, this simple *RC* filter (with $v_c(t)$ as output) is a nonideal lowpass filter.

To provide a first glimpse at the trade-offs involved in filter design, let us briefly consider the time-domain behavior of the circuit. In particular, the impulse response of the system described by eq. (3.141) is

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t), \tag{3.146}$$

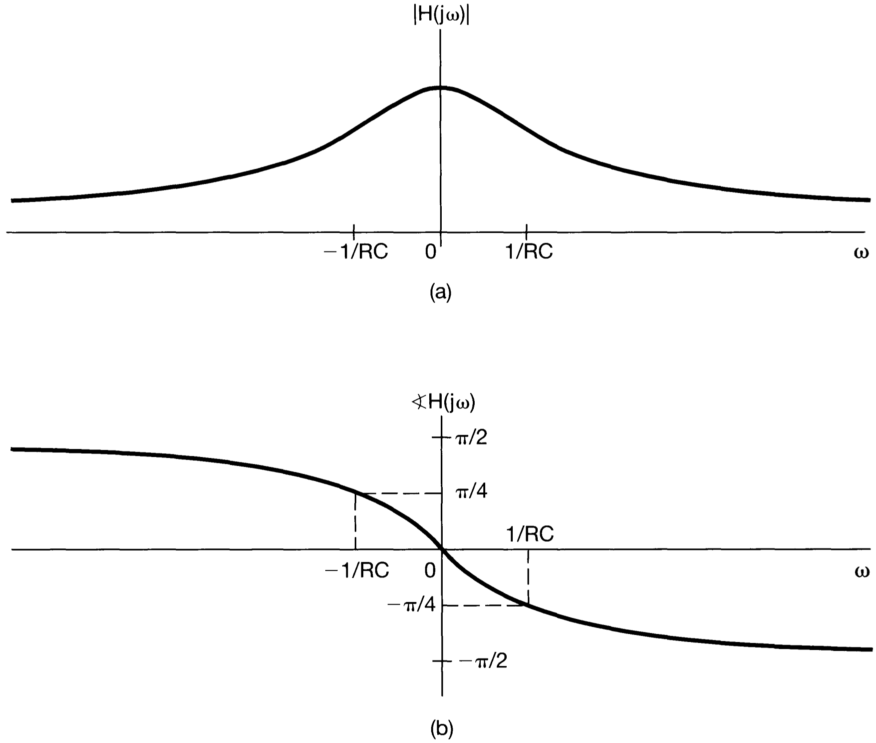


Figure 3.30 (a) Magnitude and (b) phase plots for the frequency response for the *RC* circuit of Figure 3.29 with output $v_c(t)$.

and the step response is

$$s(t) = [1 - e^{-t/RC}]u(t), \tag{3.147}$$

both of which are plotted in Figure 3.31 (where $\tau = RC$). Comparing Figures 3.30 and 3.31, we see a fundamental trade-off. Specifically, suppose that we would like our filter to pass only very low frequencies. From Figure 3.30(a), this implies that $1/RC$ must be small, or equivalently, that RC is large, so that frequencies other than the low ones of interest will be attentuated sufficiently. However, looking at Figure 3.31(b), we see that if RC is large, then the step response will take a considerable amount of time to reach its long-term value of 1. That is, the system responds sluggishly to the step input. Conversely, if we wish to have a faster step response, we need a smaller value of RC, which in turn implies that the filter will pass higher frequencies. This type of trade-off between behavior in the frequency domain and in the time domain is typical of the issues arising in the design and analysis of LTI systems and filters and is a subject we will look at more carefully in Chapter 6.

3.10.2 A Simple RC Highpass Filter

As an alternative to choosing the capacitor voltage as the output in our *RC* circuit, we can choose the voltage across the resistor. In this case, the differential equation relating input

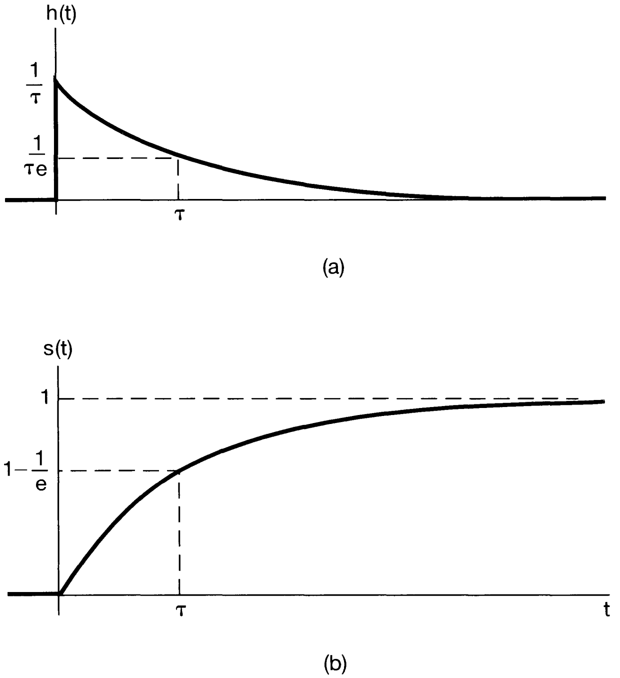


Figure 3.31 (a) Impulse response of the first-order *RC* lowpass filter with $\tau = RC$; (b) step response of *RC* lowpass filter with $\tau = RC$.

and output is

$$RC\frac{dv_r(t)}{dt} + v_r(t) = RC\frac{dv_s(t)}{dt}. \tag{3.148}$$

We can find the frequency response $G(j\omega)$ of this system in exactly the same way we did in the previous case: If $v_s(t) = e^{j\omega t}$, then we must have $v_r(t) = G(j\omega)e^{j\omega t}$; substituting these expressions into eq. (3.148) and performing a bit of algebra, we find that

$$G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}. \tag{3.149}$$

The magnitude and phase of this frequency response are shown in Figure 3.32. From the figure, we see that the system attenuates lower frequencies and passes higher frequencies—i.e., those for which $|\omega| \gg 1/RC$—with minimal attenuation. That is, this system acts as a nonideal highpass filter.

As with the lowpass filter, the parameters of the circuit control both the frequency response of the highpass filter and its time response characteristics. For example, consider the step response for the filter. From Figure 3.29, we see that $v_r(t) = v_s(t) - v_c(t)$. Thus, if $v_s(t) = u(t)$, $v_c(t)$ must be given by eq. (3.147). Consequently, the step response of the highpass filter is

$$v_r(t) = e^{-t/RC}u(t), \tag{3.150}$$

which is depicted in Figure 3.33. Consequently, as RC is increased, the response becomes more sluggish—i.e., the step response takes a longer time to reach its long-term value

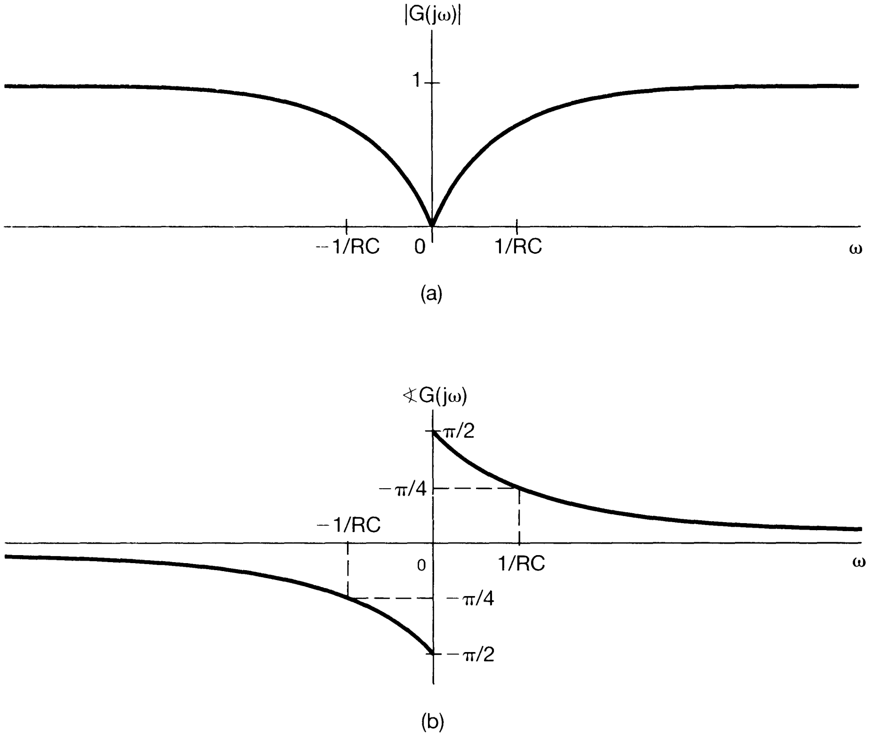


Figure 3.32 (a) Magnitude and (b) phase plots for the frequency response of the *RC* circuit of Figure 3.29 with output $v_r(t)$.

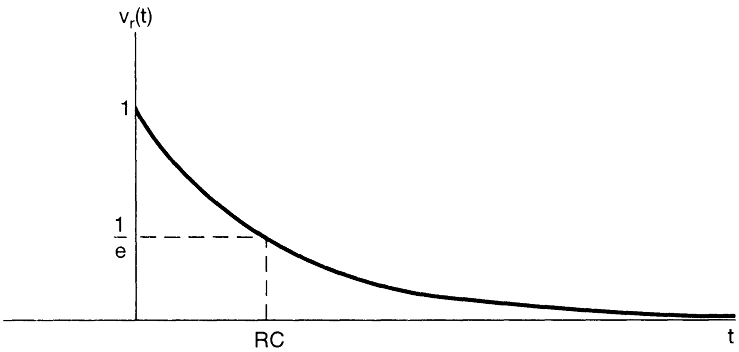


Figure 3.33 Step response of the first-order *RC* highpass filter with $\tau = RC$.

of 0. From Figure 3.32, we see that increasing RC (or equivalently, decreasing $1/RC$) also affects the frequency response, specifically, it extends the passband down to lower frequencies.

We observe from the two examples in this section that a simple RC circuit can serve as a rough approximation to a highpass or a lowpass filter, depending upon the choice of the physical output variable. As illustrated in Problem 3.71, a simple mechanical system using a mass and a mechanical damper can also serve as a lowpass or highpass filter described by analogous first-order differential equations. Because of their simplicity, these examples of electrical and mechanical filters do not have a sharp transition from passband to stopband and, in fact, have only a single parameter (namely, RC in the electrical case) that controls both the frequency response and time response behavior of the system. By designing more complex filters, implemented using more energy storage elements (capacitances and inductances in electrical filters and springs and damping devices in mechanical filters), we obtain filters described by higher order differential equations. Such filters offer considerably more flexibility in terms of their characteristics, allowing, for example, sharper passband-stopband transition or more control over the trade-offs between time response and frequency response.

3.11 EXAMPLES OF DISCRETE-TIME FILTERS DESCRIBED BY DIFFERENCE EQUATIONS

As with their continuous-time counterparts, discrete-time filters described by linear constant-coefficient difference equations are of considerable importance in practice. Indeed, since they can be efficiently implemented in special- or general-purpose digital systems, filters described by difference equations are widely used in practice. As in almost all aspects of signal and system analysis, when we examine discrete-time filters described by difference equations, we find both strong similarities and important differences with the continuous-time case. In particular, discrete-time LTI systems described by difference equations can either be recursive and have impulse responses of infinite length (IIR systems) or be nonrecursive and have finite-length impulse responses (FIR systems). The former are the direct counterparts of continuous-time systems described by differential equations illustrated in the previous section, while the latter are also of considerable practical importance in digital systems. These two classes have distinct sets of advantages and disadvantages in terms of ease of implementation and in terms of the order of filter or the complexity required to achieve particular design objectives. In this section we limit ourselves to a few simple examples of recursive and nonrecursive filters, while in Chapters 5 and 6 we develop additional tools and insights that allow us to analyze and understand the properties of these systems in more detail.

3.11.1 First-Order Recursive Discrete-Time Filters

The discrete-time counterpart of each of the first-order filters considered in Section 3.10 is the LTI system described by the first-order difference equation

$$y[n] - ay[n-1] = x[n]. \tag{3.151}$$

From the eigenfunction property of complex exponential signals, we know that if $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega})e^{j\omega n}$, where $H(e^{j\omega})$ is the frequency response of the system. Substituting into eq. (3.151), we obtain

$$H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}, \tag{3.152}$$

or

$$[1 - ae^{-j\omega}]H(e^{j\omega})e^{j\omega n} = e^{j\omega n}, \tag{3.153}$$

so that

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{3.154}$$

The magnitude and phase of $H(e^{j\omega})$ are shown in Figure 3.34(a) for $a = 0.6$ and in Figure 3.34(b) for $a = -0.6$. We observe that, for the positive value of a, the difference equation (3.151) behaves like a lowpass filter with minimal attenuation of low frequencies near $\omega = 0$ and increasing attenuation as we increase ω toward $\omega = \pi$. For the negative value of a, the system is a highpass filter, passing frequencies near $\omega = \pi$ and attenuating lower frequencies. In fact, for any positive value of $a < 1$, the system approximates a lowpass filter, and for any negative value of $a > -1$, the system approximates a highpass filter, where $|a|$ controls the size of the filter passband, with broader passbands as $|a|$ is decreased.

As with the continuous-time examples, we again have a trade-off between time domain and frequency domain characteristics. In particular, the impulse response of the system described by eq. (3.151) is

$$h[n] = a^n u[n]. \tag{3.155}$$

The step response $s[n] = u[n] * h[n]$ is

$$s[n] = \frac{1 - a^{n+1}}{1 - a} u[n]. \tag{3.156}$$

From these expressions, we see that $|a|$ also controls the speed with which the impulse and step responses approach their long-term values, with faster responses for smaller values of $|a|$, and hence, for filters with smaller passbands. Just as with differential equations, higher order recursive difference equations can be used to provide sharper filter characteristics and to provide more flexibility in balancing time-domain and frequency-domain constraints.

Finally, note from eq. (3.155) that the system described by eq. (3.151) is unstable if $|a| \geq 1$ and thus does not have a finite response to complex exponential inputs. As we stated previously, Fourier-based methods and frequency domain analysis focus on systems with finite responses to complex exponentials; hence, for examples such as eq. (3.151), we restrict ourselves to stable systems.

3.11.2 Nonrecursive Discrete-Time Filters

The general form of an FIR nonrecursive difference equation is

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]. \tag{3.157}$$

That is, the output $y[n]$ is a *weighted average* of the $(N + M + 1)$ values of $x[n]$ from $x[n - M]$ through $x[n + N]$, with the weights given by the coefficients b_k. Systems of this form can be used to meet a broad array of filtering objectives, including frequency-selective filtering.

One frequently used example of such a filter is a *moving-average filter*, where the output $y[n]$ for any n—say, n_0—is an average of values of $x[n]$ in the vicinity of n_0. The

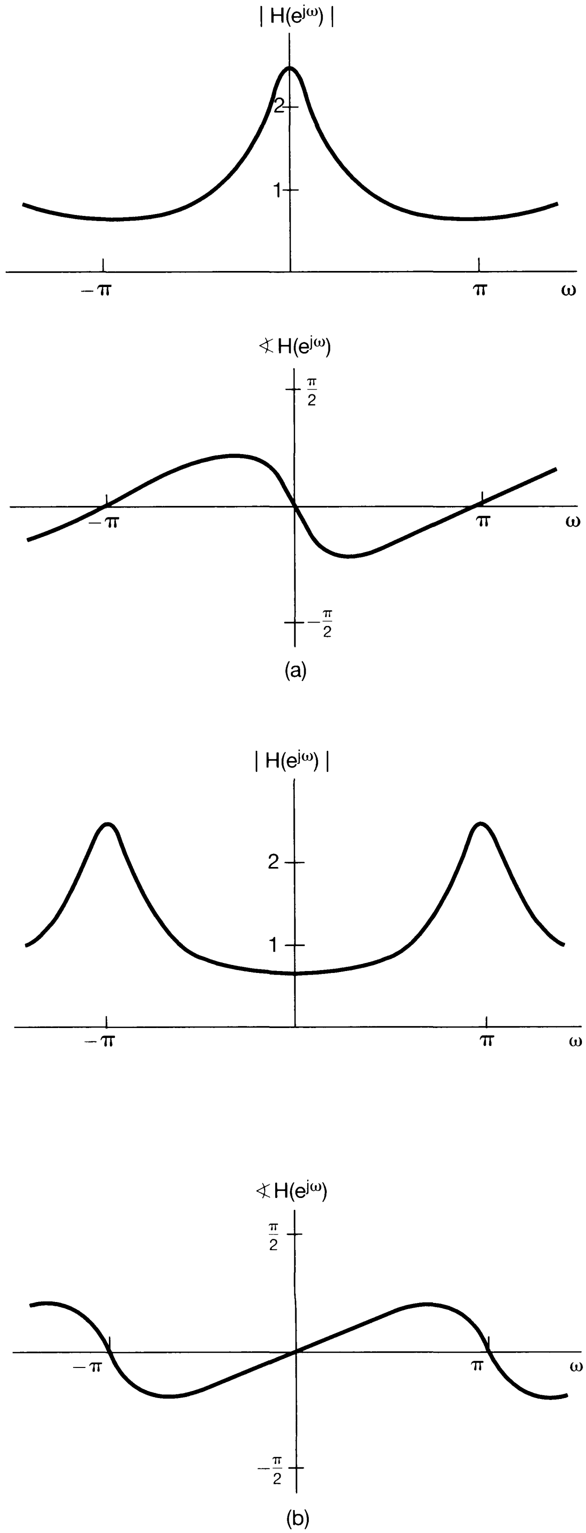


Figure 3.34 Frequency response of the first-order recursive discrete-time filter of eq. (3.151): (a) $a = 0.6$; (b) $a = -0.6$.

basic idea is that by averaging values locally, rapid high-frequency components of the input will be averaged out and lower frequency variations will be retained, corresponding to smoothing or lowpass filtering the original sequence. A simple two-point moving-average filter was briefly introduced in Section 3.9 [eq. (3.138)]. An only slightly more complex example is the three-point moving-average filter, which is of the form

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1]), \tag{3.158}$$

so that each output $y[n]$ is the average of three consecutive input values. In this case,

$$h[n] = \frac{1}{3}[\delta[n+1] + \delta[n] + \delta[n-1]],$$

and thus, from eq. (3.122), the corresponding frequency response is

$$H(e^{j\omega}) = \frac{1}{3}[e^{j\omega} + 1 + e^{-j\omega}] = \frac{1}{3}(1 + 2\cos\omega). \tag{3.159}$$

The magnitude of $H(e^{j\omega})$ is sketched in Figure 3.35. We observe that the filter has the general characteristics of a lowpass filter, although, as with the first-order recursive filter, it does not have a sharp transition from passband to stopband.

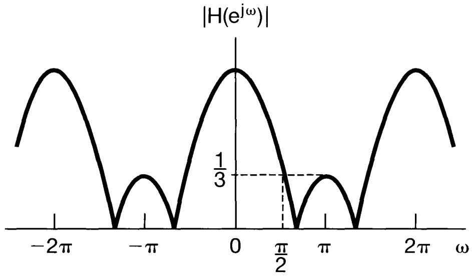


Figure 3.35 Magnitude of the frequency response of a three-point moving-average lowpass filter.

The three-point moving-average filter in eq. (3.158) has no parameters that can be changed to adjust the effective cutoff frequency. As a generalization of this moving-average filter, we can consider averaging over $N + M + 1$ neighboring points—that is, using a difference equation of the form

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]. \tag{3.160}$$

The corresponding impulse response is a rectangular pulse (i.e., $h[n] = 1/(N+M+1)$ for $-N \leq n \leq M$, and $h[n] = 0$ otherwise). The filter's frequency response is

$$H(e^{j\omega}) = \frac{1}{N+M+1}\sum_{k=-N}^{M} e^{-j\omega k}. \tag{3.161}$$

The summation in eq. (3.161) can be evaluated by performing calculations similar to those in Example 3.12, yielding

$$H(e^{j\omega}) = \frac{1}{N+M+1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}. \tag{3.162}$$

By adjusting the size, $N + M + 1$, of the averaging window we can vary the cutoff frequency. For example, the magnitude of $H(e^{j\omega})$ is shown in Figure 3.36 for $M+N+1 = 33$ and $M + N + 1 = 65$.

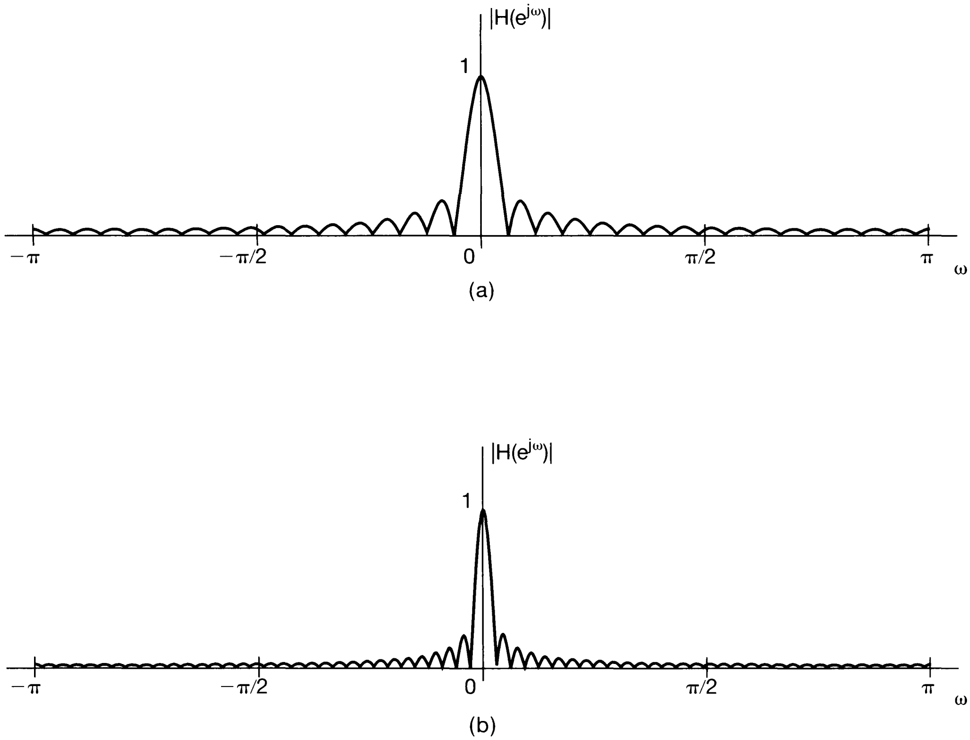


Figure 3.36 Magnitude of the frequency response for the lowpass moving-average filter of eq. (3.162): (a) $M = N = 16$; (b) $M = N = 32$.

Nonrecursive filters can also be used to perform highpass filtering operations. To illustrate this, again with a simple example, consider the difference equation

$$y[n] = \frac{x[n] - x[n-1]}{2}. \tag{3.163}$$

For input signals that are approximately constant, the value of $y[n]$ is close to zero. For input signals that vary greatly from sample to sample, the values of $y[n]$ can be expected to have larger amplitude. Thus, the system described by eq. (3.163) approximates a highpass filtering operation, attenuating slowly varying low-frequency components and passing rapidly varying higher frequency components with little attenuation. To see this more precisely we need to look at the system's frequency response. In this case, $h[n] = \frac{1}{2}\{\delta[n] - \delta[n-1]\}$, so that direct application of eq. (3.122) yields

$$H(e^{j\omega}) = \frac{1}{2}[1 - e^{-j\omega}] = je^{j\omega/2}\sin(\omega/2). \tag{3.164}$$

In Figure 3.37 we have plotted the magnitude of $H(e^{j\omega})$, showing that this simple system approximates a highpass filter, albeit one with a very gradual transition from passband to stopband. By considering more general nonrecursive filters, we can achieve far sharper transitions in lowpass, highpass, and other frequency-selective filters.

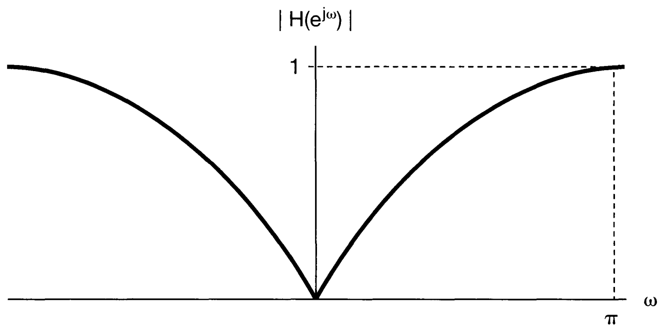


Figure 3.37 Frequency response of a simple highpass filter.

Note that, since the impulse response of any FIR system is of finite length (i.e., from eq. (3.157), $h[n] = b_n$ for $-N \le n \le M$ and 0 otherwise), it is always absolutely summable for any choices of the b_n. Hence, all such filters are stable. Also, if $N > 0$ in eq. (3.157), the system is noncausal, since $y[n]$ then depends on future values of the input. In some applications, such as those involving the processing of previously recorded signals, causality is not a necessary constraint, and thus, we are free to use filters with $N > 0$. In others, such as many involving real-time processing, causality is essential, and in such cases we must take $N \le 0$.

3.12 SUMMARY

In this chapter, we have introduced and developed Fourier series representations for both continuous-time and discrete-time systems and have used these representations to take a first look at one of the very important applications of the methods of signal and system analysis, namely, filtering. In particular, as we discussed in Section 3.2, one of the primary motivations for the use of Fourier series is the fact that complex exponential signals are eigenfunctions of LTI systems. We have also seen, in Sections 3.3–3.7, that any periodic signal of practical interest can be represented in a Fourier series—i.e., as a weighted sum of harmonically related complex exponentials that share a common period with the signal being represented. In addition, we have seen that the Fourier series representation has a number of important properties which describe how different characteristics of signals are reflected in their Fourier series coefficients.

One of the most important properties of Fourier series is a direct consequence of the eigenfunction property of complex exponentials. Specifically, if a periodic signal is applied to an LTI system, then the output will be periodic with the same period, and each of the Fourier coefficients of the output is the corresponding Fourier coefficient of the input multiplied by a complex number whose value is a function of the frequency corresponding to that Fourier coefficient. This function of frequency is characteristic of the LTI system and is referred to as the frequency response of the system. By examining the frequency response, we were led directly to the idea of filtering of signals using LTI systems, a concept that has numerous applications, including several that we have described. One important class of applications involves the notion of frequency-selective filtering—i.e., the idea of using an LTI system to pass certain specified bands of frequencies and stop or significantly attentuate others. We introduced the concept of ideal frequency-selective filters and also gave several examples of frequency-selective filters described by linear constant-coefficient differential or difference equations.

The purpose of this chapter has been to begin the process of developing both the tools of Fourier analysis and an appreciation for the utility of these tools in applications. In the chapters that follow, we continue with this agenda by developing the Fourier transform representations for aperiodic signals in continuous and discrete time and by taking a deeper look not only at filtering, but also at other important applications of Fourier methods.

Chapter 3 Problems

The first section of problems belongs to the basic category and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

3.1. A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T = 8$. The nonzero Fourier series coefficients for $x(t)$ are

$$a_1 = a_{-1} = 2, a_3 = a_{-3}^* = 4j.$$

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k).$$

3.2. A discrete-time periodic signal $x[n]$ is real valued and has a fundamental period $N = 5$. The nonzero Fourier series coefficients for $x[n]$ are

$$a_0 = 1, a_2 = a_{-2}^* = e^{j\pi/4}, a_4 = a_{-4}^* = 2e^{j\pi/3}.$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(\omega_k n + \phi_k).$$

3.3. For the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right),$$

determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

3.4. Use the Fourier series analysis equation (3.39) to calculate the coefficients a_k for the continuous-time periodic signal

$$x(t) = \begin{cases} 1.5, & 0 \le t < 1 \\ -1.5, & 1 \le t < 2 \end{cases}$$

with fundamental frequency $\omega_0 = \pi$.

3.5. Let $x_1(t)$ be a continuous-time periodic signal with fundamental frequency ω_1 and Fourier coefficients a_k. Given that

$$x_2(t) = x_1(1-t) + x_1(t-1),$$

how is the fundamental frequency ω_2 of $x_2(t)$ related to ω_1? Also, find a relationship between the Fourier series coefficients b_k of $x_2(t)$ and the coefficients a_k. You may use the properties listed in Table 3.1.

3.6. Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\frac{2\pi}{50}t},$$

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk\frac{2\pi}{50}t},$$

$$x_3(t) = \sum_{k=-100}^{100} j\sin\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi}{50}t}.$$

Use Fourier series properties to help answer the following questions:

(a) Which of the three signals is/are real valued?

(b) Which of the three signals is/are even?

3.7. Suppose the periodic signal $x(t)$ has fundamental period T and Fourier coefficients a_k. In a variety of situations, it is easier to calculate the Fourier series coefficients

b_k for $g(t) = dx(t)/dt$, as opposed to calculating a_k directly. Given that

$$\int_T^{2T} x(t)\,dt = 2,$$

find an expression for a_k in terms of b_k and T. You may use any of the properties listed in Table 3.1 to help find the expression.

3.8. Suppose we are given the following information about a signal $x(t)$:

1. $x(t)$ is real and odd.
2. $x(t)$ is periodic with period $T = 2$ and has Fourier coefficients a_k.
3. $a_k = 0$ for $|k| > 1$.
4. $\frac{1}{2}\int_0^2 |x(t)|^2\,dt = 1$.

Specify two different signals that satisfy these conditions.

3.9. Use the analysis equation (3.95) to evaluate the numerical values of one period of the Fourier series coefficients of the periodic signal

$$x[n] = \sum_{m=-\infty}^{\infty} \{4\delta[n-4m] + 8\delta[n-1-4m]\}.$$

3.10. Let $x[n]$ be a real and odd periodic signal with period $N = 7$ and Fourier coefficients a_k. Given that

$$a_{15} = j,\, a_{16} = 2j,\, a_{17} = 3j,$$

determine the values of a_0, a_{-1}, a_{-2}, and a_{-3}.

3.11. Suppose we are given the following information about a signal $x[n]$:

1. $x[n]$ is a real and even signal.
2. $x[n]$ has period $N = 10$ and Fourier coefficients a_k.
3. $a_{11} = 5$.
4. $\frac{1}{10}\sum_{n=0}^{9} |x[n]|^2 = 50$.

Show that $x[n] = A\cos(Bn + C)$, and specify numerical values for the constants A, B, and C.

3.12. Each of the two sequences $x_1[n]$ and $x_2[n]$ has a period $N = 4$, and the corresponding Fourier series coefficients are specified as

$$x_1[n] \longleftrightarrow a_k, \quad x_2[n] \longleftrightarrow b_k,$$

where

$$a_0 = a_3 = \frac{1}{2}a_1 = \frac{1}{2}a_2 = 1 \quad \text{and} \quad b_0 = b_1 = b_2 = b_3 = 1.$$

Using the multiplication property in Table 3.1, determine the Fourier series coefficients c_k for the signal $g[n] = x_1[n]x_2[n]$.

3.13. Consider a continuous-time LTI system whose frequency response is

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = \frac{\sin(4\omega)}{\omega}.$$

If the input to this system is a periodic signal

$$x(t) = \begin{cases} 1, & 0 \le t < 4 \\ -1, & 4 \le t < 8 \end{cases}$$

with period $T = 8$, determine the corresponding system output $y(t)$.

3.14. When the impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]$$

is the input to a particular LTI system with frequency response $H(e^{j\omega})$, the output of the system is found to be

$$y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right).$$

Determine the values of $H(e^{jk\pi/2})$ for $k = 0, 1, 2,$ and 3.

3.15. Consider a continuous-time ideal lowpass filter S whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \le 100 \\ 0, & |\omega| > 100 \end{cases}.$$

When the input to this filter is a signal $x(t)$ with fundamental period $T = \pi/6$ and Fourier series coefficients a_k, it is found that

$$x(t) \xrightarrow{S} y(t) = x(t).$$

For what values of k is it guaranteed that $a_k = 0$?

3.16. Determine the output of the filter shown in Figure P3.16 for the following periodic inputs:

(a) $x_1[n] = (-1)^n$

(b) $x_2[n] = 1 + \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right)$

(c) $x_3[n] = \sum_{k=-\infty}^{\infty}\left(\frac{1}{2}\right)^{n-4k} u[n - 4k]$

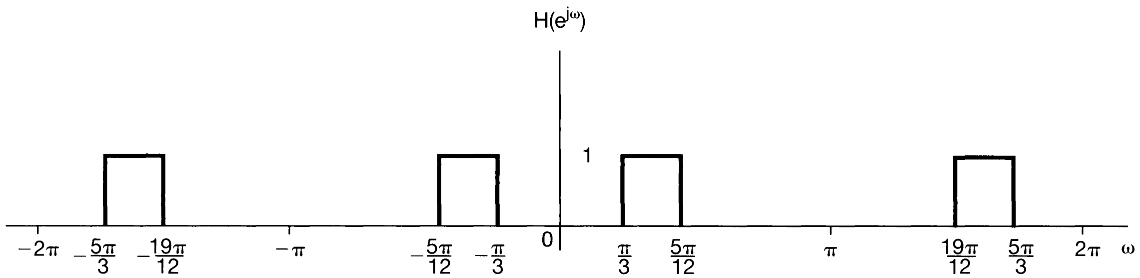


Figure P3.16

3.17. Consider three continuous-time systems S_1, S_2, and S_3 whose responses to a complex exponential input e^{j5t} are specified as

$$S_1 : e^{j5t} \longrightarrow te^{j5t},$$
$$S_2 : e^{j5t} \longrightarrow e^{j5(t-1)},$$
$$S_3 : e^{j5t} \longrightarrow \cos(5t).$$

For each system, determine whether the given information is sufficient to conclude that the system is definitely *not* LTI.

3.18. Consider three discrete-time systems S_1, S_2, and S_3 whose respective responses to a complex exponential input $e^{j\pi n/2}$ are specified as

$$S_1 : e^{j\pi n/2} \longrightarrow e^{j\pi n/2}u[n],$$
$$S_2 : e^{j\pi n/2} \longrightarrow e^{j3\pi n/2},$$
$$S_3 : e^{j\pi n/2} \longrightarrow 2e^{j5\pi n/2}.$$

For each system, determine whether the given information is sufficient to conclude that the system is definitely *not* LTI.

3.19. Consider a causal LTI system implemented as the *RL* circuit shown in Figure P3.19. A current source produces an input current $x(t)$, and the system output is considered to be the current $y(t)$ flowing through the inductor.

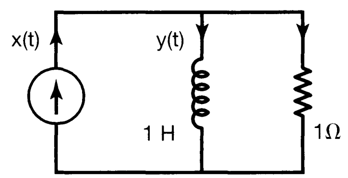


Figure P3.19

(a) Find the differential equation relating $x(t)$ and $y(t)$.
(b) Determine the frequency response of this system by considering the output of the system to inputs of the form $x(t) = e^{j\omega t}$.
(c) Determine the output $y(t)$ if $x(t) = \cos(t)$.

3.20. Consider a causal LTI system implemented as the *RLC* circuit shown in Figure P3.20. In this circuit, $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered the system output.

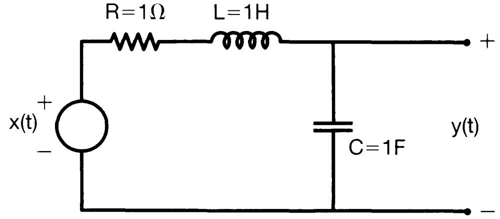


Figure P3.20

(a) Find the differential equation relating $x(t)$ and $y(t)$.
(b) Determine the frequency response of this system by considering the output of the system to inputs of the form $x(t) = e^{j\omega t}$.
(c) Determine the output $y(t)$ if $x(t) = \sin(t)$.

BASIC PROBLEMS

3.21. A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T = 8$. The nonzero Fourier series coefficients for $x(t)$ are specified as

$$a_1 = a_{-1}^* = j, a_5 = a_{-5} = 2.$$

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k).$$

3.22. Determine the Fourier series representations for the following signals:
(a) Each $x(t)$ illustrated in Figure P3.22(a)–(f).
(b) $x(t)$ periodic with period 2 and

$$x(t) = e^{-t} \quad \text{for} \quad -1 < t < 1$$

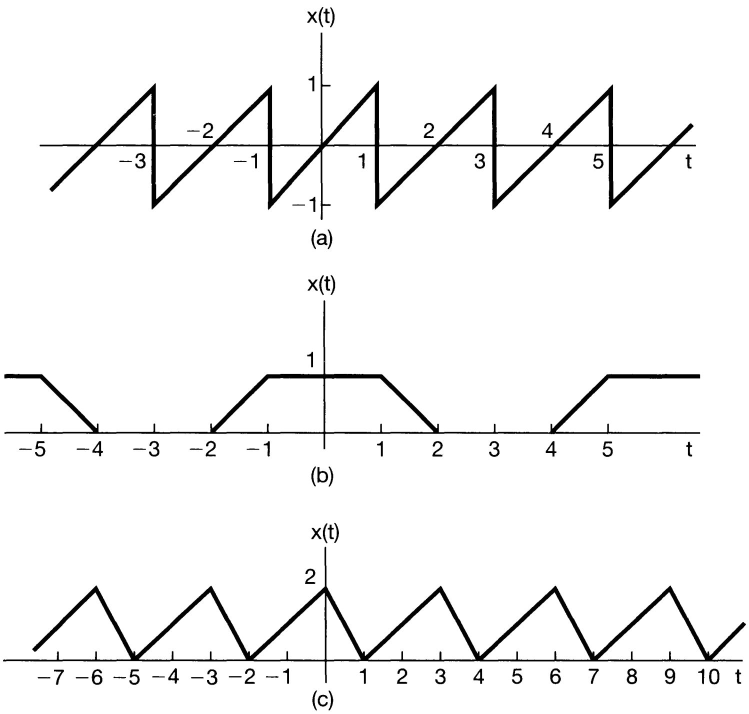


Figure P3.22

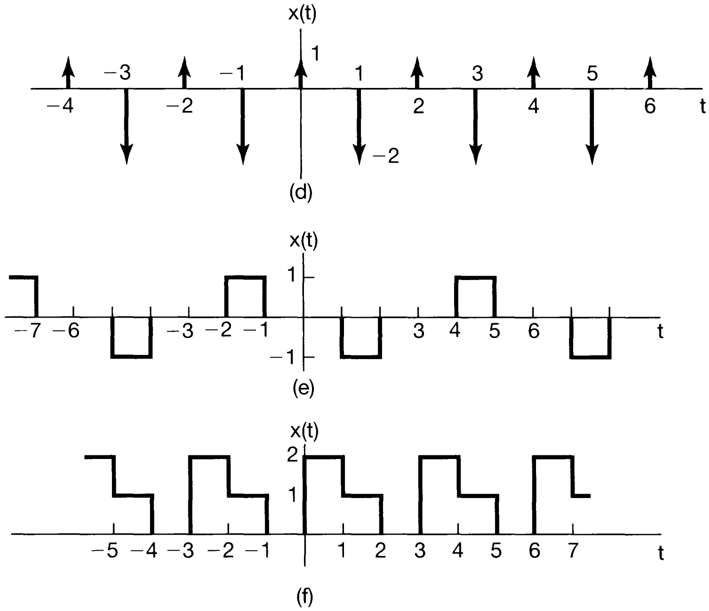


Figure P3.22 *Continued*

(c) $x(t)$ periodic with period 4 and

$$x(t) = \begin{cases} \sin \pi t, & 0 \leq t \leq 2 \\ 0, & 2 < t \leq 4 \end{cases}$$

3.23. In each of the following, we specify the Fourier series coefficients of a continuous-time signal that is periodic with period 4. Determine the signal $x(t)$ in each case.

(a) $a_k = \begin{cases} 0, & k = 0 \\ (j)^k \frac{\sin k\pi/4}{k\pi}, & \text{otherwise} \end{cases}$

(b) $a_k = (-1)^k \frac{\sin k\pi/8}{2k\pi}, \quad a_0 = \frac{1}{16}$

(c) $a_k = \begin{cases} jk, & |k| < 3 \\ 0, & \text{otherwise} \end{cases}$

(d) $a_k = \begin{cases} 1, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}$

3.24. Let

$$x(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2 - t, & 1 \leq t \leq 2 \end{cases}$$

be a periodic signal with fundamental period $T = 2$ and Fourier coefficients a_k.

(a) Determine the value of a_0.

(b) Determine the Fourier series representation of $dx(t)/dt$.

(c) Use the result of part (b) and the differentiation property of the continuous-time Fourier series to help determine the Fourier series coefficients of $x(t)$.

3.25. Consider the following three continuous-time signals with a fundamental period of $T = 1/2$:

$$x(t) = \cos(4\pi t),$$
$$y(t) = \sin(4\pi t),$$
$$z(t) = x(t)y(t).$$

(a) Determine the Fourier series coefficients of $x(t)$.
(b) Determine the Fourier series coefficients of $y(t)$.
(c) Use the results of parts (a) and (b), along with the multiplication property of the continuous-time Fourier series, to determine the Fourier series coefficients of $z(t) = x(t)y(t)$.
(d) Determine the Fourier series coefficients of $z(t)$ through direct expansion of $z(t)$ in trigonometric form, and compare your result with that of part (c).

3.26. Let $x(t)$ be a periodic signal whose Fourier series coefficients are

$$a_k = \begin{cases} 2, & k = 0 \\ j(\frac{1}{2})^{|k|}, & \text{otherwise} \end{cases}.$$

Use Fourier series properties to answer the following questions:
(a) Is $x(t)$ real?
(b) Is $x(t)$ even?
(c) Is $dx(t)/dt$ even?

3.27. A discrete-time periodic signal $x[n]$ is real valued and has a fundamental period $N = 5$. The nonzero Fourier series coefficients for $x[n]$ are

$$a_0 = 2, a_2 = a_{-2}^* = 2e^{j\pi/6}, \quad a_4 = a_{-4}^* = e^{j\frac{\pi}{3}}.$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(\omega_k n + \phi_k).$$

3.28. Determine the Fourier series coefficients for each of the following discrete-time periodic signals. Plot the magnitude and phase of each set of coefficients a_k.
(a) Each $x[n]$ depicted in Figure P3.28(a)–(c)
(b) $x[n] = \sin(2\pi n/3)\cos(\pi n/2)$
(c) $x[n]$ periodic with period 4 and

$$x[n] = 1 - \sin\frac{\pi n}{4} \quad \text{for } 0 \leq n \leq 3$$

(d) $x[n]$ periodic with period 12 and

$$x[n] = 1 - \sin\frac{\pi n}{4} \quad \text{for } 0 \leq n \leq 11$$

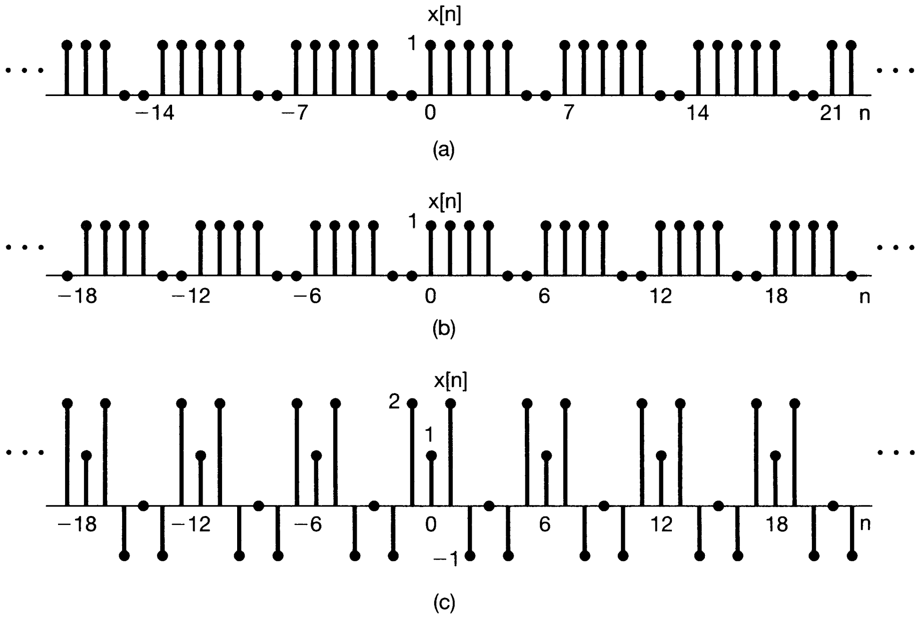


Figure P3.28

3.29. In each of the following, we specify the Fourier series coefficients of a signal that is periodic with period 8. Determine the signal $x[n]$ in each case.

(a) $a_k = \cos\left(\frac{k\pi}{4}\right) + \sin\left(\frac{3k\pi}{4}\right)$ **(b)** $a_k = \begin{cases} \sin\left(\frac{k\pi}{3}\right), & 0 \leq k \leq 6 \\ 0, & k = 7 \end{cases}$

(c) a_k as in Figure P3.29(a) **(d)** a_k as in Figure P3.29(b)

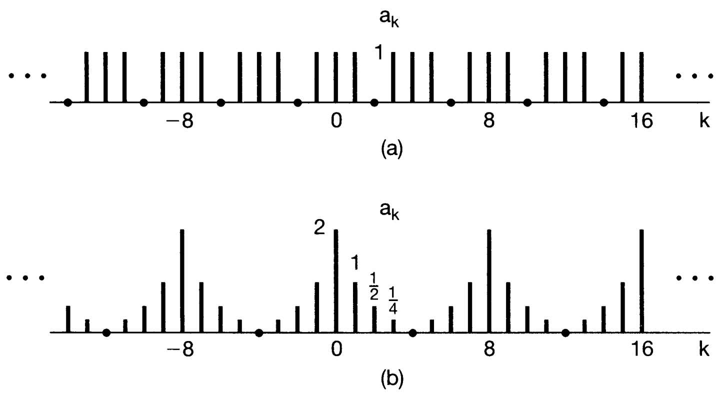


Figure P3.29

3.30. Consider the following three discrete-time signals with a fundamental period of 6:

$$x[n] = 1 + \cos\left(\frac{2\pi}{6}n\right), \qquad y[n] = \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right), \qquad z[n] = x[n]y[n].$$

(a) Determine the Fourier series coefficients of $x[n]$.
(b) Determine the Fourier series coefficients of $y[n]$.
(c) Use the results of parts (a) and (b), along with the multiplication property of the discrete-time Fourier series, to determine the Fourier series coefficients of $z[n] = x[n]y[n]$.
(d) Determine the Fourier series coefficients of $z[n]$ through direct evaluation, and compare your result with that of part (c).

3.31. Let

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 7 \\ 0, & 8 \leq n \leq 9 \end{cases}$$

be a periodic signal with fundamental period $N = 10$ and Fourier series coefficients a_k. Also, let

$$g[n] = x[n] - x[n-1].$$

(a) Show that $g[n]$ has a fundamental period of 10.
(b) Determine the Fourier series coefficients of $g[n]$.
(c) Using the Fourier series coefficients of $g[n]$ and the First-Difference property in Table 3.2, determine a_k for $k \neq 0$.

3.32. Consider the signal $x[n]$ depicted in Figure P3.32. This signal is periodic with period $N = 4$. The signal can be expressed in terms of a discrete-time Fourier series as

$$x[n] = \sum_{k=0}^{3} a_k e^{jk(2\pi/4)n}. \qquad \text{(P3.32–1)}$$

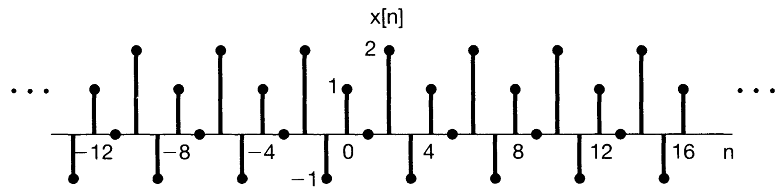


Figure P3.32

As mentioned in the text, one way to determine the Fourier series coefficients is to treat eq. (P3.32–1) as a set of four linear equations (for $n = 0, 1, 2, 3$) in four unknowns (a_0, a_1, a_2, and a_3).

(a) Write out these four equations explicitly, and solve them directly using any standard technique for solving four equations in four unknowns. (Be sure first to reduce the foregoing complex exponentials to the simplest form.)
(b) Check your answer by calculating the a_k directly, using the discrete-time Fourier series analysis equation

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/4)n}.$$

3.33. Consider a causal continuous-time LTI system whose input $x(t)$ and output $y(t)$ are related by the following differential equation:

$$\frac{d}{dt}y(t) + 4y(t) = x(t).$$

Find the Fourier series representation of the output $y(t)$ for each of the following inputs:

(a) $x(t) = \cos 2\pi t$

(b) $x(t) = \sin 4\pi t + \cos(6\pi t + \pi/4)$

3.34. Consider a continuous-time LTI system with impulse response

$$h(t) = e^{-4|t|}.$$

Find the Fourier series representation of the output $y(t)$ for each of the following inputs:

(a) $x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$

(b) $x(t) = \sum_{n=-\infty}^{+\infty} (-1)^n \delta(t-n)$

(c) $x(t)$ is the periodic wave depicted in Figure P3.34.

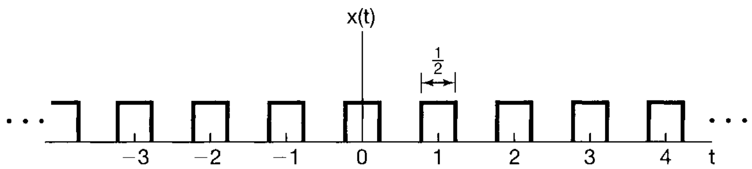


Figure P3.34

3.35. Consider a continuous-time LTI system S whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \geq 250 \\ 0, & \text{otherwise} \end{cases}.$$

When the input to this system is a signal $x(t)$ with fundamental period $T = \pi/7$ and Fourier series coefficients a_k, it is found that the output $y(t)$ is identical to $x(t)$. For what values of k is it guaranteed that $a_k = 0$?

3.36. Consider a causal discrete-time LTI system whose input $x[n]$ and output $y[n]$ are related by the following difference equation:

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

Find the Fourier series representation of the output $y[n]$ for each of the following inputs:

(a) $x[n] = \sin(\frac{3\pi}{4}n)$

(b) $x[n] = \cos(\frac{\pi}{4}n) + 2\cos(\frac{\pi}{2}n)$

3.37. Consider a discrete-time LTI system with impulse response

$$h[n] = \left(\frac{1}{2}\right)^{|n|}.$$

Find the Fourier series representation of the output $y[n]$ for each of the following inputs:

(a) $x[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]$

(b) $x[n]$ is periodic with period 6 and

$$x[n] = \begin{cases} 1, & n = 0, \pm 1 \\ 0, & n = \pm 2, \pm 3 \end{cases}$$

3.38. Consider a discrete-time LTI system with impulse response

$$h[n] = \begin{cases} 1, & 0 \leq n \leq 2 \\ -1, & -2 \leq n \leq -1 \\ 0, & \text{otherwise} \end{cases}.$$

Given that the input to this system is

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - 4k],$$

determine the Fourier series coefficients of the output $y[n]$.

3.39. Consider a discrete-time LTI system S whose frequency response is

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{8} \\ 0, & \frac{\pi}{8} < |\omega| < \pi \end{cases}.$$

Show that if the input $x[n]$ to this system has a period $N = 3$, the output $y[n]$ has only one nonzero Fourier series coefficient per period.

ADVANCED PROBLEMS

3.40. Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficients a_k. Derive the Fourier series coefficients of each of the following signals in terms of a_k:

(a) $x(t - t_0) + x(t + t_0)$

(b) $\mathcal{E}v\{x(t)\}$

(c) $\mathcal{R}e\{x(t)\}$

(d) $\frac{d^2x(t)}{dt^2}$

(e) $x(3t - 1)$ [for this part, first determine the period of $x(3t - 1)$]

3.41. Suppose we are given the following information about a continuous-time periodic signal with period 3 and Fourier coefficients a_k:

1. $a_k = a_{k+2}$.
2. $a_k = a_{-k}$.
3. $\int_{-0.5}^{0.5} x(t)dt = 1$.
4. $\int_1^2 x(t)dt = 2$.

Determine $x(t)$.

3.42. Let $x(t)$ be a real-valued signal with fundamental period T and Fourier series coefficients a_k.

(a) Show that $a_k = a^*_{-k}$ and a_0 must be real.

(b) Show that if $x(t)$ is even, then its Fourier series coefficients must be real and even.

(c) Show that if $x(t)$ is odd, then its Fourier series coefficients are imaginary and odd and $a_0 = 0$.

(d) Show that the Fourier coefficients of the even part of $x(t)$ are equal to $\mathcal{R}e\{a_k\}$.

(e) Show that the Fourier coefficients of the odd part of $x(t)$ are equal to $j\mathcal{I}m\{a_k\}$.

3.43. **(a)** A continuous-time periodic signal $x(t)$ with period T is said to be *odd harmonic* if, in its Fourier series representation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \qquad \text{(P3.43–1)}$$

$a_k = 0$ for every non-zero even integer k.

(i) Show that if $x(t)$ is odd harmonic, then

$$x(t) = -x\left(t + \frac{T}{2}\right). \qquad \text{(P3.43–2)}$$

(ii) Show that if $x(t)$ satisfies eq. (P3.43–2), then it is odd harmonic.

(b) Suppose that $x(t)$ is an odd-harmonic periodic signal with period 2 such that

$$x(t) = t \quad \text{for } 0 < t < 1.$$

Sketch $x(t)$ and find its Fourier series coefficients.

(c) Analogously, to an odd-harmonic signal, we could define an even-harmonic signal as a signal for which $a_k = 0$ for k odd in the representation in eq. (P3.43–1). Could T be the fundamental period for such a signal? Explain your answer.

(d) More generally, show that T is the fundamental period of $x(t)$ in eq. (P3.43–1) if one of two things happens:

(1) Either a_1 or a_{-1} is nonzero;

or

(2) There are two integers k and l that have no common factors and are such that both a_k and a_l are nonzero.

3.44. Suppose we are given the following information about a signal $x(t)$:

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 6$ and has Fourier coefficients a_k.
3. $a_k = 0$ for $k = 0$ and $k > 2$.
4. $x(t) = -x(t-3)$.
5. $\frac{1}{6}\int_{-3}^{3}|x(t)|^2\,dt = \frac{1}{2}$.
6. a_1 is a positive real number.

Show that $x(t) = A\cos(Bt + C)$, and determine the values of the constants A, B, and C.

3.45. Let $x(t)$ be a real periodic signal with Fourier series representation given in the sine-cosine form of eq. (3.32); i.e.,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty}[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]. \qquad \text{(P3.45–1)}$$

(a) Find the exponential Fourier series representation of the even and odd parts of $x(t)$; that is, find the coefficients α_k and β_k in terms of the coefficients in eq. (P3.45–1) so that

$$\mathcal{E}v\{x(t)\} = \sum_{k=-\infty}^{+\infty} \alpha_k e^{jk\omega_0 t},$$

$$\mathcal{O}d\{x(t)\} = \sum_{k=-\infty}^{+\infty} \beta_k e^{jk\omega_0 t}.$$

(b) What is the relationship between α_k and α_{-k} in part (a)? What is the relationship between β_k and β_{-k}?

(c) Suppose that the signals $x(t)$ and $z(t)$ shown in Figure P3.45 have the sine-cosine series representations

$$x(t) = a_0 + 2\sum_{k=1}^{\infty}\left[B_k \cos\left(\frac{2\pi kt}{3}\right) - C_k \sin\left(\frac{2\pi kt}{3}\right)\right],$$

$$z(t) = d_0 + 2\sum_{k=1}^{\infty}\left[E_k \cos\left(\frac{2\pi kt}{3}\right) - F_k \sin\left(\frac{2\pi kt}{3}\right)\right].$$

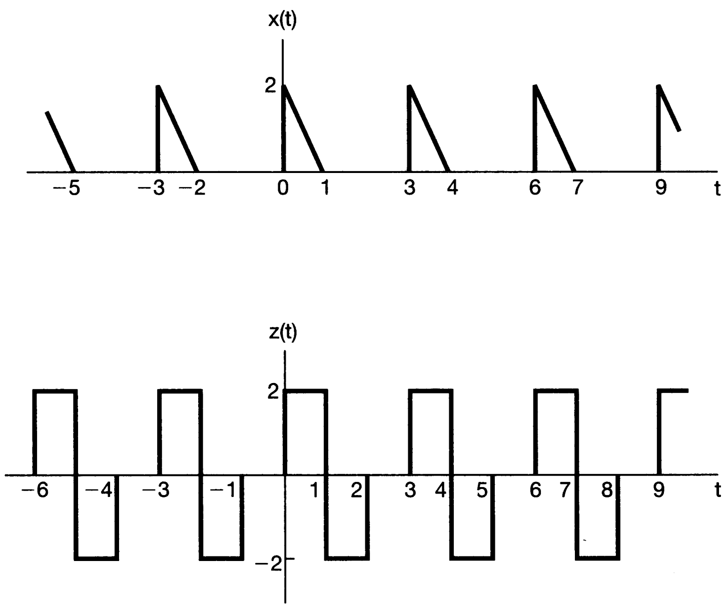


Figure P3.45

Sketch the signal

$$y(t) = 4(a_0 + d_0) + 2\sum_{k=1}^{\infty}\left\{\left[B_k + \frac{1}{2}E_k\right]\cos\left(\frac{2\pi kt}{3}\right) + F_k \sin\left(\frac{2\pi kt}{3}\right)\right\}.$$

3.46 In this problem, we derive two important properties of the continuous-time Fourier series: the multiplication property and Parseval's relation. Let $x(t)$ and $y(t)$ both be continuous-time periodic signals having period T_0 and with Fourier series representations given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \quad y(t) = \sum_{k=-\infty}^{+\infty} b_k e^{jk\omega_0 t}. \tag{P3.46–1}$$

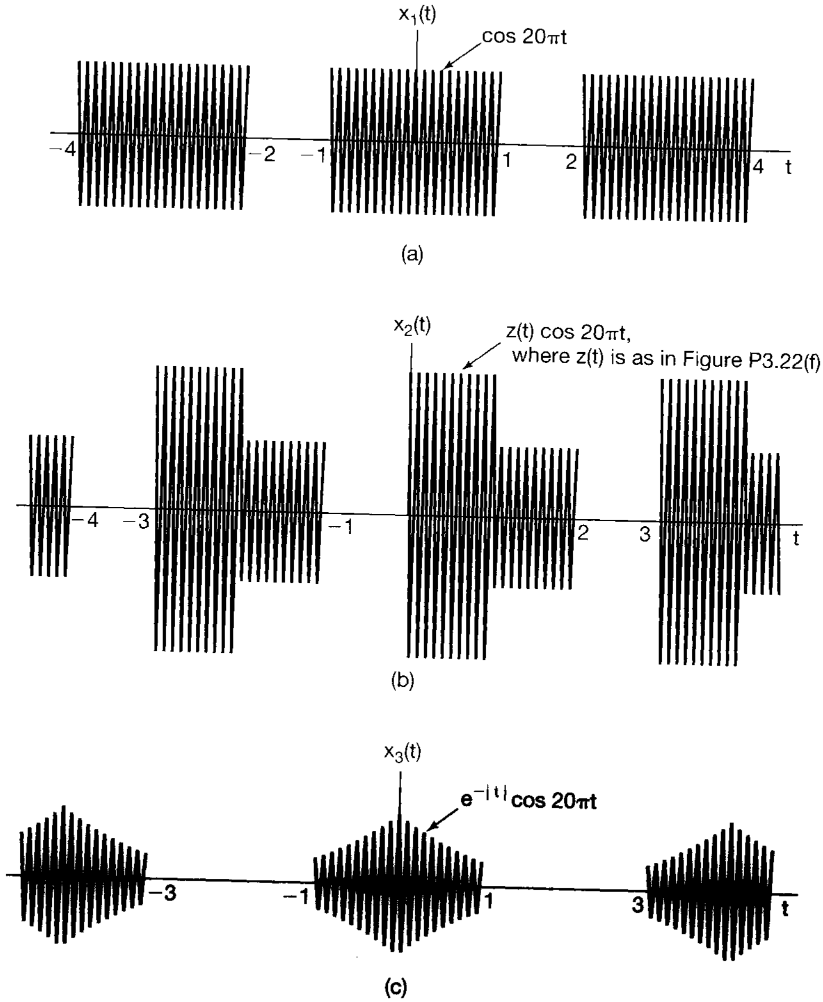


Figure P3.46

(a) Show that the Fourier series coefficients of the signal

$$z(t) = x(t)y(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

are given by the discrete convolution

$$c_k = \sum_{n=-\infty}^{+\infty} a_n b_{k-n}.$$

(b) Use the result of part (a) to compute the Fourier series coefficients of the signals $x_1(t)$, $x_2(t)$, and $x_3(t)$ depicted in Figure P3.46.

(c) Suppose that $y(t)$ in eq. (P3.46–1) equals $x^*(t)$. Express the b_k in the equation in terms of a_k, and use the result of part (a) to prove Parseval's relation for periodic signals—that is,

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{+\infty} |a_k|^2.$$

3.47 Consider the signal

$$x(t) = \cos 2\pi t.$$

Since $x(t)$ is periodic with a fundamental period of 1, it is also periodic with a period of N, where N is any positive integer. What are the Fourier series coefficients of $x(t)$ if we regard it as a periodic signal with period 3?

3.48. Let $x[n]$ be a periodic sequence with period N and Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}. \qquad \text{(P3.48–1)}$$

The Fourier series coefficients for each of the following signals can be expressed in terms of a_k in eq. (P3.48–1). Derive the expressions.

(a) $x[n - n_0]$

(b) $x[n] - x[n-1]$

(c) $x[n] - x[n - \frac{N}{2}]$ (assume that N is even)

(d) $x[n] + x[n + \frac{N}{2}]$ (assume that N is even; note that this signal is periodic with period $N/2$)

(e) $x^*[-n]$

(f) $(-1)^n x[n]$ (assume that N is even)

(g) $(-1)^n x[n]$ (assume that N is odd; note that this signal is periodic with period $2N$)

(h) $y[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

3.49. Let $x[n]$ be a periodic sequence with period N and Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}. \qquad \text{(P3.49–1)}$$

(a) Suppose that N is even and that $x[n]$ in eq. (P3.49–1) satisfies

$$x[n] = -x\left[n + \frac{N}{2}\right] \text{ for all } n.$$

Show that $a_k = 0$ for all even integers k.

(b) Suppose that N is divisible by 4. Show that if

$$x[n] = -x\left[n + \frac{N}{4}\right] \text{ for all } n,$$

then $a_k = 0$ for every value of k that is a multiple of 4.

(c) More generally, suppose that N is divisible by an integer M. Show that if

$$\sum_{r=0}^{(N/M)-1} x\left[n + r\frac{N}{M}\right] = 0 \text{ for all } n,$$

then $a_k = 0$ for every value of k that is a multiple of M.

3.50. Suppose we are given the following information about a periodic signal $x[n]$ with period 8 and Fourier coefficients a_k:

1. $a_k = -a_{k-4}$.
2. $x[2n+1] = (-1)^n$.

Sketch one period of $x[n]$.

3.51. Let $x[n]$ be a periodic signal with period $N = 8$ and Fourier series coefficients $a_k = -a_{k-4}$. A signal

$$y[n] = \left(\frac{1+(-1)^n}{2}\right)x[n-1]$$

with period $N = 8$ is generated. Denoting the Fourier series coefficients of $y[n]$ by b_k, find a function $f[k]$ such that

$$b_k = f[k]a_k.$$

3.52. $x[n]$ is a real periodic signal with period N and complex Fourier series coefficients a_k. Let the Cartesian form for a_k be denoted by

$$a_k = b_k + jc_k,$$

where b_k and c_k are both real.

(a) Show that $a_{-k} = a_k^*$. What is the relation between b_k and b_{-k}? What is the relation between c_k and c_{-k}?

(b) Suppose that N is even. Show that $a_{N/2}$ is real.

(c) Show that $x[n]$ can also be expressed as a trigonometric Fourier series of the form

$$x[n] = a_0 + 2\sum_{k=1}^{(N-1)/2} b_k \cos\left(\frac{2\pi kn}{N}\right) - c_k \sin\left(\frac{2\pi kn}{N}\right)$$

if N is odd or as

$$x[n] = (a_0 + a_{N/2}(-1)^n) + 2\sum_{k=1}^{(N-2)/2} b_k \cos\left(\frac{2\pi kn}{N}\right) - c_k \sin\left(\frac{2\pi kn}{N}\right)$$

if N is even.

(d) Show that if the polar form of a_k is $A_k e^{j\theta k}$, then the Fourier series representation for $x[n]$ can also be written as

$$x[n] = a_0 + 2\sum_{k=1}^{(N-1)/2} A_k \cos\left(\frac{2\pi kn}{N} + \theta_k\right)$$

if N is odd or as

$$x[n] = (a_0 + a_{N/2}(-1)^n) + 2\sum_{k=1}^{(N/2)-1} A_k \cos\left(\frac{2\pi kn}{N} + \theta_k\right)$$

if N is even.

(e) Suppose that $x[n]$ and $z[n]$, as depicted in Figure P3.52, have the sine-cosine series representations

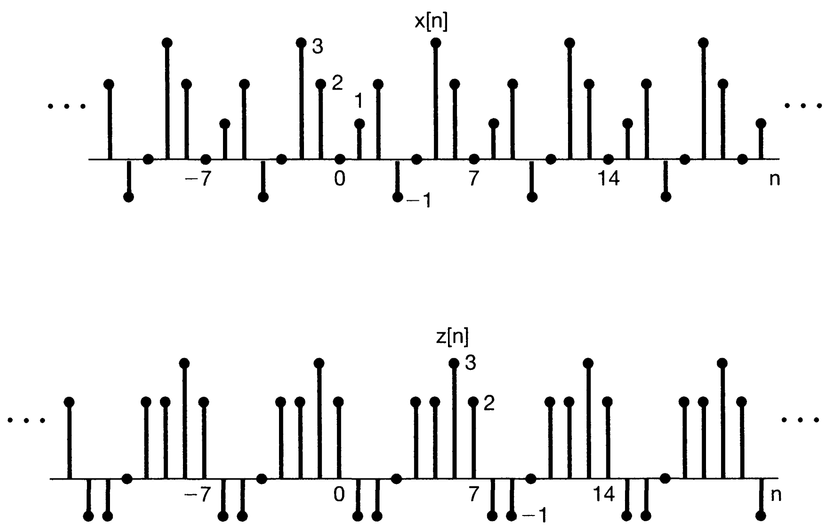


Figure P3.52

$$x[n] = a_0 + 2\sum_{k=1}^{3}\left\{b_k \cos\left(\frac{2\pi kn}{7}\right) - c_k \sin\left(\frac{2\pi kn}{7}\right)\right\},$$

$$z[n] = d_0 + 2\sum_{k=1}^{3}\left\{d_k \cos\left(\frac{2\pi kn}{7}\right) - f_k \sin\left(\frac{2\pi kn}{7}\right)\right\}.$$

Sketch the signal

$$y[n] = a_0 - d_0 + 2\sum_{k=1}^{3}\left\{d_k \cos\left(\frac{2\pi kn}{7}\right) + (f_k - c_k)\sin\left(\frac{2\pi kn}{7}\right)\right\}.$$

3.53. Let $x[n]$ be a real periodic signal with period N and Fourier coefficients a_k.

(a) Show that if N is even, at least two of the Fourier coefficients within one period of a_k are real.

(b) Show that if N is odd, at least one of the Fourier coefficients within one period of a_k is real.

3.54. Consider the function

$$a[k] = \sum_{n=0}^{N-1} e^{j(2\pi/N)kn}.$$

(a) Show that $a[k] = N$ for $k = 0, \pm N, \pm 2N, \pm 3N, \ldots$.

(b) Show that $a[k] = 0$ whenever k is not an integer multiple of N. (*Hint:* Use the finite sum formula.)

(c) Repeat parts (a) and (b) if

$$a[k] = \sum_{n=\langle N \rangle} e^{j(2\pi/N)kn}.$$

3.55. Let $x[n]$ be a periodic signal with fundamental period N and Fourier series coefficients a_k. In this problem, we derive the time-scaling property

$$x_{(m)}[n] = \begin{cases} x[\frac{n}{m}], & n = 0, \pm m, \pm 2m, \cdots \\ 0, & \text{elsewhere} \end{cases}$$

listed in Table 3.2.

(a) Show that $x_{(m)}[n]$ has period of mN.

(b) Show that if

$$x[n] = v[n] + w[n],$$

then

$$x_{(m)}[n] = v_{(m)}[n] + w_{(m)}[n].$$

(c) Assuming that $x[n] = e^{j2\pi k_0 n/N}$ for some integer k_0, verify that

$$x_{(m)}[n] = \frac{1}{m}\sum_{l=0}^{m-1} e^{j2\pi(k_0+lN)n/(mN)}.$$

That is, one complex exponential in $x[n]$ becomes a linear combination of m complex exponentials in $x_{(m)}[n]$.

(d) Using the results of parts (a), (b), and (c), show that if $x[n]$ has the Fourier coefficients a_k, then $x_{(m)}[n]$ must have the Fourier coefficients $\frac{1}{m}a_k$.

3.56. Let $x[n]$ be a periodic signal with period N and Fourier coefficients a_k.

(a) Express the Fourier coefficients b_k of $|x[n]|^2$ in terms of a_k.

(b) If the coefficients a_k are real, is it guaranteed that the coefficients b_k are also real?

3.57. **(a)** Let

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk(2\pi/N)n} \qquad \text{(P3.57–1)}$$

and

$$y[n] = \sum_{k=0}^{N-1} b_k e^{jk(2\pi/N)n}$$

be periodic signals. Show that

$$x[n]y[n] = \sum_{k=0}^{N-1} c_k e^{jk(2\pi/N)n},$$

where

$$c_k = \sum_{l=0}^{N-1} a_l b_{k-l} = \sum_{l=0}^{N-1} a_{k-l} b_l.$$

(b) Generalize the result of part (a) by showing that

$$c_k = \sum_{l=<N>} a_l b_{k-l} = \sum_{l=<N>} a_{k-l} b_l.$$

(c) Use the result of part (b) to find the Fourier series representation of the following signals, where $x[n]$ is given by eq. (P3.57–1).

(i) $x[n]\cos\left(\frac{6\pi n}{N}\right)$

(ii) $x[n]\sum_{r=-\infty}^{+\infty}\delta[n-rN]$

(iii) $x[n]\left(\sum_{r=-\infty}^{+\infty}\delta\left[n-\frac{rN}{3}\right]\right)$ (assume that N is divisible by 3)

(d) Find the Fourier series representation for the signal $x[n]y[n]$, where

$$x[n] = \cos(\pi n/3)$$

and

$$y[n] = \begin{cases} 1, & |n| \leq 3 \\ 0, & 4 \leq |n| \leq 6 \end{cases}$$

is periodic with period 12.

(e) Use the result of part (b) to show that

$$\sum_{n=<N>} x[n]y[n] = N \sum_{l=<N>} a_l b_{-l},$$

and from this expression, derive Parseval's relation for discrete-time periodic signals.

3.58. Let $x[n]$ and $y[n]$ be periodic signals with common period N, and let

$$z[n] = \sum_{r=<N>} x[r]y[n-r]$$

be their periodic convolution.

(a) Show that $z[n]$ is also periodic with period N.

(b) Verify that if a_k, b_k, and c_k are the Fourier coefficients of $x[n]$, $y[n]$, and $z[n]$, respectively, then

$$c_k = Na_k b_k.$$

(c) Let

$$x[n] = \sin\left(\frac{3\pi n}{4}\right)$$

and

$$y[n] = \begin{cases} 1, & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 7 \end{cases}$$

be two signals that are periodic with period 8. Find the Fourier series representation for the periodic convolution of these signals.

(d) Repeat part (c) for the following two periodic signals that also have period 8:

$$x[n] = \begin{cases} \sin\left(\frac{3\pi n}{4}\right), & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 7 \end{cases},$$

$$y[n] = \left(\frac{1}{2}\right)^n, 0 \leq n \leq 7.$$

3.59. **(a)** Suppose $x[n]$ is a periodic signal with period N. Show that the Fourier series coefficients of the periodic signal

$$g(t) = \sum_{k=-\infty}^{\infty} x[k]\,\delta(t-kT)$$

are periodic with period N.

(b) Suppose that $x(t)$ is a periodic signal with period T and Fourier series coefficients a_k with period N. Show that there must exist a periodic sequence $g[n]$ such that

$$x(t) = \sum_{k=-\infty}^{\infty} g[k]\,\delta(t - kT/N).$$

(c) Can a continuous periodic signal have periodic Fourier coefficients?

3.60. Consider the following pairs of signals $x[n]$ and $y[n]$. For each pair, determine whether there is a discrete-time LTI system for which $y[n]$ is the output when the corresponding $x[n]$ is the input. If such a system exists, determine whether the system is unique (i.e., whether there is more than one LTI system with the given input-output pair). Also, determine the frequency response of an LTI system with the desired behavior. If no such LTI system exists for a given $x[n]$, $y[n]$ pair, explain why.

(a) $x[n] = (\frac{1}{2}^n)$, $y[n] = (\frac{1}{4}^n)$

(b) $x[n] = (\frac{1}{2}^n)u[n]$, $y[n] = (\frac{1}{4})^n u[n]$

(c) $x[n] = (\frac{1}{2}^n)u[n]$, $y[n] = 4^n u[-n]$

(d) $x[n] = e^{jn/8}$, $y[n] = 2e^{jn/8}$

(e) $x[n] = e^{jn/8}u[n]$, $y[n] = 2e^{jn/8}u[n]$

(f) $x[n] = j^n$, $y[n] = 2j^n(1 - j)$

(g) $x[n] = \cos(\pi n/3)$, $y[n] = \cos(\pi n/3) + \sqrt{3}\sin(\pi n/3)$

(h) $x[n]$ and $y_1[n]$ as in Figure P3.60

(i) $x[n]$ and $y_2[n]$ as in Figure P3.60

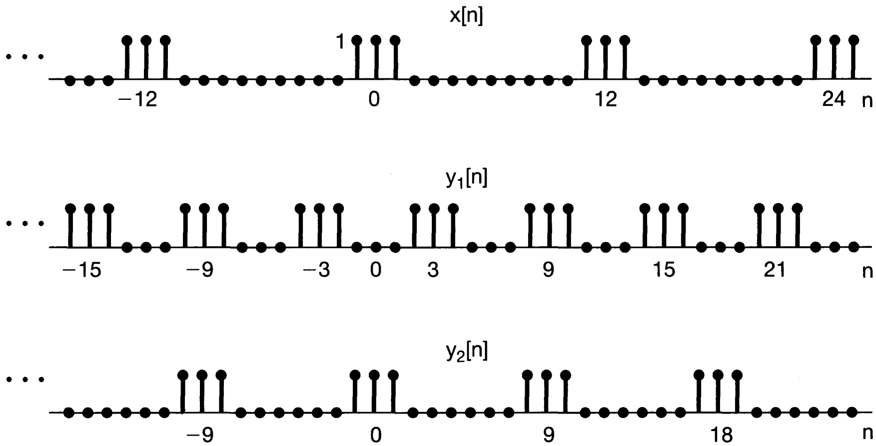


Figure P3.60

3.61. As we have seen, the techniques of Fourier analysis are of value in examining continuous-time LTI systems because periodic complex exponentials are eigenfunctions for LTI systems. In this problem, we wish to substantiate the following statement: Although some LTI systems may have additional eigenfunctions, the complex exponentials are the *only* signals that are eigenfunctions of *every* LTI system.

(a) What are the eigenfunctions of the LTI system with unit impulse response $h(t) = \delta(t)$? What are the associated eigenvalues?

(b) Consider the LTI system with unit impulse response $h(t) = \delta(t - T)$. Find a signal that is not of the form e^{st}, but that is an eigenfunction of the system with eigenvalue 1. Similarly, find the eigenfunctions with eigenvalues 1/2 and 2 that are not complex exponentials. (*Hint:* You can find impulse trains that meet these requirements.)

(c) Consider a stable LTI system with impulse response $h(t)$ that is real and even. Show that $\cos \omega t$ and $\sin \omega t$ are eigenfunctions of this system.

(d) Consider the LTI system with impulse response $h(t) = u(t)$. Suppose that $\phi(t)$ is an eigenfunction of this system with eigenvalue λ. Find the differential equation that $\phi(t)$ must satisfy, and solve the equation. This result, together with those of parts (a) through (c), should prove the validity of the statement made at the beginning of the problem.

3.62. One technique for building a dc power supply is to take an ac signal and full-wave rectify it. That is, we put the ac signal $x(t)$ through a system that produces $y(t) = |x(t)|$ as its output.

(a) Sketch the input and output waveforms if $x(t) = \cos t$. What are the fundamental periods of the input and output?

(b) If $x(t) = \cos t$, determine the coefficients of the Fourier series for the output $y(t)$.

(c) What is the amplitude of the dc component of the input signal? What is the amplitude of the dc component of the output signal?

3.63. Suppose that a continuous-time periodic signal is the input to an LTI system. The signal has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{jk(\pi/4)t},$$

where α is a real number between 0 and 1, and the frequency response of the system is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq W \\ 0, & |\omega| > W \end{cases}.$$

How large must W be in order for the output of the system to have at least 90% of the average energy per period of $x(t)$?

3.64. As we have seen in this chapter, the concept of an eigenfunction is an extremely important tool in the study of LTI systems. The same can be said for linear, but time-varying, systems. Specifically, consider such a system with input $x(t)$ and output $y(t)$. We say that a signal $\phi(t)$ is an *eigenfunction* of the system if

$$\phi(t) \longrightarrow \lambda \phi(t).$$

That is, if $x(t) = \phi(t)$, then $y(t) = \lambda \phi(t)$, where the complex constant λ is called the *eigenvalue associated with* $\phi(t)$.

(a) Suppose that we can represent the input $x(t)$ to our system as a linear combination of eigenfunctions $\phi_k(t)$, each of which has a corresponding eigenvalue λ_k; that is,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \phi_k(t).$$

Express the output $y(t)$ of the system in terms of $\{c_k\}$, $\{\phi_k(t)\}$, and $\{\lambda_k\}$.

(b) Consider the system characterized by the differential equation

$$y(t) = t^2 \frac{d^2x(t)}{dt^2} + t\frac{dx(t)}{dt}.$$

Is this system linear? Is it time invariant?

(c) Show that the functions

$$\phi_k(t) = t^k$$

are eigenfunctions of the system in part (b). For each $\phi_k(t)$, determine the corresponding eigenvalue λ_k.

(d) Determine the output of the system if

$$x(t) = 10t^{-10} + 3t + \frac{1}{2}t^4 + \pi.$$

EXTENSION PROBLEMS

3.65. Two functions $u(t)$ and $v(t)$ are said to be *orthogonal over the interval (a,b)* if

$$\int_a^b u(t)v^*(t)\,dt = 0. \qquad \text{(P3.65–1)}$$

If, in addition,

$$\int_a^b |u(t)|^2\,dt = 1 = \int_a^b |v(t)|^2\,dt,$$

the functions are said to be *normalized* and hence are called *orthonormal*. A set of functions $\{\phi_k(t)\}$ is called an *orthogonal (orthonormal) set* if each pair of functions in the set is orthogonal (orthonormal).

(a) Consider the pairs of signals $u(t)$ and $v(t)$ depicted in Figure P3.65. Determine whether each pair is orthogonal over the interval (0, 4).

(b) Are the functions $\sin m\omega_0 t$ and $\sin n\omega_0 t$ orthogonal over the interval $(0, T)$, where $T = 2\pi/\omega_0$? Are they also orthonormal?

(c) Repeat part (b) for the functions $\phi_m(t)$ and $\phi_n(t)$, where

$$\phi_k(t) = \frac{1}{\sqrt{T}}[\cos k\omega_0 t + \sin k\omega_0 t].$$

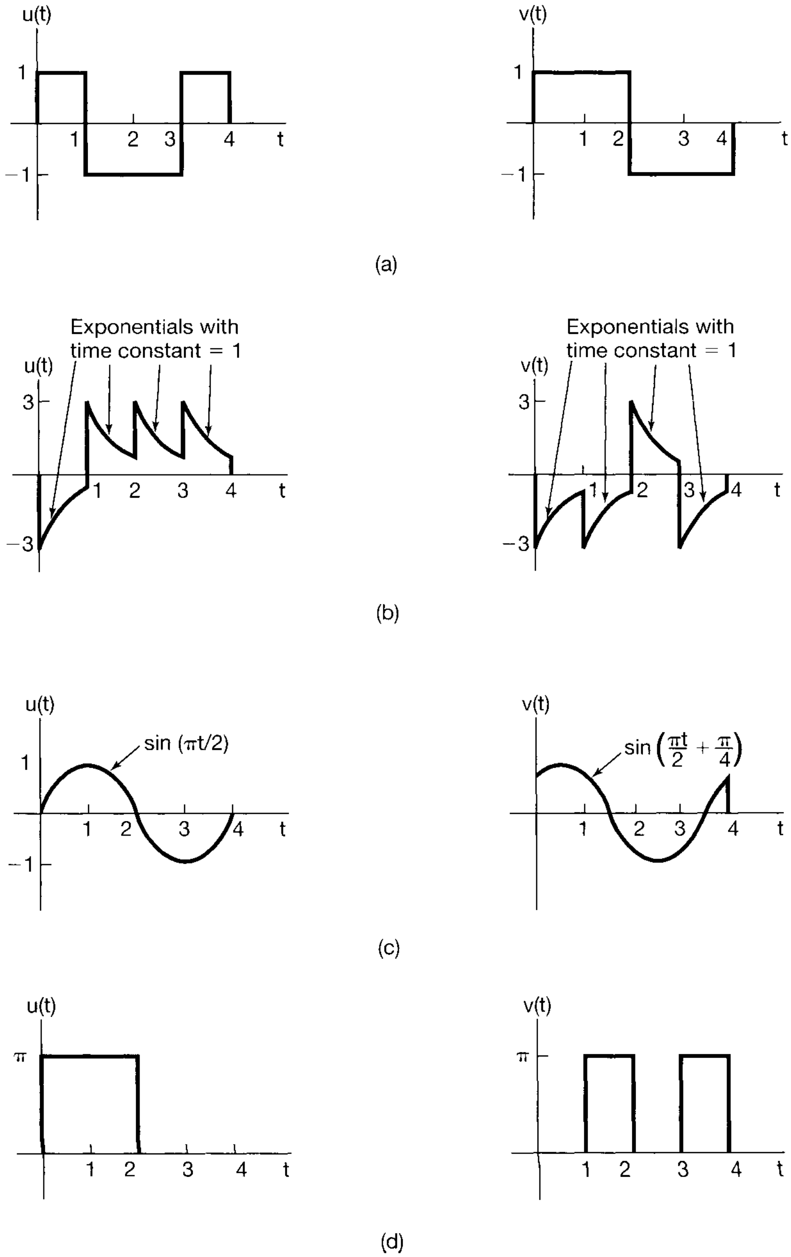


Figure P3.65

(d) Show that the functions $\phi_k(t) = e^{jk\omega_0 t}$ are orthogonal over *any* interval of length $T = 2\pi/\omega_0$. Are they orthonormal?

(e) Let $x(t)$ be an arbitrary signal, and let $x_o(t)$ and $x_e(t)$ be, respectively, the odd and even parts of $x(t)$. Show that $x_o(t)$ and $x_e(t)$ are orthogonal over the interval $(-T, T)$ for any T.

(f) Show that if $\{\phi_k(t)\}$ is a set of orthogonal signals over the interval (a, b), then the set $\{(1/\sqrt{A_k})\phi_k(t)\}$, where

$$A_k = \int_a^b |\phi_k(t)|^2 \, dt,$$

is orthonormal.

(g) Let $\{\phi_i(t)\}$ be a set of orthonormal signals on the interval (a, b), and consider a signal of the form

$$x(t) = \sum_i a_i \phi_i(t),$$

where the a_i are complex constants. Show that

$$\int_a^b |x(t)|^2 \, dt = \sum_i |a_i|^2.$$

(h) Suppose that $\phi_1(t), \ldots, \phi_N(t)$ are nonzero only in the time interval $0 \leq t \leq T$ and that they are orthonormal over this time interval. Let L_i denote the LTI system with impulse response

$$h_i(t) = \phi_i(T - t). \qquad \text{(P3.65–2)}$$

Show that if $\phi_j(t)$ is applied to this system, then the output at time T is 1 if $i = j$ and 0 if $i \neq j$. The system with impulse response given by eq. (P3.65–2) was referred to in Problems 2.66 and 2.67 as the *matched filter* for the signal $\phi_i(t)$.

3.66. The purpose of this problem is to show that the representation of an arbitrary periodic signal by a Fourier series or, more generally, as a linear combination of any set of orthogonal functions is computationally efficient and in fact very useful for obtaining good approximations of signals.[12]

Specifically, let $\{\phi_i(t)\}$, $i = 0, \pm 1, \pm 2, \ldots$ be a set of orthonormal functions on the interval $a \leq t \leq b$, and let $x(t)$ be a given signal. Consider the following approximation of $x(t)$ over the interval $a \leq t \leq b$:

$$\hat{x}_n(t) = \sum_{i=-N}^{+N} a_i \phi_i(t). \qquad \text{(P3.66–1)}$$

Here, the a_i are (in general, complex) constants. To measure the deviation between $x(t)$ and the series approximation $\hat{x}_N(t)$, we consider the error $e_N(t)$ defined as

$$e_N(t) = x(t) - \hat{x}_N(t). \qquad \text{(P3.66–2)}$$

A reasonable and widely used criterion for measuring the quality of the approximation is the energy in the error signal over the interval of interest—that is, the integral

[12] See Problem 3.65 for the definitions of orthogonal and orthonormal functions.

of the square of the magnitude of the error over the interval $a \leq t \leq b$:

$$E = \int_a^b |e_N(t)|^2 \, dt. \tag{P3.66–3}$$

(a) Show that E is minimized by choosing

$$a_i = \int_a^b x(t)\phi_i^*(t)\, dt. \tag{P3.66–4}$$

[*Hint:* Use eqs. (P3.66–1)–(P3.66–3) to express E in terms of a_i, $\phi_i(t)$, and $x(t)$. Then express a_i in rectangular coordinates as $a_i = b_i + jc_i$, and show that the equations

$$\frac{\partial E}{\partial b_i} = 0 \quad \text{and} \quad \frac{\partial E}{\partial c_i} = 0,\, i = 0, \pm 1, \pm 2, \ldots, N$$

are satisfied by the a_i as given by eq. (P3.66–4).]

(b) How does the result of part (a) change if

$$A_i = \int_a^b |\phi_i(t)|^2 \, dt$$

and the $\{\phi_i(t)\}$ are orthogonal but not orthonormal?

(c) Let $\phi_n(t) = e^{jn\omega_0 t}$, and choose any interval of length $T_0 = 2\pi/\omega_0$. Show that the a_i that minimize E are as given in eq. (3.50).

(d) The set of *Walsh functions* is an often-used set of orthonormal functions. (See Problem 2.66.) The set of five Walsh functions, $\phi_0(t), \phi_1(t), \ldots, \phi_4(t)$, is illustrated in Figure P3.66, where we have scaled time so that the $\phi_i(t)$ are nonzero and orthonormal over the interval $0 \leq t \leq 1$. Let $x(t) = \sin \pi t$. Find the approximation of $x(t)$ of the form

$$\hat{x}(t) = \sum_{i=0}^{4} a_i \phi_i(t)$$

such that

$$\int_0^1 |x(t) - \hat{x}(t)|^2 \, dt$$

is minimized.

(e) Show that $\hat{x}_N(t)$ in eq. (P3.66–1) and $e_N(t)$ in eq. (P3.66–2) are orthogonal if the a_i are chosen as in eq. (P3.66–4).

The results of parts (a) and (b) are extremely important in that they show that each coefficient a_i is *independent* of all the other a_j's, $i \neq j$. Thus, if we add more terms to the approximation [e.g., if we compute the approximation $\hat{x}_{N+1}(t)$], the coefficients of $\phi_i(t)$, $i = 1, \ldots, N$, that were previously determined will not change. In contrast to this, consider another type of se-

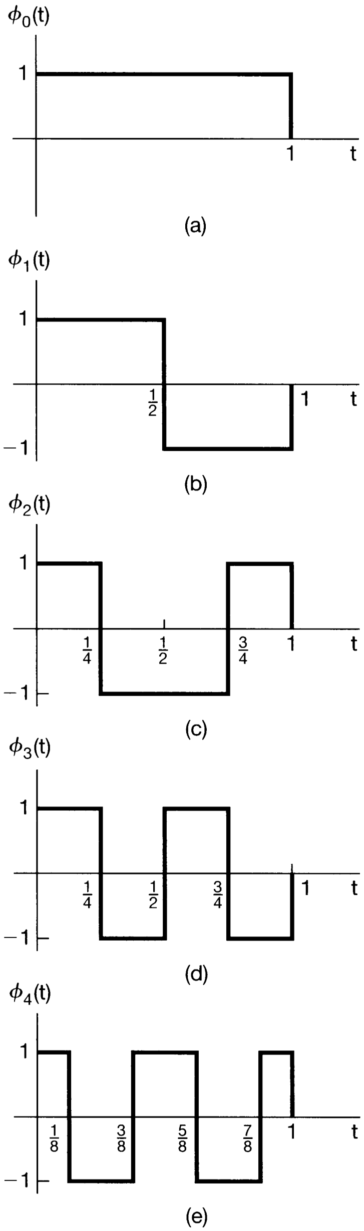


Figure P3.66

ries expansion, the polynomial Taylor series. The *infinite* Taylor series for e^t is $e^t = 1 + t + t^2/2! + \ldots$, but as we shall show, when we consider a *finite* polynomial series and the error criterion of eq. (P3.66–3), we get a very different result.

Specifically, let $\phi_0(t) = 1$, $\phi_1(t) = t$, $\phi_2(t) = t^2$, and so on.

(f) Are the $\phi_i(t)$ orthogonal over the interval $0 \leq t \leq 1$?

(g) Consider an approximation of $x(t) = e^t$ over the interval $0 \leq t \leq 1$ of the form

$$\hat{x}_0(t) = a_0\phi_0(t).$$

Find the value of a_0 that minimizes the energy in the error signal over the interval.

(h) We now wish to approximate e^t by a Taylor series using two terms—i.e., $\hat{x}_1(t) = a_0 + a_1 t$. Find the optimum values for a_0 and a_1. [*Hint:* Compute E in terms of a_0 and a_1, and then solve the simultaneous equations

$$\frac{\partial E}{\partial a_0} = 0 \quad \text{and} \quad \frac{\partial E}{\partial a_1} = 0.$$

Note that your answer for a_0 has changed from its value in part (g), where there was only one term in the series. Further, as you increase the number of terms in the series, that coefficient and all others will continue to change. We can thus see the advantage to be gained in expanding a function using orthogonal terms.]

3.67 As we discussed in the text, the origins of Fourier analysis can be found in problems of mathematical physics. In particular, the work of Fourier was motivated by his investigation of heat diffusion. In this problem, we illustrate how the Fourier series enter into the investigation.[13]

Consider the problem of determining the temperature at a given depth beneath the surface of the earth as a function of time, where we assume that the temperature at the surface is a given function of time $T(t)$ that is periodic with period 1. (The unit of time is one year.) Let $T(x, t)$ denote the temperature at a depth x below the surface at time t. This function obeys the heat diffusion equation

$$\frac{\partial T(x,t)}{\partial t} = \frac{1}{2}k^2\frac{\partial^2 T(x,t)}{\partial x^2} \qquad \text{(P3.67–1)}$$

with auxiliary condition

$$T(0, t) = T(t). \qquad \text{(P3.67–2)}$$

Here, k is the heat diffusion constant for the earth ($k > 0$). Suppose that we expand $T(t)$ in a Fourier series:

$$T(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jn2\pi t}. \qquad \text{(P3.67–3)}$$

Similarly, let us expand $T(x, t)$ at any given depth x in a Fourier series in t. We obtain

$$T(x,t) = \sum_{n=-\infty}^{+\infty} b_n(x)e^{jn2\pi t}, \qquad \text{(P3.67– 4)}$$

where the Fourier coefficients $b_n(x)$ depend upon the depth x.

[13]The problem has been adapted from A. Sommerfeld, *Partial Differential Equations in Physics* (N. York: Academic Press, 1949), pp 68–71.

(a) Use eqs. (P3.67–1)–(P3.67–4) to show that $b_n(x)$ satisfies the differential equation

$$\frac{d^2 b_n(x)}{dx^2} = \frac{4\pi jn}{k^2} b_n(x) \qquad \text{(P3.67–5a)}$$

with auxiliary condition

$$b_n(0) = a_n. \qquad \text{(P3.67–5b)}$$

Since eq. (P3.67–5a) is a second-order equation, we need a second auxiliary condition. We argue on physical grounds that, far below the earth's surface, the variations in temperature due to surface fluctuations should disappear. That is,

$$\lim_{x\to\infty} T(x,t) = \text{a constant.} \qquad \text{(P3.67–5c)}$$

(b) Show that the solution of eqs. (P3.67–5) is

$$b_n(x) = \begin{cases} a_n \exp[-\sqrt{2\pi|n|}(1+j)x/k], & n \geq 0 \\ a_n \exp[-\sqrt{2\pi|n|}(1-j)x/k], & n \leq 0 \end{cases}.$$

(c) Thus, the temperature oscillations at depth x are damped and phase-shifted versions of the temperature oscillations at the surface. To see this more clearly, let

$$T(t) = a_0 + a_1 \sin 2\pi t$$

(so that a_0 represents the mean yearly temperature). Sketch $T(t)$ and $T(x,t)$ over a one-year period for

$$x = k\sqrt{\frac{\pi}{2}},$$

$a_0 = 2$, and $a_1 = 1$. Note that at this depth not only are the temperature oscillations significantly damped, but the phase shift is such that it is warmest in winter and coldest in summer. This is exactly the reason why vegetable cellars are constructed!

3.68. Consider the closed contour shown in Figure P3.68. As illustrated, we can view this curve as being traced out by the tip of a rotating vector of varying length. Let $r(\theta)$ denote the length of the vector as a function of the angle θ. Then $r(\theta)$ is periodic in θ with period 2π and thus has a Fourier series representation. Let $\{a_k\}$ denote the Fourier coefficients of $r(\theta)$.

(a) Consider now the projection $x(\theta)$ of the vector $r(\theta)$ onto the x-axis, as indicated in the figure. Determine the Fourier coefficients for $x(\theta)$ in terms of the a_k's.

(b) Consider the sequence of coefficients

$$b_k = a_k e^{jk\pi/4}.$$

Sketch the figure in the plane that corresponds to this set of coefficients.

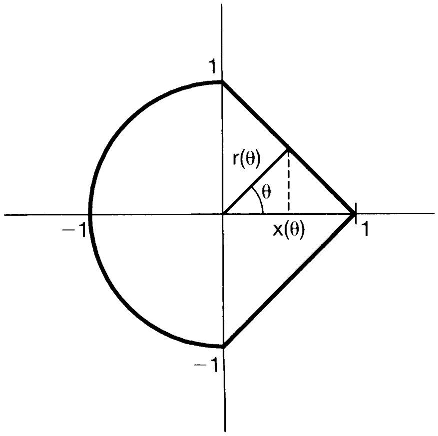


Figure P3.68

(c) Repeat part (b) with

$$b_k = a_k\delta[k].$$

(d) Sketch figures in the plane such that $r(\theta)$ is *not* constant, but does have each of the following properties:
(i) $r(\theta)$ is even.
(ii) The fundamental period of $r(\theta)$ is π.
(iii) The fundamental period of $r(\theta)$ is $\pi/2$.

3.69. In this problem, we consider the discrete-time counterpart of the concepts introduced in Problems 3.65 and 3.66. In analogy with the continuous-time case, two discrete-time signals $\phi_k[n]$ and $\phi_m[n]$ are said to be *orthogonal* over the interval (N_1, N_2) if

$$\sum_{n=N_1}^{N_2} \phi_k[n]\phi_m^*[n] = \begin{cases} A_k, & k = m \\ 0, & k \neq m \end{cases}. \qquad \text{(P3.69–1)}$$

If the value of the constants A_k and A_m are both 1, then the signals are said to be *orthonormal*.

(a) Consider the signals

$$\phi_k[n] = \delta[n-k],\ k = 0, \pm 1, \pm 2, \ldots, \pm N.$$

Show that these signals are orthonormal over the interval $(-N, N)$.

(b) Show that the signals

$$\phi_k[n] = e^{jk(2\pi/N)n},\ k = 0, 1, \ldots, N-1,$$

are orthogonal over any interval of length N.

(c) Show that if

$$x[n] = \sum_{i=1}^{M} a_i\phi_i[n],$$

where the $\phi_i[n]$ are orthogonal over the interval (N_1, N_2), then

$$\sum_{n=N_1}^{N_2} |x[n]|^2 = \sum_{i=1}^{M} |a_i|^2 A_i.$$

(d) Let $\phi_i[n], i = 0, 1, \ldots, M$, be a set of orthogonal functions over the interval (N_1, N_2), and let $x[n]$ be a given signal. Suppose that we wish to approximate $x[n]$ as a linear combination of the $\phi_i[n]$; that is,

$$\hat{x}[n] = \sum_{i=0}^{M} a_i \phi_i[n],$$

where the a_i are constant coefficients. Let

$$e[n] = x[n] - \hat{x}[n],$$

and show that if we wish to minimize

$$E = \sum_{n=N_1}^{N_2} |e[n]|^2,$$

then the a_i are given by

$$a_i = \frac{1}{A_i} \sum_{n=N_1}^{N_2} x[n] \phi_i^*[n]. \qquad \text{(P3.69–2)}$$

[*Hint:* As in Problem 3.66, express E in terms of a_i, $\phi_i[n]$, A_i, and $x[n]$, write $a_i = b_i + jc_i$, and show that the equations

$$\frac{\partial E}{\partial b_i} = 0 \quad \text{and} \quad \frac{\partial E}{\partial c_i} = 0$$

are satisfied by the a_i given by eq. (P3.69–2). Note that applying this result when the $\phi_i[n]$ are as in part (b) yields eq. (3.95) for a_k.]

(e) Apply the result of part (d) when the $\phi_i[n]$ are as in part (a) to determine the coefficients a_i in terms of $x[n]$.

3.70. (a) In this problem, we consider the definition of the two-dimensional Fourier series for periodic signals with two independent variables. Specifically, consider a signal $x(t_1, t_2)$ that satisfies the equation

$$x(t_1, t_2) = x(t_1 + T_1, t_2 + T_2), \text{ for all } t_1, t_2.$$

This signal is periodic with period T_1 in the t_1 direction and with period T_2 in the t_2 direction. Such a signal has a series representation of the form

$$x(t_1, t_2) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} a_{mn} e^{j(m\omega_1 t_1 + n\omega_2 t_2)},$$

where

$$\omega_1 = 2\pi/T_1, \quad \omega_2 = 2\pi/T_2.$$

Find an expression for a_{mn} in terms of $x(t_1, t_2)$.

(b) Determine the Fourier series coefficients a_{mn} for the following signals:

(i) $\cos(2\pi t_1 + 2t_2)$

(ii) the signal illustrated in Figure P3.70

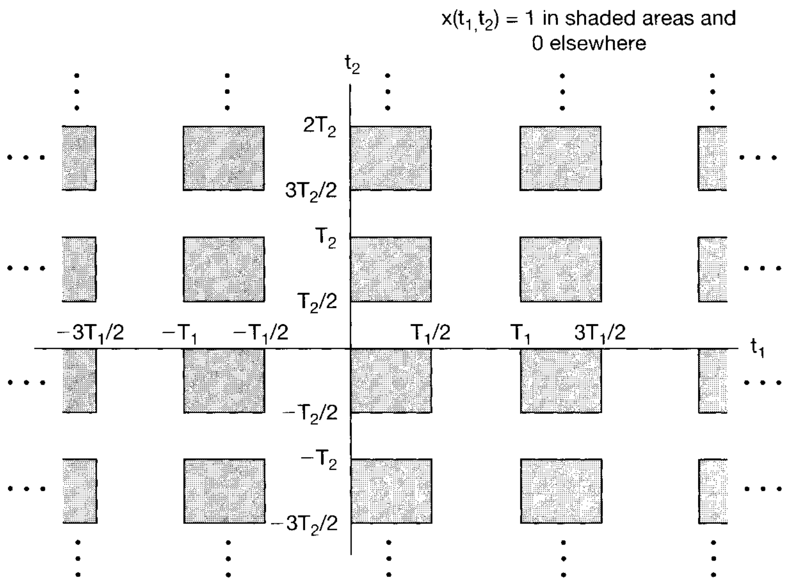


Figure P3.70

3.71. Consider the mechanical system shown in Figure P3.71. The differential equation relating velocity $v(t)$ and the input force $f(t)$ is given by

$$Bv(t) + K\int v(t)\,dt = f(t).$$

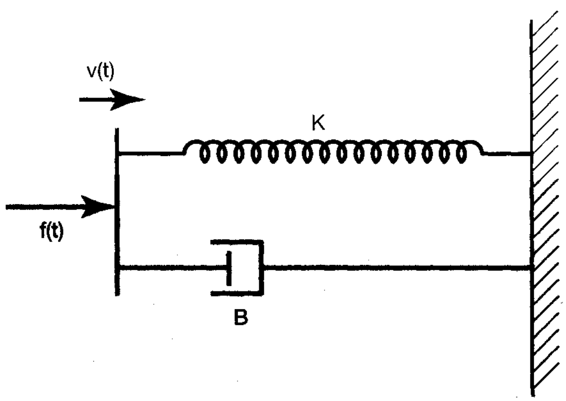


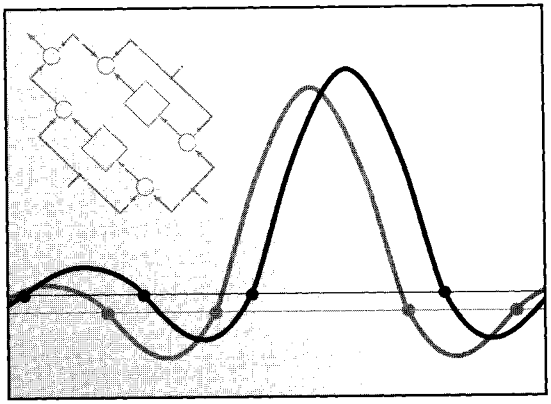
Figure P3.71

(a) Assuming that the output is $f_s(t)$, the compressive force acting on the spring, write the differential equation relating $f_s(t)$ and $f(t)$. Obtain the frequency response of the system, and argue that it approximates that of a lowpass filter.

(b) Assuming that the output is $f_d(t)$, the compressive force acting on the dashpot, write the differential equation relating $f_d(t)$ and $f(t)$. Obtain the frequency response of the system, and argue that it approximates that of a highpass filter.

4

THE CONTINUOUS-TIME FOURIER TRANSFORM



4.0 INTRODUCTION

In Chapter 3, we developed a representation of periodic signals as linear combinations of complex exponentials. We also saw how this representation can be used in describing the effect of LTI systems on signals.

In this and the following chapter, we extend these concepts to apply to signals that are not periodic. As we will see, a rather large class of signals, including all signals with finite energy, can also be represented through a linear combination of complex exponentials. Whereas for periodic signals the complex exponential building blocks are harmonically related, for aperiodic signals they are infinitesimally close in frequency, and the representation in terms of a linear combination takes the form of an integral rather than a sum. The resulting spectrum of coefficients in this representation is called the Fourier transform, and the synthesis integral itself, which uses these coefficients to represent the signal as a linear combination of complex exponentials, is called the inverse Fourier transform.

The development of this representation for aperiodic signals in continuous time is one of Fourier's most important contributions, and our development of the Fourier transform follows very closely the approach he used in his original work. In particular, Fourier reasoned that an aperiodic signal can be viewed as a periodic signal with an infinite period. More precisely, in the Fourier series representation of a periodic signal, as the period increases the fundamental frequency decreases and the harmonically related components become closer in frequency. As the period becomes infinite, the frequency components form a continuum and the Fourier series sum becomes an integral. In the next section we develop the Fourier series representation for continuous-time periodic signals, and in the sections that follow we build on this foundation as we explore many of the important properties of the continuous-time Fourier transform that form the foundation of frequency-domain methods for continuous-time signals and systems. In Chapter 5, we parallel this development for discrete-time signals.

4.1 REPRESENTATION OF APERIODIC SIGNALS: THE CONTINUOUS-TIME FOURIER TRANSFORM

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

To gain some insight into the nature of the Fourier transform representation, we begin by revisiting the Fourier series representation for the continuous-time periodic square wave examined in Example 3.5. Specifically, over one period,

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

and periodically repeats with period T, as shown in Figure 4.1.

As determined in Example 3.5, the Fourier series coefficients a_k for this square wave are

[eq. (3.44)]
$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}, \tag{4.1}$$

where $\omega_0 = 2\pi/T$. In Figure 3.7, bar graphs of these coefficients were shown for a fixed value of T_1 and several different values of T.

An alternative way of interpreting eq. (4.1) is as samples of an envelope function, specifically,

$$Ta_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_0}. \tag{4.2}$$

That is, with ω thought of as a continuous variable, the function $(2\sin\omega T_1)/\omega$ represents the envelope of Ta_k, and the coefficients a_k are simply equally spaced samples of this envelope. Also, for fixed T_1, the envelope of Ta_k is independent of T. In Figure 4.2, we again show the Fourier series coefficients for the periodic square wave, but this time as samples of the envelope of Ta_k, as specified in eq. (4.2). From the figure, we see that as

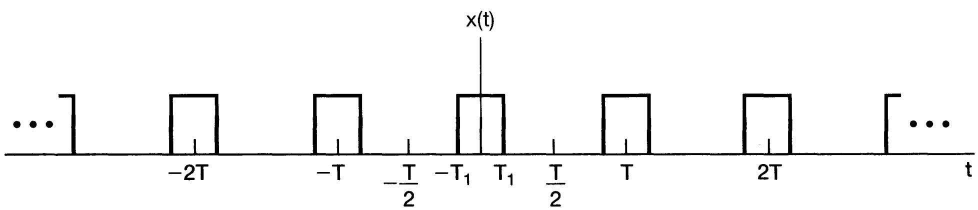


Figure 4.1 A continuous-time periodic square wave.

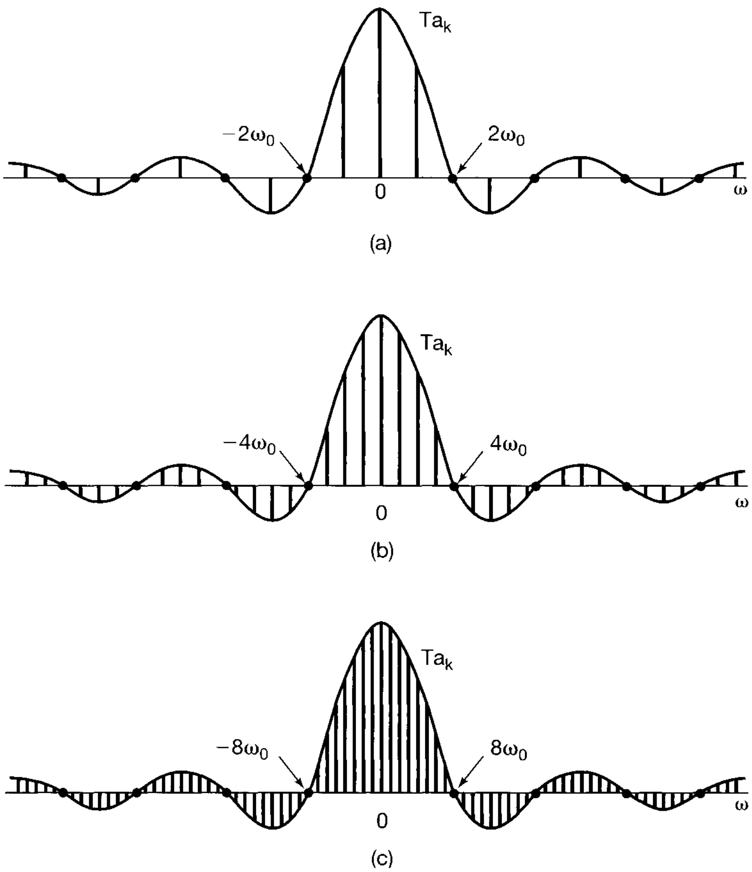


Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 for several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

T increases, or equivalently, as the fundamental frequency $\omega_0 = 2\pi/T$ decreases, the envelope is sampled with a closer and closer spacing. As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse (i.e., all that remains in the time domain is an aperiodic signal corresponding to one period of the square wave). Also, the Fourier series coefficients, multiplied by T, become more and more closely spaced samples of the envelope, so that in some sense (which we will specify shortly) the set of Fourier series coefficients approaches the envelope function as $T \to \infty$.

This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals. Specifically, we think of an aperiodic signal as the limit of a periodic signal as the period becomes arbitrarily large, and we examine the limiting behavior of the Fourier series representation for this signal. In particular, consider a signal $x(t)$ that is of finite duration. That is, for some number T_1, $x(t) = 0$ if $|t| > T_1$, as illustrated in Figure 4.3(a). From this aperiodic signal, we can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period, as indicated in Figure 4.3(b). As we choose the period T to be larger, $\tilde{x}(t)$ is identical to $x(t)$ over a longer interval, and as $T \to \infty$, $\tilde{x}(t)$ is equal to $x(t)$ for any finite value of t.

Let us now examine the effect of this on the Fourier series representation of $\tilde{x}(t)$. Rewriting eqs. (3.38) and (3.39) here for convenience, with the integral in eq. (3.39)

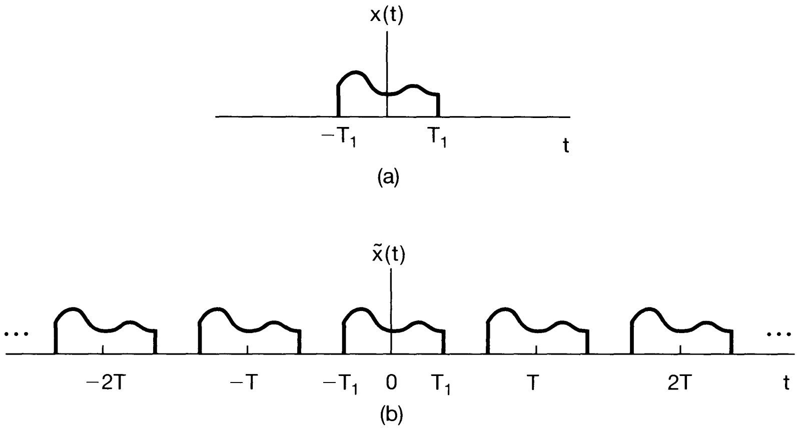


Figure 4.3 (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

carried out over the interval $-T/2 \leq t \leq T/2$, we have

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \tag{4.3}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}dt, \tag{4.4}$$

where $\omega_0 = 2\pi/T$. Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, eq. (4.4) can be rewritten as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

Therefore, defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt, \tag{4.5}$$

we have, for the coefficients a_k,

$$a_k = \frac{1}{T}X(jk\omega_0). \tag{4.6}$$

Combining eqs. (4.6) and (4.3), we can express $\tilde{x}(t)$ in terms of $X(j\omega)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t},$$

or equivalently, since $2\pi/T = \omega_0$,

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0. \tag{4.7}$$

As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$, and consequently, in the limit eq. (4.7) becomes a representation of $x(t)$. Furthermore, $\omega_0 \to 0$ as $T \to \infty$, and the right-hand side of eq. (4.7) passes to an integral. This can be seen by considering the graphical interpretation of the equation, illustrated in Figure 4.4. Each term in the summation on the right-hand side is the area of a rectangle of height $X(jk\omega_0)e^{jk\omega_0 t}$ and width ω_0. (Here, t is regarded as fixed.) As $\omega_0 \to 0$, the summation converges to the integral of $X(j\omega)e^{j\omega t}$. Therefore, using the fact that $\tilde{x}(t) \to x(t)$ as $T \to \infty$, we see that eqs. (4.7) and (4.5) respectively become

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega} \tag{4.8}$$

and

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.} \tag{4.9}$$

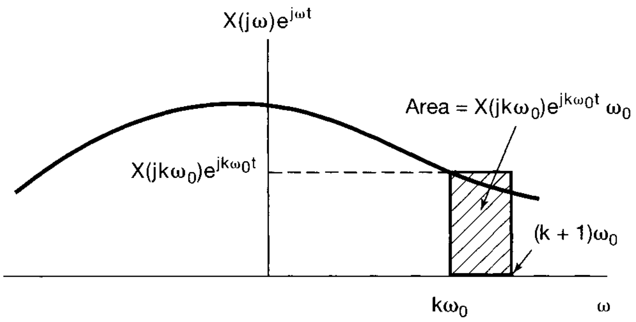


Figure 4.4 Graphical interpretation of eq. (4.7).

Equations (4.8) and (4.9) are referred to as the *Fourier transform pair,* with the function $X(j\omega)$ referred to as the *Fourier Transform* or *Fourier integral* of $x(t)$ and eq. (4.8) as the *inverse Fourier transform* equation. The *synthesis* equation (4.8) plays a role for aperiodic signals similar to that of eq. (3.38) for periodic signals, since both represent a signal as a linear combination of complex exponentials. For periodic signals, these complex exponentials have amplitudes $\{a_k\}$, as given by eq. (3.39), and occur at a discrete set of harmonically related frequencies $k\omega_0$, $k = 0, \pm 1, \pm 2, \ldots$. For aperiodic signals, the complex exponentials occur at a continuum of frequencies and, according to the synthesis equation (4.8), have "amplitude" $X(j\omega)(d\omega/2\pi)$. In analogy with the terminology used for the Fourier series coefficients of a periodic signal, the transform $X(j\omega)$ of an aperiodic signal $x(t)$ is commonly referred to as the *spectrum* of $x(t)$, as it provides us with the information needed for describing $x(t)$ as a linear combination (specifically, an integral) of sinusoidal signals at different frequencies.

Based on the above development, or equivalently on a comparison of eq. (4.9) and eq. (3.39), we also note that the Fourier coefficients a_k of a periodic signal $\tilde{x}(t)$ can be expressed in terms of equally spaced *samples* of the Fourier transform of one period of $\tilde{x}(t)$. Specifically, suppose that $\tilde{x}(t)$ is a periodic signal with period T and Fourier coefficients

a_k. Let $x(t)$ be a finite-duration signal that is equal to $\tilde{x}(t)$ over exactly one period—say, for $s \leq t \leq s+T$ for some value of s—and that is zero otherwise. Then, since eq. (3.39) allows us to compute the Fourier coefficients of $\tilde{x}(t)$ by integrating over any period, we can write

$$a_k = \frac{1}{T}\int_s^{s+T} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_s^{s+T} x(t)e^{-jk\omega_0 t}dt.$$

Since $x(t)$ is zero outside the range $s \leq t \leq s+T$ we can equivalently write

$$a_k = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

Comparing with eq. (4.9) we conclude that

$$a_k = \frac{1}{T}X(j\omega)\bigg|_{\omega = k\omega_0}, \tag{4.10}$$

where $X(j\omega)$ is the Fourier transform of $x(t)$. Equation 4.10 states that the Fourier coefficients of $\tilde{x}(t)$ are proportional to samples of the Fourier transform of one period of $\tilde{x}(t)$. This fact, which is often of use in practice, is examined further in Problem 4.37.

4.1.2 Convergence of Fourier Transforms

Although the argument we used in deriving the Fourier transform pair assumed that $x(t)$ was of arbitrary but finite duration, eqs. (4.8) and (4.9) remain valid for an extremely broad class of signals of infinite duration. In fact, our derivation of the Fourier transform suggests that a set of conditions like those required for the convergence of Fourier series should also apply here, and indeed, that can be shown to be the case.[1] Specifically, consider $X(j\omega)$ evaluated according to eq. (4.9), and let $\hat{x}(t)$ denote the signal obtained by using $X(j\omega)$ in the right-hand side of eq. (4.8). That is,

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega.$$

What we would like to know is when eq. (4.8) is valid [i.e., when is $\hat{x}(t)$ a valid representation of the original signal $x(t)$?]. If $x(t)$ has finite energy, i.e., if it is square integrable, so that

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty, \tag{4.11}$$

then we are guaranteed that $X(j\omega)$ is finite [i.e., eq. (4.9) converges] and that, with $e(t)$ denoting the error between $\hat{x}(t)$ and $x(t)$ [i.e., $e(t) = \hat{x}(t) - x(t)$],

[1]For a mathematically rigorous discussion of the Fourier transform and its properties and applications, see R. Bracewell, *The Fourier Transform and Its Applications,* 2nd ed. (New York: McGraw-Hill Book Company, 1986); A. Papoulis, *The Fourier Integral and Its Applications* (New York: McGraw-Hill Book Company, 1987); E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford: Clarendon Press, 1948); and the book by Dym and McKean referenced in footnote 2 of Chapter 3.

$$\int_{-\infty}^{+\infty} |e(t)|^2 dt = 0. \tag{4.12}$$

Equations (4.11) and (4.12) are the aperiodic counterparts of eqs. (3.51) and (3.54) for periodic signals. Thus, in a manner similar to that for periodic signals, if $x(t)$ has finite energy, then, although $x(t)$ and its Fourier representation $\hat{x}(t)$ may differ significantly at individual values of t, there is no energy in their difference.

Just as with periodic signals, there is an alternative set of conditions which are sufficient to ensure that $\hat{x}(t)$ is equal to $x(t)$ for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity. These conditions, again referred to as the Dirichlet conditions, require that:

1. $x(t)$ be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |x(t)| dt < \infty. \tag{4.13}$$

2. $x(t)$ have a finite number of maxima and minima within any finite interval.
3. $x(t)$ have a finite number of discontinuities within any finite interval. Futhermore, each of these discontinuities must be finite.

Therefore, absolutely integrable signals that are continuous or that have a finite number of discontinuities have Fourier transforms.

Although the two alternative sets of conditions that we have given are sufficient to guarantee that a signal has a Fourier transform, we will see in the next section that periodic signals, which are neither absolutely integrable nor square integrable over an *infinite* interval, can be considered to have Fourier transforms if impulse functions are permitted in the transform. This has the advantage that the Fourier series and Fourier transform can be incorporated into a common framework, which we will find to be very convenient in subsequent chapters. Before examining the point further in Section 4.2, however, let us consider several examples of the Fourier transform.

4.1.3 Examples of Continuous-Time Fourier Transforms

Example 4.1

Consider the signal

$$x(t) = e^{-at}u(t) \quad a > 0.$$

From eq. (4.9),

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty}.$$

That is,

$$X(j\omega) = \frac{1}{a + j\omega}, \quad a > 0.$$

Since this Fourier transform is complex valued, to plot it as a function of ω, we express $X(j\omega)$ in terms of its magnitude and phase:

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \quad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

Each of these components is sketched in Figure 4.5.

Note that if a is complex rather than real, then $x(t)$ is absolutely integrable as long as $\Re e\{a\} > 0$, and in this case the preceding calculation yields the same form for $X(j\omega)$. That is,

$$X(j\omega) = \frac{1}{a + j\omega}, \quad \Re e\{a\} > 0.$$

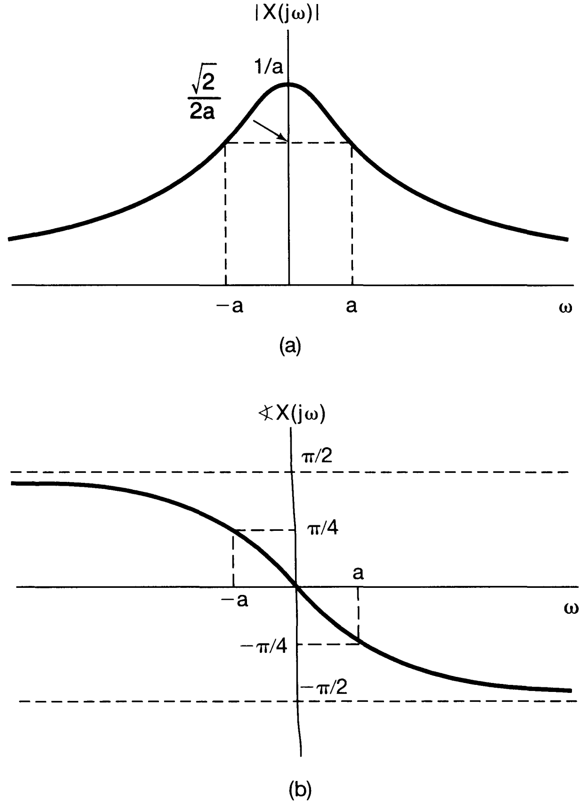


Figure 4.5 Fourier transform of the signal $x(t) = e^{-at}u(t), a > 0$, considered in Example 4.1.

Example 4.2

Let

$$x(t) = e^{-a|t|}, \quad a > 0.$$

This signal is sketched in Figure 4.6. The Fourier transform of the signal is

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|}e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$$
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$
$$= \frac{2a}{a^2+\omega^2}.$$

In this case $X(j\omega)$ is real, and it is illustrated in Figure 4.7.

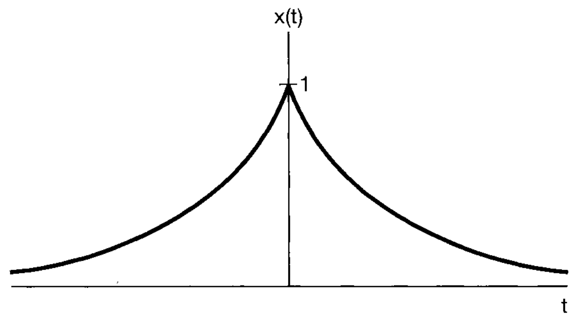


Figure 4.6 Signal $x(t) = e^{-a|t|}$ of Example 4.2.

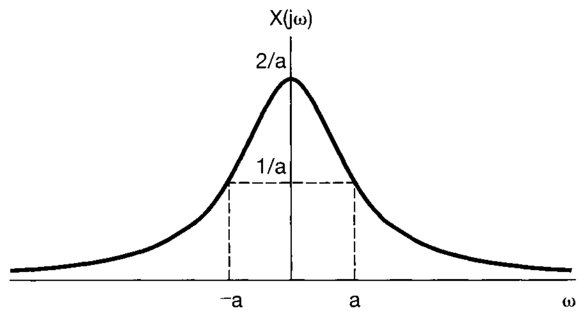


Figure 4.7 Fourier transform of the signal considered in Example 4.2 and depicted in Figure 4.6.

Example 4.3

Now let us determine the Fourier transform of the unit impulse

$$x(t) = \delta(t). \tag{4.14}$$

Substituting into eq. (4.9) yields

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}\,dt = 1. \tag{4.15}$$

That is, the unit impulse has a Fourier transform consisting of equal contributions at *all* frequencies.

Example 4.4

Consider the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}, \tag{4.16}$$

as shown in Figure 4.8(a). Applying eq. (4.9), we find that the Fourier transform of this signal is

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = 2\frac{\sin\omega T_1}{\omega}, \tag{4.17}$$

as sketched in Figure 4.8(b).

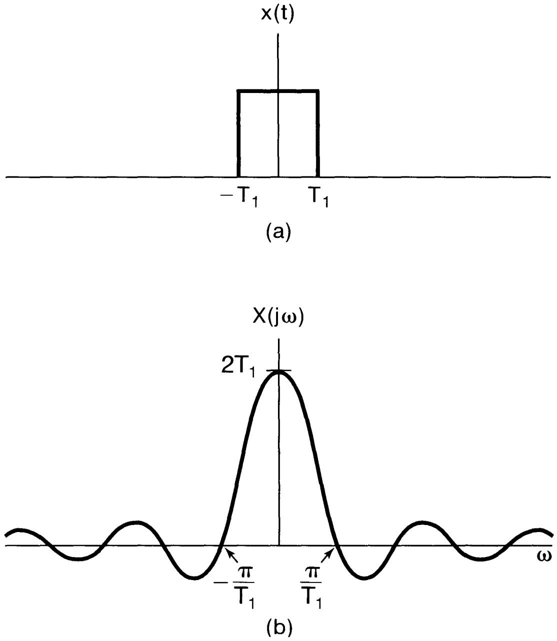


Figure 4.8 (a) The rectangular pulse signal of Example 4.4 and (b) its Fourier transform.

As we discussed at the beginning of this section, the signal given by eq. (4.16) can be thought of as the limiting form of a periodic square wave as the period becomes arbitrarily large. Therefore, we might expect that the convergence of the synthesis equation for this signal would behave in a manner similar to that observed in Example 3.5 for the square wave. This is, in fact, the case. Specifically, consider the inverse Fourier transform for the rectangular pulse signal:

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\frac{\sin\omega T_1}{\omega}e^{j\omega t}d\omega.$$

Then, since $x(t)$ is square integrable,

$$\int_{-\infty}^{+\infty} |x(t) - \hat{x}(t)|^2 \, dt = 0.$$

Furthermore, because $x(t)$ satisfies the Dirichlet conditions, $\hat{x}(t) = x(t)$, except at the points of discontinuity, $t = \pm T_1$, where $\hat{x}(t)$ converges to 1/2, which is the average of the values of $x(t)$ on both sides of the discontinuity. In addition, the convergence of $\hat{x}(t)$ to $x(t)$ exhibits the Gibbs phenomenon, much as was illustrated for the periodic square wave in Figure 3.9. Specifically, in analogy with the finite Fourier series approximation, eq. (3.47), consider the following integral over a finite-length interval of frequencies:

$$\frac{1}{2\pi} \int_{-W}^{W} 2 \frac{\sin \omega T_1}{\omega} e^{j\omega t} \, d\omega.$$

As $W \to \infty$, this signal converges to $x(t)$ everywhere, except at the discontinuities. Moreover, the signal exhibits ripples near the discontinuities. The peak amplitude of these ripples does not decrease as W increases, although the ripples do become compressed toward the discontinuity, and the energy in the ripples converges to zero.

Example 4.5

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}. \qquad (4.18)$$

This transform is illustrated in Figure 4.9(a). Using the synthesis equation (4.8), we can

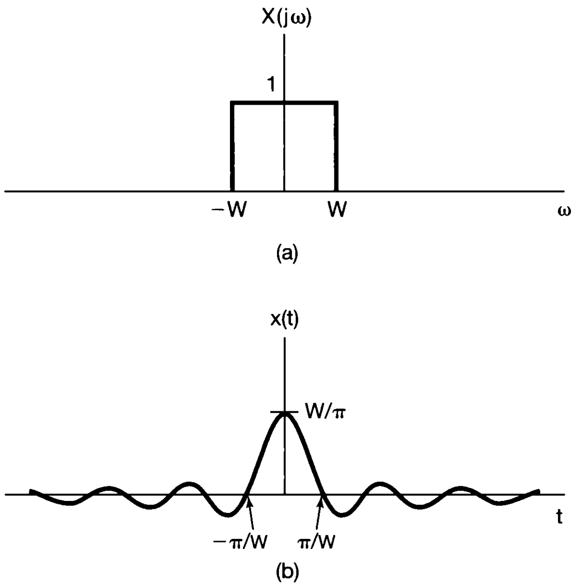


Figure 4.9 Fourier transform pair of Example 4.5: (a) Fourier transform for Example 4.5 and (b) the corresponding time function.

then determine

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{\sin Wt}{\pi t}, \tag{4.19}$$

which is depicted in Figure 4.9(b).

Comparing Figures 4.8 and 4.9 or, equivalently, eqs. (4.16) and (4.17) with eqs. (4.18) and (4.19), we see an interesting relationship. In each case, the Fourier transform pair consists of a function of the form $(\sin a\theta)/b\theta$ and a rectangular pulse. However, in Example 4.4, it is the *signal* $x(t)$ that is a pulse, while in Example 4.5, it is the *transform* $X(j\omega)$. The special relationship that is apparent here is a direct consequence of the *duality property* for Fourier transforms, which we discuss in detail in Section 4.3.6.

Functions of the form given in eqs. (4.17) and (4.19) arise frequently in Fourier analysis and in the study of LTI systems and are referred to as *sinc functions.* A commonly used precise form for the sinc function is

$$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{4.20}$$

The sinc function is plotted in Figure 4.10. Both of the signals in eqs. (4.17) and (4.19) can be expressed in terms of the sinc function:

$$\frac{2\sin\omega T_1}{\omega} = 2T_1 \,\text{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\,\text{sinc}\left(\frac{Wt}{\pi}\right).$$

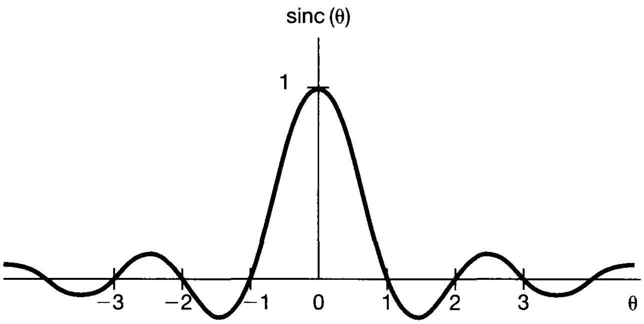


Figure 4.10 The sinc function.

Finally, we can gain insight into one other property of the Fourier transform by examining Figure 4.9, which we have redrawn as Figure 4.11 for several different values of W. From this figure, we see that as W increases, $X(j\omega)$ becomes broader, while the main peak of $x(t)$ at $t = 0$ becomes higher and the width of the first lobe of this signal (i.e., the part of the signal for $|t| < \pi/W$) becomes narrower. In fact, in the limit as $W \to \infty$, $X(j\omega) = 1$ for all ω, and consequently, from Example 4.3, we see that $x(t)$ in eq. (4.19) converges to an impulse as $W \to \infty$. The behavior depicted in Figure 4.11 is an example of the inverse relationship that exists between the time and frequency domains,

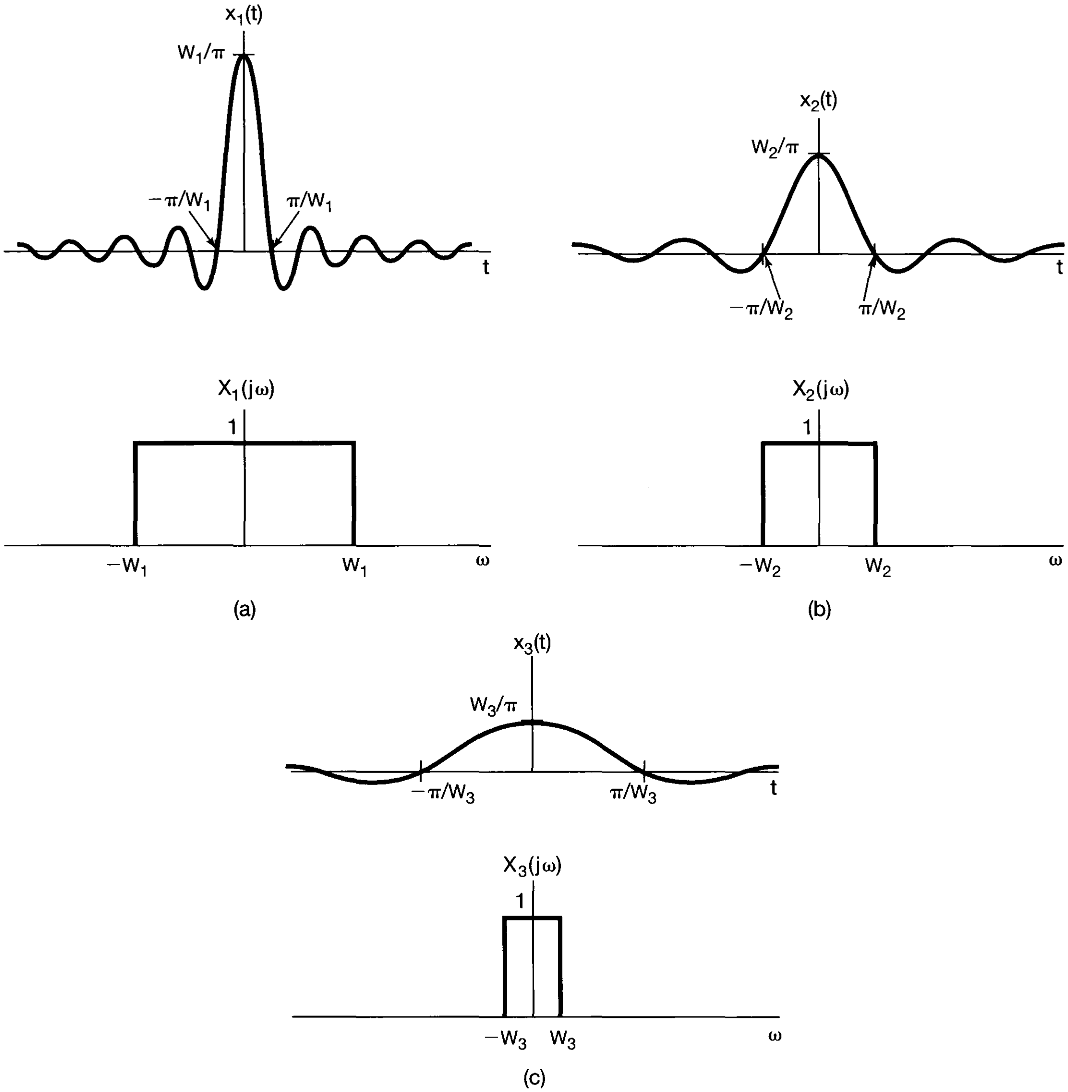


Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of W.

and we can see a similar effect in Figure 4.8, where an increase in T_1 broadens $x(t)$ but makes $X(j\omega)$ narrower. In Section 4.3.5, we provide an explanation of this behavior in the context of the scaling property of the Fourier transform.

4.2 THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

In the preceding section, we introduced the Fourier transform representation and gave several examples. While our attention in that section was focused on aperiodic signals, we can also develop Fourier transform representations for periodic signals, thus allowing us to consider both periodic and aperiodic signals within a unified context. In fact, as we will see, we can construct the Fourier transform of a periodic signal directly from its Fourier series representation. The resulting transform consists of a train of impulses in the frequency domain, with the areas of the impulses proportional to the Fourier series coefficients. This will turn out to be a very useful representation.

To suggest the general result, let us consider a signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$; that is,

$$X(j\omega) = 2\pi\delta(\omega - \omega_0). \tag{4.21}$$

To determine the signal $x(t)$ for which this is the Fourier transform, we can apply the inverse transform relation, eq. (4.8), to obtain

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega \\ &= e^{j\omega_0 t}. \end{aligned}$$

More generally, if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency, that is,

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0), \tag{4.22}$$

then the application of eq. (4.8) yields

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}. \tag{4.23}$$

We see that eq. (4.23) corresponds exactly to the Fourier *series* representation of a periodic signal, as specified by eq. (3.38). Thus, the Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the kth Fourier series coefficient a_k.

Example 4.6

Consider again the square wave illustrated in Figure 4.1. The Fourier series coefficients for this signal are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k},$$

and the Fourier transform of the signal is

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0),$$

which is sketched in Figure 4.12 for $T = 4T_1$. In comparison with Figure 3.7(a), the only differences are a proportionality factor of 2π and the use of impulses rather than a bar graph.

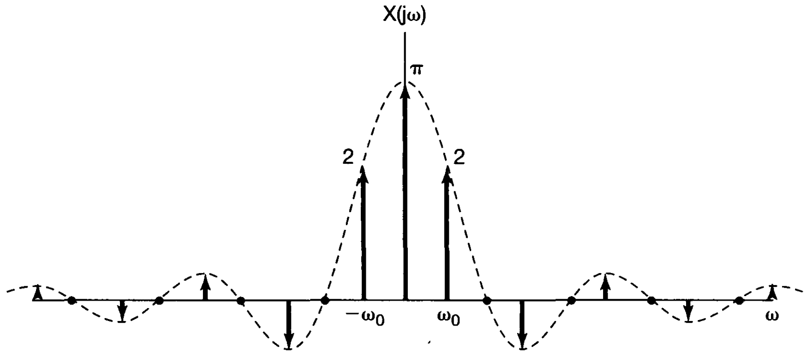


Figure 4.12 Fourier transform of a symmetric periodic square wave.

Example 4.7

Let

$$x(t) = \sin \omega_0 t.$$

The Fourier series coefficients for this signal are

$$a_1 = \frac{1}{2j},$$

$$a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, \qquad k \neq 1 \quad \text{or} \quad -1.$$

Thus, the Fourier transform is as shown in Figure 4.13(a). Similarly, for

$$x(t) = \cos \omega_0 t,$$

the Fourier series coefficients are

$$a_1 = a_{-1} = \frac{1}{2},$$

$$a_k = 0, \qquad k \neq 1 \quad \text{or} \quad -1.$$

The Fourier transform of this signal is depicted in Figure 4.13(b). These two transforms will be of considerable importance when we analyze sinusoidal modulation systems in Chapter 8.

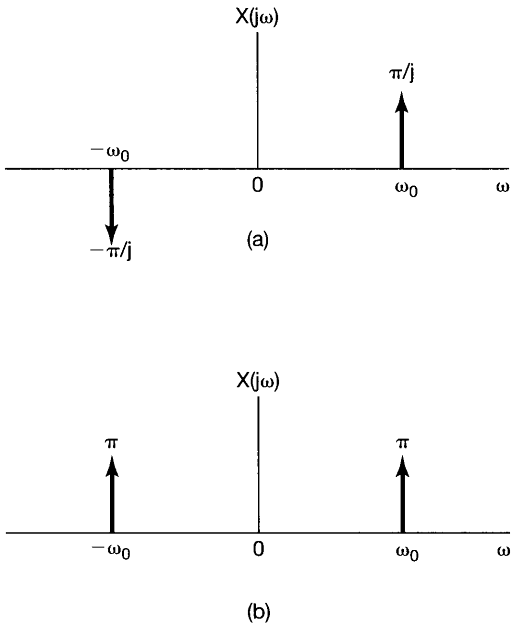


Figure 4.13 Fourier transforms of (a) $x(t) = \sin \omega_0 t$; (b) $x(t) = \cos \omega_0 t$.

Example 4.8

A signal that we will find extremely useful in our analysis of sampling systems in Chapter 7 is the impulse train

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

which is periodic with period T, as indicated in Figure 4.14(a). The Fourier series coefficients for this signal were computed in Example 3.8 and are given by

$$a_k = \frac{1}{T} \int_{-T/2}^{+T/2} \delta(t) e^{-jk\omega_0 t} \, dt = \frac{1}{T}.$$

That is, every Fourier coefficient of the periodic impulse train has the same value, $1/T$. Substituting this value for a_k in eq. (4.22) yields

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$

Thus, the Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$, as sketched in Figure 4.14(b). Here again, we see an illustration of the inverse relationship between the time and the frequency domains. As the spacing between the impulses in the time domain (i.e., the period) gets longer, the spacing between the impulses in the frequency domain (namely, the fundamental frequency) gets smaller.

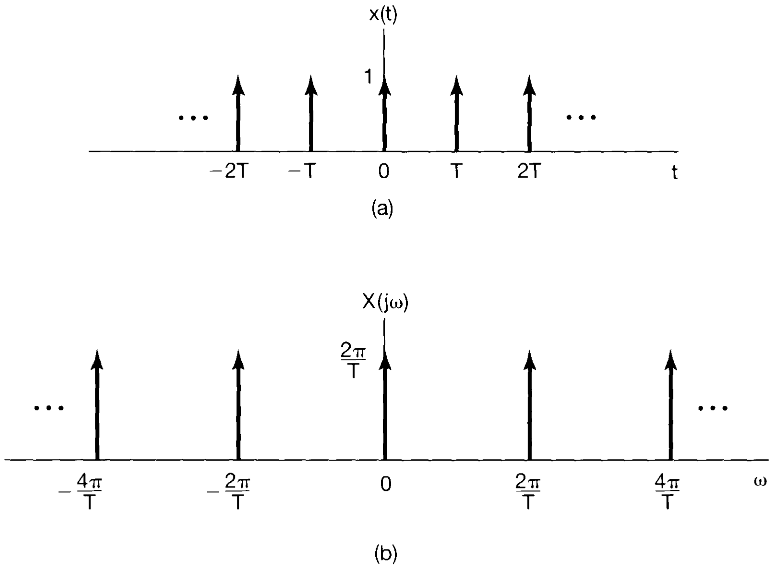


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

4.3 PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

In this and the following two sections, we consider a number of properties of the Fourier transform. A detailed listing of these properties is given in Table 4.1 in Section 4.6. As was the case for the Fourier series representation of periodic signals, these properties provide us with a significant amount of insight into the transform and into the relationship between the time-domain and frequency-domain descriptions of a signal. In addition, many of the properties are often useful in reducing the complexity of the evaluation of Fourier transforms or inverse transforms. Furthermore, as described in the preceding section, there is a close relationship between the Fourier series and Fourier transform representations of a periodic signal, and using this relationship, we can translate many of the Fourier transform properties into corresponding Fourier series properties, which we discussed independently in Chapter 3. (See, in particular, Section 3.5 and Table 3.1.)

Throughout the discussion in this section, we will be referring frequently to functions of time and their Fourier transforms, and we will find it convenient to use a shorthand notation to indicate the pairing of a signal and its transform. As developed in Section 4.1, a signal $x(t)$ and its Fourier transform $X(j\omega)$ are related by the Fourier transform synthesis and analysis equations,

[eq. (4.8)]
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \tag{4.24}$$

and

[eq. (4.9)]
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt. \tag{4.25}$$

We will sometimes find it convenient to refer to $X(j\omega)$ with the notation $\mathfrak{F}\{x(t)\}$ and to $x(t)$ with the notation $\mathfrak{F}^{-1}\{X(j\omega)\}$. We will also refer to $x(t)$ and $X(j\omega)$ as a Fourier transform pair with the notation

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega).$$

Thus, with reference to Example 4.1,

$$\frac{1}{a + j\omega} = \mathfrak{F}\{e^{-at}u(t)\},$$

$$e^{-at}u(t) = \mathfrak{F}^{-1}\left\{\frac{1}{a + j\omega}\right\},$$

and

$$e^{-at}u(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a + j\omega}.$$

4.3.1 Linearity

If

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega)$$

and

$$y(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} Y(j\omega),$$

then

$$\boxed{ax(t) + by(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega).} \tag{4.26}$$

The proof of eq. (4.26) follows directly by application of the analysis eq. (4.25) to $ax(t) + by(t)$. The linearity property is easily extended to a linear combination of an arbitrary number of signals.

4.3.2 Time Shifting

If

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

then

$$\boxed{x(t - t_0) \stackrel{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega).} \tag{4.27}$$

To establish this property, consider eq. (4.24):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Replacing t by $t - t_0$ in this equation, we obtain

$$\begin{aligned} x(t - t_0) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(e^{-j\omega t_0}X(j\omega)\right)e^{j\omega t}d\omega. \end{aligned}$$

Recognizing this as the synthesis equation for $x(t - t_0)$, we conclude that

$$\mathfrak{F}\{x(t - t_0)\} = e^{-j\omega t_0}X(j\omega).$$

One consequence of the time-shift property is that a signal which is shifted in time does not have the *magnitude* of its Fourier transform altered. That is, if we express $X(j\omega)$ in polar form as

$$\mathfrak{F}\{x(t)\} = X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)},$$

then

$$\mathfrak{F}\{x(t - t_0)\} = e^{-j\omega t_0}X(j\omega) = |X(j\omega)|e^{j[\sphericalangle X(j\omega) - \omega t_0]}.$$

Thus, the effect of a time shift on a signal is to introduce into its transform a phase shift, namely, $-\omega t_0$, which is a linear function of ω.

Example 4.9

To illustrate the usefulness of the Fourier transform linearity and time-shift properties, let us consider the evaluation of the Fourier transform of the signal $x(t)$ shown in Figure 4.15(a).

First, we observe that $x(t)$ can be expressed as the linear combination

$$x(t) = \frac{1}{2}x_1(t - 2.5) + x_2(t - 2.5),$$

where the signals $x_1(t)$ and $x_2(t)$ are the rectangular pulse signals shown in Figure 4.15(b) and (c). Then, using the result from Example 4.4, we obtain

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \quad \text{and} \quad X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}.$$

Finally, using the linearity and time-shift properties of the Fourier transform yields

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}.$$

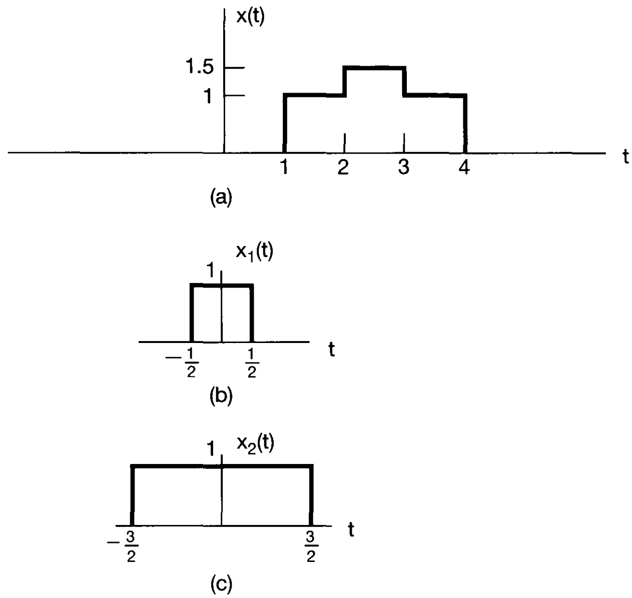


Figure 4.15 Decomposing a signal into the linear combination of two simpler signals. (a) The signal $x(t)$ for Example 4.9; (b) and (c) the two component signals used to represent $x(t)$.

4.3.3 Conjugation and Conjugate Symmetry

The conjugation property states that if

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega),$$

then

$$\boxed{x^*(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega).} \tag{4.28}$$

This property follows from the evaluation of the complex conjugate of eq. (4.25):

$$\begin{aligned} X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right]^* \\ &= \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}\,dt. \end{aligned}$$

Replacing ω by $-\omega$, we see that

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}\,dt. \tag{4.29}$$

Recognizing that the right-hand side of eq. (4.29) is the Fourier transform analysis equation for $x^*(t)$, we obtain the relation given in eq. (4.28).

The conjugation property allows us to show that if $x(t)$ is real, then $X(j\omega)$ has *conjugate symmetry;* that is,

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}].} \tag{4.30}$$

Specifically, if $x(t)$ is real so that $x^*(t) = x(t)$, we have, from eq. (4.29),

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}\,dt = X(j\omega),$$

and eq. (4.30) follows by replacing ω with $-\omega$.

From Example 4.1, with $x(t) = e^{-at}u(t)$,

$$X(j\omega) = \frac{1}{a + j\omega}$$

and

$$X(-j\omega) = \frac{1}{a - j\omega} = X^*(j\omega).$$

As one consequence of eq. (4.30), if we express $X(j\omega)$ in rectangular form as

$$X(j\omega) = \mathcal{R}e\{X(j\omega)\} + j\mathcal{I}m\{X(j\omega)\},$$

then if $x(t)$ is real,

$$\mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\}$$

and

$$\mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\}.$$

That is, the real part of the Fourier transform is an *even* function of frequency, and the imaginary part is an *odd* function of frequency. Similarly, if we express $X(j\omega)$ in polar form as

$$X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)},$$

then it follows from eq. (4.30) that $|X(j\omega)|$ is an even function of ω and $\sphericalangle X(j\omega)$ is an odd function of ω. Thus, when computing or displaying the Fourier transform of a real-valued signal, the real and imaginary parts or magnitude and phase of the transform need only be specified for positive frequencies, as the values for negative frequencies can be determined directly from the values for $\omega > 0$ using the relationships just derived.

As a further consequence of eq. (4.30), if $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even. To see this, we write

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}\,dt,$$

or, with the substitution $\tau = -t$,

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(-\tau)e^{-j\omega\tau}d\tau.$$

Since $x(-\tau) = x(\tau)$, we have

$$\begin{aligned} X(-j\omega) &= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega t}d\tau \\ &= X(j\omega). \end{aligned}$$

Thus, $X(j\omega)$ is an even function. This, together with eq. (4.30), also requires that $X^*(j\omega) = X(j\omega)$ [i.e., that $X(j\omega)$ is real]. Example 4.2 illustrates this property for the real, even signal $e^{-a|t|}$. In a similar manner, it can be shown that if $x(t)$ is a real and odd function of time, so that $x(t) = -x(-t)$, then $X(j\omega)$ is purely imaginary and odd.

Finally, as was discussed in Chapter 1, a real function $x(t)$ can always be expressed in terms of the sum of an even function $x_e(t) = \mathcal{E}v\{x(t)\}$ and an odd function $x_o(t) = \mathcal{O}d\{x(t)\}$; that is,

$$x(t) = x_e(t) + x_o(t).$$

From the linearity of the Fourier transform,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\},$$

and from the preceding discussion, $\mathcal{F}\{x_e(t)\}$ is a real function and $\mathcal{F}\{x_o(t)\}$ is purely imaginary. Thus, we can conclude that, with $x(t)$ real,

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega),$$

$$\mathcal{E}v\{x(t)\} \stackrel{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(j\omega)\},$$

$$\mathcal{O}d\{x(t)\} \stackrel{\mathcal{F}}{\longleftrightarrow} j\mathcal{I}m\{X(j\omega)\}.$$

One use of these symmetry properties is illustrated in the following example.

Example 4.10

Consider again the Fourier transform evaluation of Example 4.2 for the signal $x(t) = e^{-a|t|}$, where $a > 0$. This time we will utilize the symmetry properties of the Fourier transform to aid the evaluation process.

From Example 4.1, we have

$$e^{-at}u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{a + j\omega}.$$

Note that for $t > 0$, $x(t)$ equals $e^{-at}u(t)$, while for $t < 0$, $x(t)$ takes on mirror image values. That is,

$$\begin{aligned} x(t) &= e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) \\ &= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \\ &= 2\mathcal{E}v\{e^{-at}u(t)\}. \end{aligned}$$

Since $e^{-at}u(t)$ is real valued, the symmetry properties of the Fourier transform lead us to conclude that

$$\mathcal{E}v\{e^{-at}u(t)\} \stackrel{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\left\{\frac{1}{a + j\omega}\right\}.$$

It follows that

$$X(j\omega) = 2\mathcal{R}e\left\{\frac{1}{a + j\omega}\right\} = \frac{2a}{a^2 + \omega^2},$$

which is the same as the answer found in Example 4.2.

4.3.4 Differentiation and Integration

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Then, by differentiating both sides of the Fourier transform synthesis equation (4.24), we obtain

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega X(j\omega)e^{j\omega t}d\omega.$$

Therefore,

$$\boxed{\frac{dx(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega).} \tag{4.31}$$

This is a particularly important property, as it replaces the operation of differentiation in the time domain with that of multiplication by $j\omega$ in the frequency domain. We will find the substitution to be extremely useful in our discussion in Section 4.7 on the use of Fourier transforms for the analysis of LTI systems described by differential equations.

Since differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain, one might conclude that integration should involve division by $j\omega$ in the frequency domain. This is indeed the case, but it is only one part of the picture. The precise relationship is

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).} \tag{4.32}$$

The impulse term on the right-hand side of eq. (4.32) reflects the dc or average value that can result from integration.

The use of eqs. (4.31) and (4.32) is illustrated in the next two examples.

Example 4.11

Let us determine the Fourier transform $X(j\omega)$ of the unit step $x(t) = u(t)$, making use of eq. (4.32) and the knowledge that

$$g(t) = \delta(t) \overset{\mathfrak{F}}{\longleftrightarrow} G(j\omega) = 1.$$

Noting that

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

and taking the Fourier transform of both sides, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega),$$

where we have used the integration property listed in Table 4.1. Since $G(j\omega) = 1$, we conclude that

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega). \tag{4.33}$$

Observe that we can apply the differentiation property of eq. (4.31) to recover the transform of the impulse. That is,

$$\delta(t) = \frac{du(t)}{dt} \overset{\mathfrak{F}}{\longleftrightarrow} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1,$$

where the last equality follows from the fact that $\omega\delta(\omega) = 0$.

Example 4.12

Suppose that we wish to calculate the Fourier transform $X(j\omega)$ for the signal $x(t)$ displayed in Figure 4.16(a). Rather than applying the Fourier integral directly to $x(t)$, we instead consider the signal

$$g(t) = \frac{d}{dt}x(t).$$

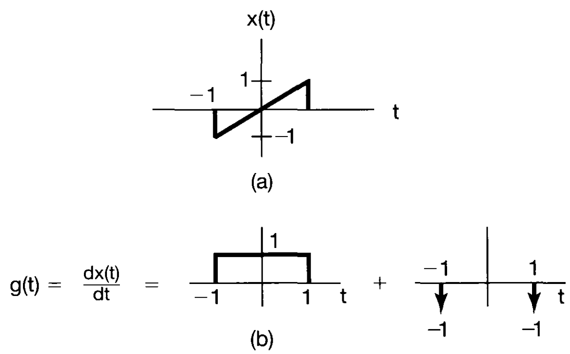


Figure 4.16 (a) A signal $x(t)$ for which the Fourier transform is to be evaluated; (b) representation of the derivative of $x(t)$ as the sum of two components.

As illustrated in Figure 4.16(b), $g(t)$ is the sum of a rectangular pulse and two impulses. The Fourier transforms of each of these component signals may be determined from Table 4.2:

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}.$$

Note that $G(0) = 0$. Using the integration property, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega).$$

With $G(0) = 0$ this becomes

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}.$$

The expression for $X(j\omega)$ is purely imaginary and odd, which is consistent with the fact that $x(t)$ is real and odd.

4.3.5 Time and Frequency Scaling

If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega),$$

then

$$\boxed{x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right),} \tag{4.34}$$

where a is a nonzero real number. This property follows directly from the definition of the Fourier transform—specifically,

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)e^{-j\omega t}dt.$$

Using the substitution $\tau = at$, we obtain

$$\mathcal{F}\{x(at)\} = \begin{cases} \dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a > 0 \\ -\dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a < 0 \end{cases},$$

which corresponds to eq. (4.34). Thus, aside from the amplitude factor $1/|a|$, a linear scaling in time by a factor of a corresponds to a linear scaling in frequency by a factor of $1/a$, and vice versa. Also, letting $a = -1$, we see from eq. (4.34) that

$$\boxed{x(-t) \overset{\mathfrak{F}}{\longleftrightarrow} X(-j\omega).} \tag{4.35}$$

That is, reversing a signal in time also reverses its Fourier transform.

A common illustration of eq. (4.34) is the effect on frequency content that results when an audiotape is recorded at one speed and played back at a different speed. If the playback speed is higher than the recording speed, corresponding to compression in time (i.e., $a > 1$), then the spectrum is expanded in frequency (i.e., the audible effect is that the playback frequencies are higher). Conversely, the signal played back will be scaled down in frequency if the playback speed is slower than the recording speed ($0 < a < 1$). For example, if a recording of the sound of a small bell ringing is played back at a reduced speed, the result will sound like the chiming of a larger and deeper sounding bell.

The scaling property is another example of the inverse relationship between time and frequency that we have already encountered on several occasions. For example, we have seen that as we increase the period of a sinusoidal signal, we decrease its frequency. Also, as we saw in Example 4.5 (see Figure 4.11), if we consider the transform

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases},$$

then as we increase W, the inverse transform of $X(j\omega)$ becomes narrower and taller and approaches an impulse as $W \to \infty$. Finally, in Example 4.8, we saw that the spacing in the frequency domain between impulses in the Fourier transform of a periodic impulse train is inversely proportional to the spacing in the time domain.

The inverse relationship between the time and frequency domains is of great importance in a variety of signal and systems contexts, including filtering and filter design, and we will encounter its consequences on numerous occasions in the remainder of the book. In addition, the reader may very well come across the implications of this property in studying a wide variety of other topics in science and engineering. One example is the uncertainty principle in physics; another is illustrated in Problem 4.49.

4.3.6 Duality

By comparing the transform and inverse transform relations given in eqs. (4.24) and (4.25), we observe that these equations are similar, but not quite identical, in form. This symmetry leads to a property of the Fourier transform referred to as *duality.* In Example 4.5, we alluded to duality when we noted the relationship that exists between the Fourier transform pairs of Examples 4.4 and 4.5. In the former example we derived the Fourier transform pair

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{\mathfrak{F}}{\longleftrightarrow} X_1(j\omega) = \frac{2 \sin \omega T_1}{\omega}, \tag{4.36}$$

while in the latter we considered the pair

$$x_2(t) \doteq \frac{\sin Wt}{\pi t} \overset{\mathfrak{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}. \tag{4.37}$$

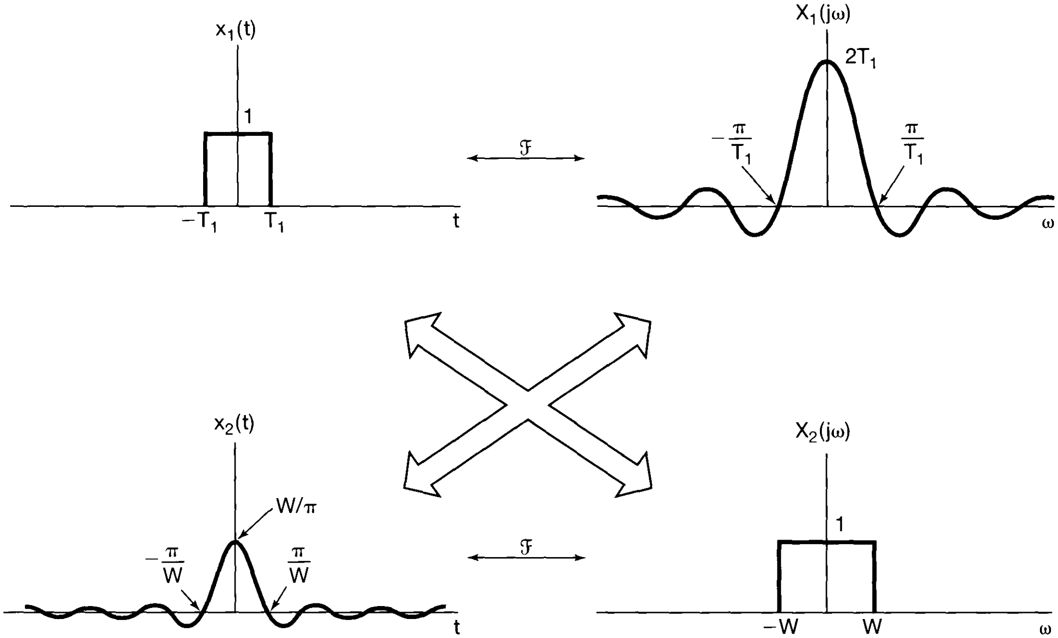


Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

The two Fourier transform pairs and the relationship between them are depicted in Figure 4.17.

The symmetry exhibited by these two examples extends to Fourier transforms in general. Specifically, because of the symmetry between eqs. (4.24) and (4.25), for any transform pair, there is a dual pair with the time and frequency variables interchanged. This is best illustrated through an example.

Example 4.13

Let us consider using duality to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

In Example 4.2 we encountered a Fourier transform pair in which the Fourier transform, as a function of ω, had a form similar to that of the signal $x(t)$. Specifically, suppose we consider a signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \frac{2}{1+\omega^2}.$$

Then, from Example 4.2,

$$x(t) = e^{-|t|} \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}.$$

The synthesis equation for this Fourier transform pair is

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega.$$

Multiplying this equation by 2π and replacing t by $-t$, we obtain

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega.$$

Now, interchanging the names of the variables t and ω, we find that

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt. \tag{4.38}$$

The right-hand side of eq. (4.38) is the Fourier transform analysis equation for $2/(1+t^2)$, and thus, we conclude that

$$\mathfrak{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}.$$

The duality property can also be used to determine or to suggest other properties of Fourier transforms. Specifically, if there are characteristics of a function of time that have implications with regard to the Fourier transform, then the same characteristics associated with a function of frequency will have *dual* implications in the time domain. For example, in Section 4.3.4, we saw that differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain. From the preceding discussion, we might then suspect that multiplication by jt in the time domain corresponds roughly to differentiation in the frequency domain. To determine the precise form of this dual property, we can proceed in a fashion exactly analogous to that used in Section 4.3.4. Thus, if we differentiate the analysis equation (4.25) with respect to ω, we obtain

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} -jtx(t)e^{-j\omega t}dt. \tag{4.39}$$

That is,

$$\boxed{-jtx(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.} \tag{4.40}$$

Similarly, we can derive the dual properties of eqs. (4.27) and (4.32):

$$\boxed{e^{j\omega_0 t}x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j(\omega-\omega_0))} \tag{4.41}$$

and

$$\boxed{-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \overset{\mathfrak{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)d\eta.} \tag{4.42}$$

4.3.7 Parseval's Relation

If $x(t)$ and $X(j\omega)$ are a Fourier transform pair, then

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega. \tag{4.43}$$

This expression, referred to as Parseval's relation, follows from direct application of the Fourier transform. Specifically,

$$\begin{aligned}\int_{-\infty}^{+\infty} |x(t)|^2 dt &= \int_{-\infty}^{+\infty} x(t)x^*(t)dt \\ &= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt.\end{aligned}$$

Reversing the order of integration gives

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)\left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]d\omega.$$

The bracketed term is simply the Fourier transform of $x(t)$; thus,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.$$

The term on the left-hand side of eq. (4.43) is the total energy in the signal $x(t)$. Parseval's relation says that this total energy may be determined either by computing the energy per unit time ($|x(t)|^2$) and integrating over all time or by computing the energy per unit frequency ($|X(j\omega)|^2/2\pi$) and integrating over all frequencies. For this reason, $|X(j\omega)|^2$ is often referred to as the *energy-density spectrum* of the signal $x(t)$. (See also Problem 4.45.) Note that Parseval's relation for finite-energy signals is the direct counterpart of Parseval's relation for periodic signals (eq. 3.67), which states that the average *power* of a periodic signal equals the sum of the average powers of its individual harmonic components, which in turn are equal to the squared magnitudes of the Fourier series coefficients.

Parseval's relation and other Fourier transform properties are often useful in determining some time domain characteristics of a signal directly from the Fourier transform. The next example is a simple illustration of this.

Example 4.14

For each of the Fourier transforms shown in Figure 4.18, we wish to evaluate the following time-domain expressions:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$D = \left.\frac{d}{dt}x(t)\right|_{t=0}$$

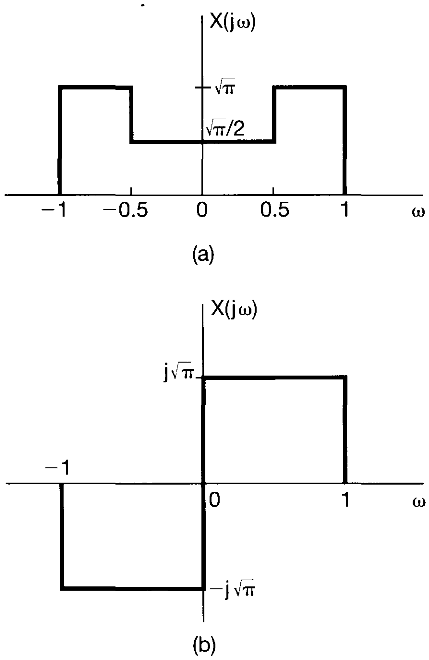


Figure 4.18 The Fourier transforms considered in Example 4.14.

To evaluate E in the frequency domain, we may use Parseval's relation. That is,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \tag{4.44}$$

which evaluates to $\frac{5}{8}$ for Figure 4.18(a) and to 1 for Figure 4.18(b).

To evaluate D in the frequency domain, we first use the differentiation property to observe that

$$g(t) = \frac{d}{dt}x(t) \overset{\mathfrak{F}}{\longleftrightarrow} j\omega X(j\omega) = G(j\omega).$$

Noting that

$$D = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)d\omega \tag{4.45}$$

we conclude:

$$D = \int_{-\infty}^{\infty} j\omega X(j\omega)\, d\omega \tag{4.46}$$

which evaluates to zero for figure 4.18(a) and to $\frac{-1}{(2\sqrt{\pi})}$ for Figure 4.18(b).

There are many other properties of the Fourier transform in addition to those we have already discussed. In the next two sections, we present two specific properties that play particularly central roles in the study of LTI systems and their applications. The first of these, discussed in Section 4.4, is referred to as the *convolution property,* which is central to many signals and systems applications, including filtering. The second, discussed in Section 4.5, is referred to as the *multiplication property,* and it provides the foundation for our discussion of sampling in Chapter 7 and amplitude modulation in Chapter 8. In Section 4.6, we summarize the properties of the Fourier transform.

4.4 THE CONVOLUTION PROPERTY

As we saw in Chapter 3, if a periodic signal is represented in a Fourier series—i.e., as a linear combination of harmonically related complex exponentials, as in eq. (3.38)—then the response of an LTI system to this input can also be represented by a Fourier series. Because complex exponentials are eigenfunctions of LTI systems, the Fourier series coefficients of the output are those of the input multiplied by the frequency response of the system evaluated at the corresponding harmonic frequencies.

In this section, we extend this result to the situation in which the signals are aperiodic. We first derive the property somewhat informally, to build on the intuition we developed for periodic signals in Chapter 3, and then provide a brief, formal derivation starting directly from the convolution integral.

Recall our interpretation of the Fourier transform synthesis equation as an expression for $x(t)$ as a linear combination of complex exponentials. Specifically, referring back to eq. (4.7), $x(t)$ is expressed as the limit of a sum; that is,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega = \lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0. \tag{4.47}$$

As developed in Sections 3.2 and 3.8, the response of a linear system with impulse response $h(t)$ to a complex exponential $e^{jk\omega_0 t}$ is $H(jk\omega_0)e^{jk\omega_0 t}$, where

$$H(jk\omega_0) = \int_{-\infty}^{+\infty} h(t)e^{-jk\omega_0 t}dt. \tag{4.48}$$

We can recognize the frequency response $H(j\omega)$, as defined in eq. (3.121), as the Fourier transform of the system impulse response. In other words, the Fourier transform of the impulse response (evaluated at $\omega = k\omega_0$) is the complex scaling factor that the LTI system applies to the eigenfunction $e^{jk\omega_0 t}$. From superposition [see eq. (3.124)], we then have

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \longrightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0,$$

and thus, from eq. (4.47), the response of the linear system to $x(t)$ is

$$\begin{aligned} y(t) &= \lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0 \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)H(j\omega)e^{j\omega t}d\omega. \end{aligned} \tag{4.49}$$

Since $y(t)$ and its Fourier transform $Y(j\omega)$ are related by

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j\omega)e^{j\omega t}d\omega, \tag{4.50}$$

we can identify $Y(j\omega)$ from eq. (4.49), yielding

$$Y(j\omega) = X(j\omega)H(j\omega). \tag{4.51}$$

As a more formal derivation, we consider the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \tag{4.52}$$

We desire $Y(j\omega)$, which is

$$Y(j\omega) = \mathfrak{F}\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt. \tag{4.53}$$

Interchanging the order of integration and noting that $x(\tau)$ does not depend on t, we have

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau. \tag{4.54}$$

By the time-shift property, eq. (4.27), the bracketed term is $e^{-j\omega\tau}H(j\omega)$. Substituting this into eq. (4.54) yields

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau = H(j\omega)\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau. \tag{4.55}$$

The integral is $X(j\omega)$, and hence,

$$Y(j\omega) = H(j\omega)X(j\omega).$$

That is,

$$\boxed{y(t) = h(t) * x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega).} \tag{4.56}$$

Equation (4.56) is of major importance in signal and system analysis. As expressed in this equation, the Fourier transform maps the convolution of two signals into the product of their Fourier transforms. $H(j\omega)$, the Fourier transform of the impulse response, is the frequency response as defined in eq. (3.121) and captures the change in complex amplitude of the Fourier transform of the input at each frequency ω. For example, in frequency-selective filtering we may want to have $H(j\omega) \approx 1$ over one range of frequencies, so that the frequency components in this band experience little or no attenuation or change due to the system, while over another range of frequencies we may want to have $H(j\omega) \approx 0$, so that components in this range are eliminated or significantly attenuated.

The frequency response $H(j\omega)$ plays as important a role in the analysis of LTI systems as does its inverse transform, the unit impulse response. For one thing, since $h(t)$ completely characterizes an LTI system, then so must $H(j\omega)$. In addition, many of the properties of LTI systems can be conveniently interpreted in terms of $H(j\omega)$. For example, in Section 2.3, we saw that the impulse response of the cascade of two LTI systems is the convolution of the impulse responses of the individual systems and that the overall impulse response does not depend on the order in which the systems are cascaded. Using eq. (4.56), we can rephrase this in terms of frequency responses. As illustrated in Figure 4.19, since the impulse response of the cascade of two LTI systems is the convolution of the individual impulse responses, the convolution property then implies that the overall frequency response of the cascade of two systems is simply the product of the individual frequency responses. From this observation, it is then clear that the overall frequency response does not depend on the order of the cascade.

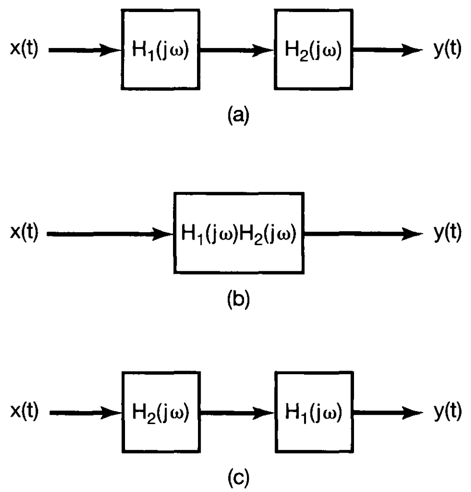


Figure 4.19 Three equivalent LTI systems. Here, each block represents an LTI system with the indicated frequency response.

As discussed in Section 4.1.2, convergence of the Fourier transform is guaranteed only under certain conditions, and consequently, the frequency response cannot be defined for every LTI system. If, however, an LTI system is stable, then, as we saw in Section 2.3.7 and Problem 2.49, its impulse response is absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |h(t)|dt < \infty. \tag{4.57}$$

Equation (4.57) is one of the three Dirichlet conditions that together guarantee the existence of the Fourier transform $H(j\omega)$ of $h(t)$. Thus, assuming that $h(t)$ satisfies the other two conditions, as essentially all signals of physical or practical significance do, we see that a stable LTI system has a frequency response $H(j\omega)$.

In using Fourier analysis to study LTI systems, we will be restricting ourselves to systems whose impulse responses possess Fourier transforms. In order to use transform techniques to examine unstable LTI systems we will develop a generalization of the continuous-time Fourier transform, the Laplace transform. We defer this discussion to Chapter 9, and until then we will consider the many problems and practical applications that we can analyze using the Fourier transform.

4.4.1 Examples

To illustrate the convolution property and its applications further, let us consider several examples.

Example 4.15

Consider a continuous-time LTI system with impulse response

$$h(t) = \delta(t - t_0). \tag{4.58}$$

The frequency response of this system is the Fourier transform of $h(t)$ and is given by

$$H(j\omega) = e^{-j\omega t_0}. \tag{4.59}$$

Thus, for any input $x(t)$ with Fourier transform $X(j\omega)$, the Fourier transform of the output is

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_0}X(j\omega). \end{aligned} \tag{4.60}$$

This result, in fact, is consistent with the time-shift property of Section 4.3.2. Specifically, a system for which the impulse response is $\delta(t - t_0)$ applies a time shift of t_0 to the input—that is,

$$y(t) = x(t - t_0).$$

Thus, the shifting property given in eq. (4.27) also yields eq. (4.60). Note that, either from our discussion in Section 4.3.2 or directly from eq. (4.59), the frequency response of a system that is a pure time shift has unity magnitude at all frequencies (i.e., $|e^{-j\omega t_0}| = 1$) and has a phase characteristic $-\omega t_0$ that is a linear function of ω.

Example 4.16

As a second example, let us examine a differentiator—that is, an LTI system for which the input $x(t)$ and the output $y(t)$ are related by

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property of Section 4.3.4,

$$Y(j\omega) = j\omega X(j\omega). \tag{4.61}$$

Consequently, from eq. (4.56), it follows that the frequency response of a differentiator is

$$H(j\omega) = j\omega. \tag{4.62}$$

Example 4.17

Consider an integrator—that is, an LTI system specified by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

The impulse response for this system is the unit step $u(t)$, and therefore, from Example 4.11 and eq. (4.33), the frequency response of the system is

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Then using eq. (4.56), we have

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(j\omega)\delta(\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega), \end{aligned}$$

which is consistent with the integration property of eq. (4.32).

Example 4.18

As we discussed in Section 3.9.2, frequency-selective filtering is accomplished with an LTI system whose frequency response $H(j\omega)$ passes the desired range of frequencies and significantly attenuates frequencies outside that range. For example, consider the ideal lowpass filter introduced in Section 3.9.2, which has the frequency reponse illustrated in Figure 4.20 and given by

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}. \tag{4.63}$$

Now that we have developed the Fourier transform representation, we know that the impulse response $h(t)$ of this ideal filter is the inverse transform of eq. (4.63). Using the result in Example 4.5, we then have

$$h(t) = \frac{\sin \omega_c t}{\pi t}, \tag{4.64}$$

which is plotted in Figure 4.21.

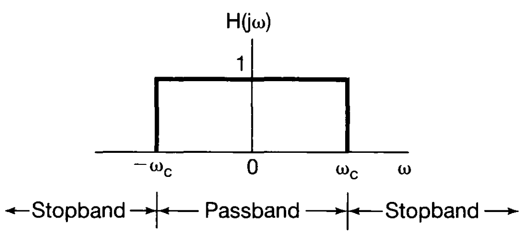


Figure 4.20 Frequency response of an ideal lowpass filter.

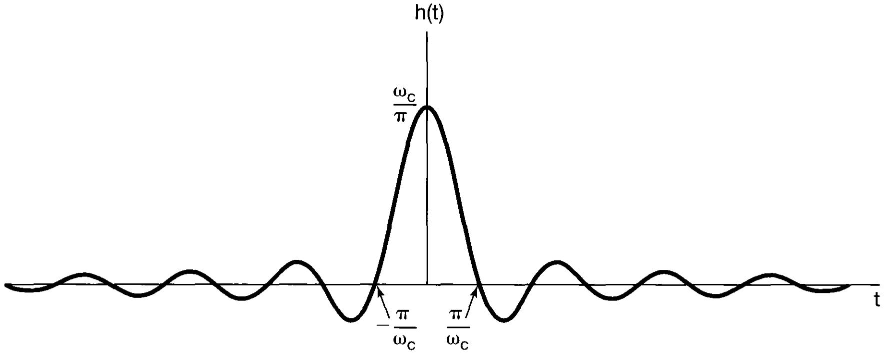


Figure 4.21 Impulse response of an ideal lowpass filter.

From Example 4.18, we can begin to see some of the issues that arise in filter design that involve looking in both the time and frequency domains. In particular, while the ideal lowpass filter does have perfect frequency selectivity, its impulse response has some characteristics that may not be desirable. First, note that $h(t)$ is not zero for $t < 0$. Consequently, the ideal lowpass filter is not causal, and thus, in applications requiring causal systems, the ideal filter is not an option. Moreover, as we discuss in Chapter 6, even if causality is not an essential constraint, the ideal filter is not easy to approximate closely, and nonideal filters that are more easily implemented are typically preferred. Furthermore, in some applications (such as the automobile suspension system discussed in Section 6.7.1), oscillatory behavior in the impulse response of a lowpass filter may be undesirable. In such applications the time domain characteristics of the ideal lowpass filter, as shown in Figure 4.21, may be unacceptable, implying that we may need to trade off frequency-domain characteristics such as ideal frequency selectivity with time-domain properties.

For example, consider the LTI system with impulse response

$$h(t) = e^{-t}u(t). \tag{4.65}$$

The frequency response of this system is

$$H(j\omega) = \frac{1}{j\omega + 1}. \tag{4.66}$$

Comparing eqs. (3.145) and (4.66), we see that this system can be implemented with the simple RC circuit discussed in Section 3.10. The impulse response and the magnitude of the frequency response are shown in Figure 4.22. While the system does not have the strong frequency selectivity of the ideal lowpass filter, it is causal and has an impulse response that decays monotonically, i.e., without oscillations. This filter or somewhat more complex ones corresponding to higher order differential equations are quite frequently preferred to ideal filters because of their causality, ease of implementation, and flexibility in allowing trade-offs, among other design considerations such as frequency selectivity and oscillatory behavior in the time domain. Many of these issues will be discussed in more detail in Chapter 6.

The convolution property is often useful in evaluating the convolution integral—i.e., in computing the response of LTI systems. This is illustrated in the next example.

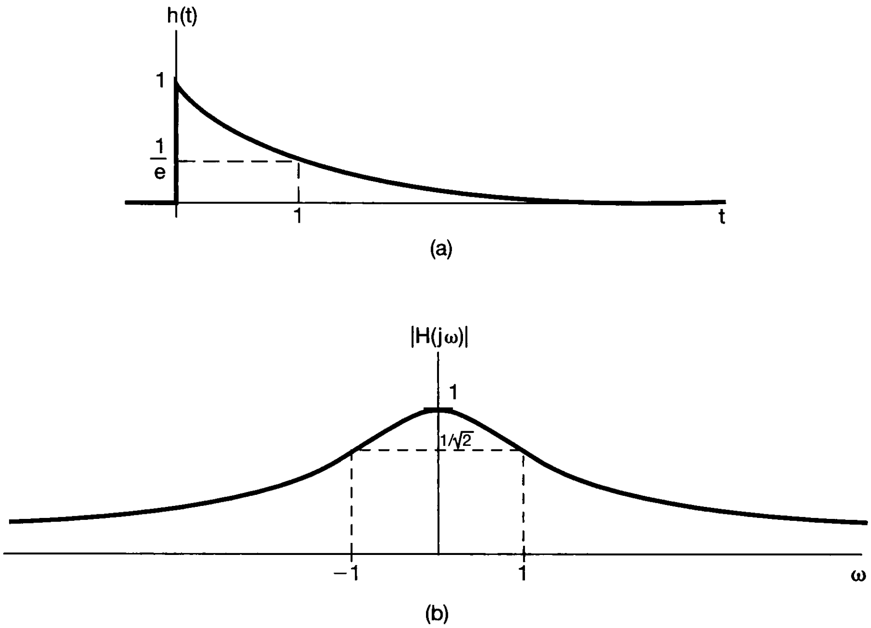


Figure 4.22 (a) Impulse response of the LTI system in eq. (4.65); (b) magnitude of the frequency response of the system.

Example 4.19

Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t), \qquad a > 0,$$

to the input signal

$$x(t) = e^{-bt}u(t), \qquad b > 0.$$

Rather than computing $y(t) = x(t) * h(t)$ directly, let us transform the problem into the frequency domain. From Example 4.1, the Fourier transforms of $x(t)$ and $h(t)$ are

$$X(j\omega) = \frac{1}{b + j\omega}$$

and

$$H(j\omega) = \frac{1}{a + j\omega}.$$

Therefore,

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}. \tag{4.67}$$

To determine the output $y(t)$, we wish to obtain the inverse transform of $Y(j\omega)$. This is most simply done by expanding $Y(j\omega)$ in a partial-fraction expansion. Such expansions are extremely useful in evaluating inverse transforms, and the general method for performing a partial-fraction expansion is developed in the appendix. For this example, assuming that $b \neq a$, the partial fraction expansion for $Y(j\omega)$ takes the form

$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega}, \tag{4.68}$$

where A and B are constants to be determined. One way to find A and B is to equate the right-hand sides of eqs. (4.67) and (4.68), multiply both sides by $(a + j\omega)(b + j\omega)$, and solve for A and B. Alternatively, in the appendix we present a more general and efficient method for computing the coefficients in partial-fraction expansions such as eq. (4.68). Using either of these approaches, we find that

$$A = \frac{1}{b - a} = -B,$$

and therefore,

$$Y(j\omega) = \frac{1}{b - a}\left[\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right]. \tag{4.69}$$

The inverse transform for each of the two terms in eq. (4.69) can be recognized by inspection. Using the linearity property of Section 4.3.1, we have

$$y(t) = \frac{1}{b - a}[e^{-at}u(t) - e^{-bt}u(t)].$$

When $b = a$, the partial fraction expansion of eq. (4.69) is not valid. However, with $b = a$, eq. (4.67) becomes

$$Y(j\omega) = \frac{1}{(a + j\omega)^2}.$$

Recognizing this as

$$\frac{1}{(a + j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right],$$

we can use the dual of the differentiation property, as given in eq. (4.40). Thus,

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a + j\omega}$$

$$te^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right] = \frac{1}{(a + j\omega)^2},$$

and consequently,

$$y(t) = te^{-at}u(t).$$

Example 4.20

As another illustration of the usefulness of the convolution property, let us consider the problem of determining the response of an ideal lowpass filter to an input signal $x(t)$ that has the form of a sinc function. That is,

$$x(t) = \frac{\sin \omega_i t}{\pi t}.$$

Of course, the impulse response of the ideal lowpass filter is of a similar form, namely,

$$h(t) = \frac{\sin \omega_c t}{\pi t}.$$

The filter output $y(t)$ will therefore be the convolution of two sinc functions, which, as we now show, also turns out to be a sinc function. A particularly convenient way of deriving this result is to first observe that

$$Y(j\omega) = X(j\omega)H(j\omega),$$

where

$$X(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_i \\ 0 & \text{elsewhere} \end{cases}$$

and

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{elsewhere} \end{cases}.$$

Therefore,

$$Y(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & \text{elsewhere} \end{cases},$$

where ω_0 is the smaller of the two numbers ω_i and ω_c. Finally, the inverse Fourier transform of $Y(j\omega)$ is given by

$$y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t} & \text{if } \omega_c \leq \omega_i \\ \dfrac{\sin \omega_i t}{\pi t} & \text{if } \omega_i \leq \omega_c \end{cases}.$$

That is, depending upon which of ω_c and ω_i is smaller, the output is equal to either $x(t)$ or $h(t)$.

4.5 THE MULTIPLICATION PROPERTY

The convolution property states that convolution in the *time* domain corresponds to multiplication in the *frequency* domain. Because of duality between the time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$\boxed{r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta} \tag{4.70}$$

This can be shown by exploiting duality as discussed in Section 4.3.6, together with the convolution property, or by directly using the Fourier transform relations in a manner analogous to the procedure used in deriving the convolution property.

Multiplication of one signal by another can be thought of as using one signal to scale or *modulate* the amplitude of the other, and consequently, the multiplication of two signals is often referred to as *amplitude modulation.* For this reason, eq. (4.70) is sometimes referred to as the *modulation property.* As we shall see in Chapters 7 and 8, this property has several very important applications. To illustrate eq. (4.70), and to suggest one of the applications that we will discuss in subsequent chapters, let us consider several examples.

Example 4.21

Let $s(t)$ be a signal whose spectrum $S(j\omega)$ is depicted in Figure 4.23(a). Also, consider the signal

$$p(t) = \cos \omega_0 t.$$

Then

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0),$$

as sketched in Figure 4.23(b), and the spectrum $R(j\omega)$ of $r(t) = s(t)p(t)$ is obtained by

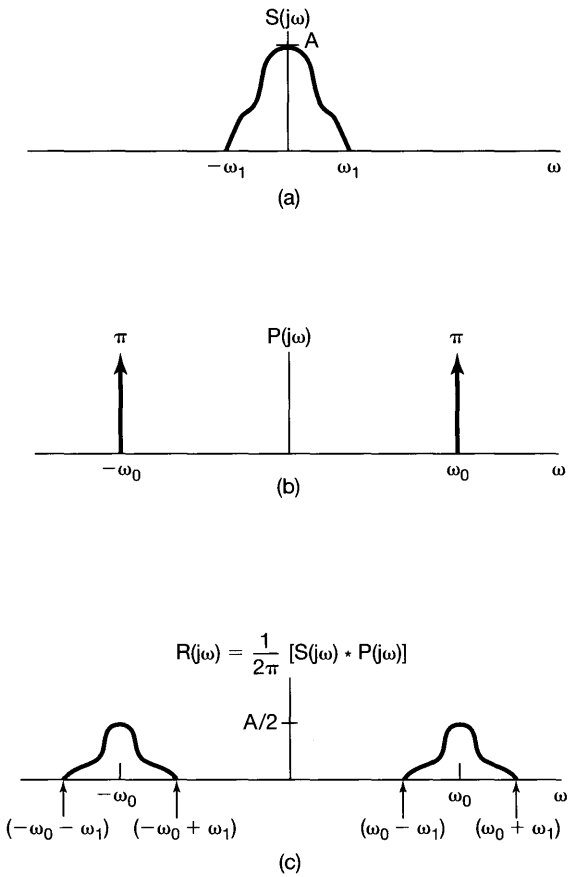


Figure 4.23 Use of the multiplication property in Example 4.21: (a) the Fourier transform of a signal $s(t)$; (b) the Fourier transform of $p(t) = \cos \omega_0 t$; (c) the Fourier transform of $r(t) = s(t)p(t)$.

an application of eq. (4.70), yielding

$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$
$$= \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0)), \quad (4.71)$$

which is sketched in Figure 4.23(c). Here we have assumed that $\omega_0 > \omega_1$, so that the two nonzero portions of $R(j\omega)$ do not overlap. Clearly, the spectrum of $r(t)$ consists of the sum of two shifted and scaled versions of $S(j\omega)$.

From eq. (4.71) and from Figure 4.23, we see that all of the information in the signal $s(t)$ is preserved when we multiply this signal by a sinusoidal signal, although the information has been shifted to higher frequencies. This fact forms the basis for sinusoidal amplitude modulation systems for communications. In the next example, we learn how we can recover the original signal $s(t)$ from the amplitude-modulated signal $r(t)$.

Example 4.22

Let us now consider $r(t)$ as obtained in Example 4.21, and let

$$g(t) = r(t)p(t),$$

where, again, $p(t) = \cos\omega_0 t$. Then, $R(j\omega)$, $P(j\omega)$, and $G(j\omega)$ are as shown in Figure 4.24.

From Figure 4.24(c) and the linearity of the Fourier transform, we see that $g(t)$ is the sum of $(1/2)s(t)$ and a signal with a spectrum that is nonzero only at higher frequen-

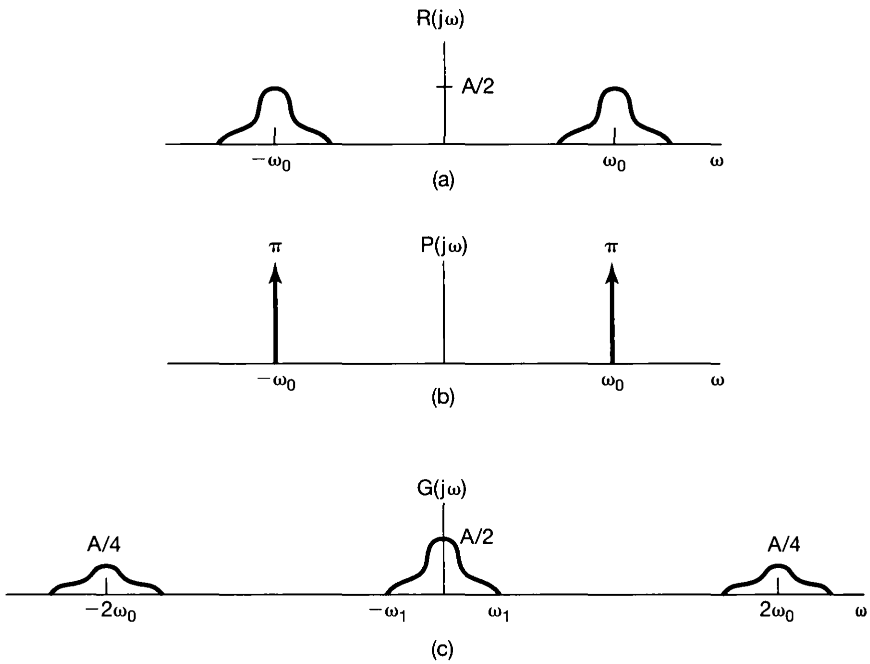


Figure 4.24 Spectra of signals considered in Example 4.22: (a) $R(j\omega)$; (b) $P(j\omega)$; (c) $G(j\omega)$.

cies (centered around $\pm 2\omega_0$). Suppose then that we apply the signal $g(t)$ as the input to a frequency-selective lowpass filter with frequency response $H(j\omega)$ that is constant at low frequencies (say, for $|\omega| < \omega_1$) and zero at high frequencies (for $|\omega| > \omega_1$). Then the output of this system will have as its spectrum $H(j\omega)G(j\omega)$, which, because of the particular choice of $H(j\omega)$, will be a scaled replica of $S(j\omega)$. Therefore, the output itself will be a scaled version of $s(t)$. In Chapter 8, we expand significantly on this idea as we develop in detail the fundamentals of amplitude modulation.

Example 4.23

Another illustration of the usefulness of the Fourier transform multiplication property is provided by the problem of determining the Fourier transform of the signal

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}.$$

The key here is to recognize $x(t)$ as the product of two sinc functions:

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right).$$

Applying the multiplication property of the Fourier transform, we obtain

$$X(j\omega) = \frac{1}{2}\mathfrak{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathfrak{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}.$$

Noting that the Fourier transform of each sinc function is a rectangular pulse, we can proceed to convolve those pulses to obtain the function $X(j\omega)$ displayed in Figure 4.25.

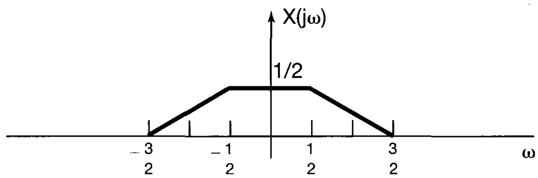


Figure 4.25 The Fourier transform of $x(t)$ in Example 4.23.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

As suggested in Examples 4.21 and 4.22 and developed more fully in Chapter 8, one of the important applications of the multiplication property is amplitude modulation in communication systems. Another important application is in the implementation of frequency-selective bandpass filters with tunable center frequencies that can be adjusted by the simple turn of a dial. In a frequency-selective bandpass filter built with elements such as resistors, operational amplifiers, and capacitors, the center frequency depends on a number of element values, all of which must be varied simultaneously in the correct way if the center frequency is to be adjusted directly. This is generally difficult and cumbersome in comparison with building a filter whose characteristics are fixed. An alternative to directly varying the filter characteristics is to use a fixed frequency-selective filter and shift the spectrum of the signal appropriately, using the principles of sinusoidal amplitude modulation.

For example, consider the system shown in Figure 4.26. Here, an input signal $x(t)$ is multiplied by the complex exponential signal $e^{j\omega_c t}$. The resulting signal is then passed through a lowpass filter with cutoff frequency ω_0, and the output is multiplied by $e^{-j\omega_c t}$. The spectra of the signals $x(t)$, $y(t)$, $w(t)$, and $f(t)$ are illustrated in Figure 4.27.

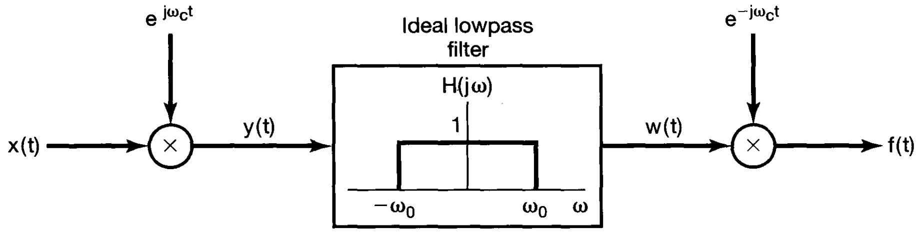


Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

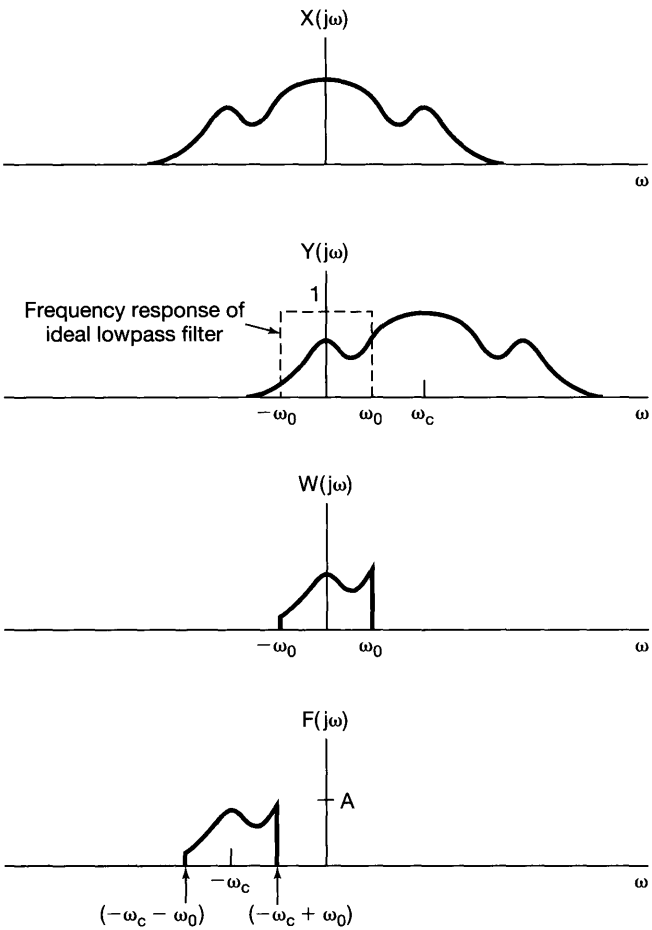


Figure 4.27 Spectra of the signals in the system of Figure 4.26.

Specifically, from either the multiplication property or the frequency-shifting property it follows that the Fourier transform of $y(t) = e^{j\omega_c t}x(t)$ is

$$Y(j\omega) = \int_{-\infty}^{+\infty} \delta(\theta - \omega_c)X(\omega - \theta)d\theta$$

so that $Y(j\omega)$ equals $X(j\omega)$ shifted to the right by ω_c and frequencies in $X(j\omega)$ near $\omega = \omega_c$ have been shifted into the passband of the lowpass filter. Similarly, the Fourier transform of $f(t) = e^{-j\omega_c t}w(t)$ is

$$F(j\omega) = W(j(\omega + \omega_0)),$$

so that the Fourier transform of $F(j\omega)$ is $W(j\omega)$ shifted to the left by ω_c. From Figure 4.27, we observe that the overall system of Figure 4.26 is equivalent to an ideal bandpass filter with center frequency $-\omega_c$ and bandwidth $2\omega_0$, as illustrated in Figure 4.28. As the frequency ω_c of the complex exponential oscillator is varied, the center frequency of the bandpass filter varies.

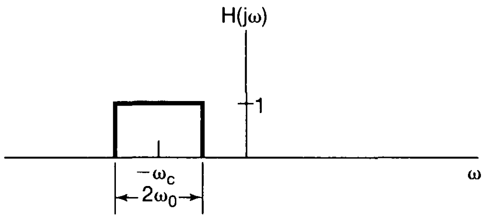


Figure 4.28 Bandpass filter equivalent of Figure 4.26.

In the system of Figure 4.26 with $x(t)$ real, the signals $y(t)$, $w(t)$, and $f(t)$ are all complex. If we retain only the real part of $f(t)$, the resulting spectrum is that shown in Figure 4.29, and the equivalent bandpass filter passes bands of frequencies centered around ω_c and $-\omega_c$, as indicated in Figure 4.30. Under certain conditions, it is also possible to use sinusoidal rather than complex exponential modulation to implement the system of the latter figure. This is explored further in Problem 4.46.

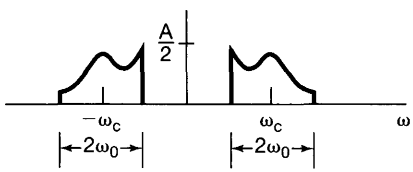


Figure 4.29 Spectrum of $\mathcal{R}e\{f(t)\}$ associated with Figure 4.26.

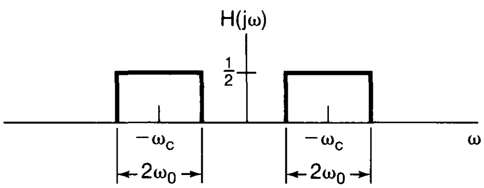


Figure 4.30 Equivalent bandpass filter for $\mathcal{R}e\{f(t)\}$ in Figure 4.29.

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty} \|x(t)\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|X(j\omega)\|^2 d\omega$		

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1,\quad a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Fourier analysis in our examination of signals and systems. All of the transform pairs, except for the last one in the table, have been considered in examples in the preceding sections. The last pair is considered in Problem 4.40. In addition, note that several of the signals in Table 4.2 are periodic, and for these we have also listed the corresponding Fourier series coefficients.

4.7 SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENTIAL EQUATIONS

As we have discussed on several occasions, a particularly important and useful class of continuous-time LTI systems is those for which the input and output satisfy a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{4.72}$$

In this section, we consider the question of determining the frequency response of such an LTI system. Throughout the discussion we will always assume that the frequency response of the system exists, i.e., that eq. (3.121) converges.

There are two closely related ways in which to determine the frequency response $H(j\omega)$ for an LTI system described by the differential equation (4.72). The first of these, which relies on the fact that complex exponential signals are eigenfunctions of LTI systems, was used in Section 3.10 in our analysis of several simple, nonideal filters. Specifically, if $x(t) = e^{j\omega t}$, then the output must be $y(t) = H(j\omega)e^{j\omega t}$. Substituting these expressions into the differential equation (4.72) and performing some algebra, we can then solve for $H(j\omega)$. In this section we use an alternative approach to arrive at the same answer, making use of the differentiation property, eq. (4.31), of Fourier transforms.

Consider an LTI system characterized by eq. (4.72). From the convolution property,

$$Y(j\omega) = H(j\omega)X(j\omega),$$

or equivalently,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}, \tag{4.73}$$

where $X(j\omega)$, $Y(j\omega)$, and $H(j\omega)$ are the Fourier transforms of the input $x(t)$, output $y(t)$, and impulse response $h(t)$, respectively. Next, consider applying the Fourier transform to both sides of eq. (4.72) to obtain

$$\mathfrak{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathfrak{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}. \tag{4.74}$$

From the linearity property, eq. (4.26), this becomes

$$\sum_{k=0}^{N} a_k \mathfrak{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathfrak{F}\left\{\frac{d^k x(t)}{dt^k}\right\}, \tag{4.75}$$

and from the differentiation property, eq. (4.31),

$$\sum_{k=0}^{N} a_k(j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k(j\omega)^k X(j\omega),$$

or equivalently,

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k(j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k(j\omega)^k\right].$$

Thus, from eq. (4.73),

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k(j\omega)^k}{\sum_{k=0}^{N} a_k(j\omega)^k}. \tag{4.76}$$

Observe that $H(j\omega)$ is thus a rational function; that is, it is a ratio of polynomials in $(j\omega)$. The coefficients of the numerator polynomial are the same coefficients as those that appear on the right-hand side of eq. (4.72), and the coefficients of the denominator polynomial are the same coefficients as appear on the left side of eq. (4.72). Hence, the frequency response given in eq. (4.76) for the LTI system characterized by eq. (4.72) can be written down directly by inspection.

The differential equation (4.72) is commonly referred to as an Nth-order differential equation, as the equation involves derivatives of the output $y(t)$ up through the Nth derivative. Also, the denominator of $H(j\omega)$ in eq. (4.76) is an Nth-order polynomial in $(j\omega)$.

Example 4.24

Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), \tag{4.77}$$

with $a > 0$. From eq. (4.76), the frequency response is

$$H(j\omega) = \frac{1}{j\omega + a}. \tag{4.78}$$

Comparing this with the result of Example 4.1, we see that eq. (4.78) is the Fourier transform of $e^{-at}u(t)$. The impulse response of the system is then recognized as

$$h(t) = e^{-at}u(t).$$

Example 4.25

Consider a stable LTI system that is characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

From eq. (4.76), the frequency response is

$$H(j\omega) = \frac{(j\omega)+2}{(j\omega)^2+4(j\omega)+3}. \tag{4.79}$$

To determine the corresponding impulse response, we require the inverse Fourier transform of $H(j\omega)$. This can be found using the technique of partial-fraction expansion employed in Example 4.19 and discussed in detail in the appendix. (In particular, see Example A.1, in which the details of the calculations for the partial-fraction expansion of eq. (4.79) are worked out.) As a first step, we factor the denominator of the right-hand side of eq. (4.79) into a product of lower order terms:

$$H(j\omega) = \frac{j\omega+2}{(j\omega+1)(j\omega+3)}. \tag{4.80}$$

Then, using the method of partial-fraction expansion, we find that

$$H(j\omega) = \frac{\frac{1}{2}}{j\omega+1} + \frac{\frac{1}{2}}{j\omega+3}.$$

The inverse transform of each term can be recognized from Example 4.24, with the result that

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

The procedure used in Example 4.25 to obtain the inverse Fourier transform is generally useful in inverting transforms that are ratios of polynomials in $j\omega$. In particular, we can use eq. (4.76) to determine the frequency response of any LTI system described by a linear constant-coefficient differential equation and then can calculate the impulse response by performing a partial-fraction expansion that puts the frequency response into a form in which the inverse transform of each term can be recognized by inspection. In addition, if the Fourier transform $X(j\omega)$ of the input to such a system is also a ratio of polynomials in $j\omega$, then so is $Y(j\omega) = H(j\omega)X(j\omega)$. In this case we can use the same technique to solve the differential equation—that is, to find the response $y(t)$ to the input $x(t)$. This is illustrated in the next example.

Example 4.26

Consider the system of Example 4.25, and suppose that the input is

$$x(t) = e^{-t}u(t).$$

Then, using eq. (4.80), we have

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega+2}{(j\omega+1)(j\omega+3)}\right]\left[\frac{1}{j\omega+1}\right]$$

$$= \frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}. \tag{4.81}$$

As discussed in the appendix, in this case the partial-fraction expansion takes the form

$$Y(j\omega) = \frac{A_{11}}{j\omega + 1} + \frac{A_{12}}{(j\omega + 1)^2} + \frac{A_{21}}{j\omega + 3}, \tag{4.82}$$

where A_{11}, A_{12}, and A_{21} are constants to be determined. In Example A.2 in the appendix, the technique of partial-fraction expansion is used to determine these constants. The values obtained are

$$A_{11} = \frac{1}{4}, \quad A_{12} = \frac{1}{2}, \quad A_{21} = -\frac{1}{4},$$

so that

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega + 1)^2} - \frac{\frac{1}{4}}{j\omega + 3}. \tag{4.83}$$

Again, the inverse Fourier transform for each term in eq. (4.83) can be obtained by inspection. The first and third terms are of the same type that we have encountered in the preceding two examples, while the inverse transform of the second term can be obtained from Table 4.2 or, as was done in Example 4.19, by applying the dual of the differentiation property, as given in eq. (4.40), to $1/(j\omega + 1)$. The inverse transform of eq. (4.83) is then found to be

$$y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$

From the preceding examples, we see how the techniques of Fourier analysis allow us to reduce problems concerning LTI systems characterized by differential equations to straightforward algebraic problems. This important fact is illustrated further in a number of the problems at the end of the chapter. In addition (see Chapter 6), the algebraic structure of the rational transforms encountered in dealing with LTI systems described by differential equations greatly facilitate the analysis of their frequency-domain properties and the development of insights into both the time-domain and frequency-domain characteristics of this important class of systems.

4.8 SUMMARY

In this chapter, we have developed the Fourier transform representation for continous-time signals and have examined many of the properties that make this transform so useful. In particular, by viewing an aperiodic signal as the limit of a periodic signal as the period becomes arbitrarily large, we derived the Fourier transform representation for aperiodic signals from the Fourier series representation for periodic signals developed in Chapter 3. In addition, periodic signals themselves can be represented using Fourier transforms consisting of trains of impulses located at the harmonic frequencies of the periodic signal and with areas proportional to the corresponding Fourier series coefficients.

The Fourier transform possesses a wide variety of important properties that describe how different characteristics of signals are reflected in their transforms, and in this chapter we have derived and examined many of these properties. Among them are two that have particular significance for our study of signals and systems. The first is the convolution property, which is a direct consequence of the eigenfunction property of complex exponential signals and which leads to the description of an LTI system in terms of its frequency response. This description plays a fundamental role in the frequency-domain approach to the analysis of LTI systems, which we will continue to explore in subsequent chapters. The second property of the Fourier transform that has extremely important implications is the multiplication property, which provides the basis for the frequency-domain analysis of sampling and modulation systems. We examine these systems further in Chapters 7 and 8.

We have also seen that the tools of Fourier analysis are particularly well suited to the examination of LTI systems characterized by linear constant-coefficient differential equations. Specifically, we have found that the frequency response for such a system can be determined by inspection and that the technique of partial-fraction expansion can then be used to facilitate the calculation of the impulse response of the system. In subsequent chapters, we will find that the convenient algebraic structure of the frequency responses of these systems allows us to gain considerable insight into their characteristics in both the time and frequency domains.

Chapter 4 Problems

The first section of problems belongs to the basic category and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

4.1. Use the Fourier transform analysis equation (4.9) to calculate the Fourier transforms of:

(a) $e^{-2(t-1)}u(t-1)$ **(b)** $e^{-2|t-1|}$

Sketch and label the magnitude of each Fourier transform.

4.2. Use the Fourier transform analysis equation (4.9) to calculate the Fourier transforms of:

(a) $\delta(t+1)+\delta(t-1)$ **(b)** $\frac{d}{dt}\{u(-2-t)+u(t-2)\}$

Sketch and label the magnitude of each Fourier transform.

4.3. Determine the Fourier transform of each of the following periodic signals:

(a) $\sin(2\pi t+\frac{\pi}{4})$ **(b)** $1+\cos(6\pi t+\frac{\pi}{8})$

4.4. Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transforms of:

(a) $X_1(j\omega) = 2\pi\,\delta(\omega)+\pi\,\delta(\omega-4\pi)+\pi\,\delta(\omega+4\pi)$

(b) $X_2(j\omega) = \begin{cases} 2, & 0 \le \omega \le 2 \\ -2, & -2 \le \omega < 0 \\ 0, & |\omega| > 2 \end{cases}$

4.5. Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transform of $X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$, where

$$|X(j\omega)| = 2\{u(\omega + 3) - u(\omega - 3)\},$$

$$\sphericalangle X(j\omega) = -\frac{3}{2}\omega + \pi.$$

Use your answer to determine the values of t for which $x(t) = 0$.

4.6. Given that $x(t)$ has the Fourier transform $X(j\omega)$, express the Fourier transforms of the signals listed below in terms of $X(j\omega)$. You may find useful the Fourier transform properties listed in Table 4.1.

(a) $x_1(t) = x(1 - t) + x(-1 - t)$

(b) $x_2(t) = x(3t - 6)$

(c) $x_3(t) = \frac{d^2}{dt^2}x(t - 1)$

4.7. For each of the following Fourier transforms, use Fourier transform properties (Table 4.1) to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

(a) $X_1(j\omega) = u(\omega) - u(\omega - 2)$

(b) $X_2(j\omega) = \cos(2\omega)\sin(\frac{\omega}{2})$

(c) $X_3(j\omega) = A(\omega)e^{jB(\omega)}$, where $A(\omega) = (\sin 2\omega)/\omega$ and $B(\omega) = 2\omega + \frac{\pi}{2}$

(d) $X(j\omega) = \sum_{k=-\infty}^{\infty}(\frac{1}{2})^{|k|}\,\delta(\omega - \frac{k\pi}{4})$

4.8. Consider the signal

$$x(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ t + \frac{1}{2}, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 1, & t > \frac{1}{2} \end{cases}.$$

(a) Use the differentiation and integration properties in Table 4.1 and the Fourier transform pair for the rectangular pulse in Table 4.2 to find a closed-form expression for $X(j\omega)$.

(b) What is the Fourier transform of $g(t) = x(t) - \frac{1}{2}$?

4.9. Consider the signal

$$x(t) = \begin{cases} 0, & |t| > 1 \\ (t + 1)/2, & -1 \le t \le 1 \end{cases}.$$

(a) With the help of Tables 4.1 and 4.2, determine the closed-form expression for $X(j\omega)$.

(b) Take the real part of your answer to part (a), and verify that it is the Fourier transform of the even part of $x(t)$.

(c) What is the Fourier transform of the odd part of $x(t)$?

4.10. **(a)** Use Tables 4.1 and 4.2 to help determine the Fourier transform of the following signal:

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$$

(b) Use Parseval's relation and the result of the previous part to determine the numerical value of

$$A = \int_{-\infty}^{+\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$$

4.11. Given the relationships

$$y(t) = x(t) * h(t)$$

and

$$g(t) = x(3t) * h(3t),$$

and given that $x(t)$ has Fourier transform $X(j\omega)$ and $h(t)$ has Fourier transform $H(j\omega)$, use Fourier transform properties to show that $g(t)$ has the form

$$g(t) = Ay(Bt).$$

Determine the values of A and B.

4.12. Consider the Fourier transform pair

$$e^{-|t|} \overset{\mathfrak{F}}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$

(a) Use the appropriate Fourier transform properties to find the Fourier transform of $te^{-|t|}$.

(b) Use the result from part (a), along with the duality property, to determine the Fourier transform of

$$\frac{4t}{(1+t^2)^2}.$$

Hint: See Example 4.13.

4.13. Let $x(t)$ be a signal whose Fourier transform is

$$X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5),$$

and let

$$h(t) = u(t) - u(t-2).$$

(a) Is $x(t)$ periodic?
(b) Is $x(t) * h(t)$ periodic?
(c) Can the convolution of two aperiodic signals be periodic?

4.14. Consider a signal $x(t)$ with Fourier transform $X(j\omega)$. Suppose we are given the following facts:

1. $x(t)$ is real and nonnegative.
2. $\mathfrak{F}^{-1}\{(1 + j\omega)X(j\omega)\} = Ae^{-2t}u(t)$, where A is independent of t.
3. $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi$.

Determine a closed-form expression for $x(t)$.

4.15. Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Suppose we are given the following facts:

1. $x(t)$ is real.
2. $x(t) = 0$ for $t \leq 0$.
3. $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{R}e\{X(j\omega)\}e^{j\omega t} d\omega = |t|e^{-|t|}$.

Determine a closed-form expression for $x(t)$.

4.16. Consider the signal

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{\pi}{4})}{(k\frac{\pi}{4})} \delta(t - k\frac{\pi}{4}).$$

(a) Determine $g(t)$ such that

$$x(t) = \left(\frac{\sin t}{\pi t}\right) g(t).$$

(b) Use the multiplication property of the Fourier transform to argue that $X(j\omega)$ is periodic. Specify $X(j\omega)$ over one period.

4.17. Determine whether each of the following statements is true or false. Justify your answers.

(a) An odd and imaginary signal always has an odd and imaginary Fourier transform.

(b) The convolution of an odd Fourier transform with an even Fourier transform is always odd.

4.18. Find the impulse response of a system with the frequency response

$$H(j\omega) = \frac{(\sin^2(3\omega))\cos\omega}{\omega^2}.$$

4.19. Consider a causal LTI system with frequency response

$$H(j\omega) = \frac{1}{j\omega + 3}.$$

For a particular input $x(t)$ this system is observed to produce the output

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t).$$

Determine $x(t)$.

4.20. Find the impulse response of the causal LTI system represented by the *RLC* circuit considered in Problem 3.20. Do this by taking the inverse Fourier transform of the circuit's frequency response. You may use Tables 4.1 and 4.2 to help evaluate the inverse Fourier transform.

BASIC PROBLEMS

4.21. Compute the Fourier transform of each of the following signals:

(a) $[e^{-\alpha t}\cos\omega_0 t]u(t)$, $\alpha > 0$

(b) $e^{-3|t|}\sin 2t$

(c) $x(t) = \begin{cases} 1+\cos\pi t, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$

(d) $\sum_{k=0}^{\infty} \alpha^k \delta(t-kT)$, $|\alpha| < 1$

(e) $[te^{-2t}\sin 4t]u(t)$

(f) $\left[\frac{\sin \pi t}{\pi t}\right]\left[\frac{\sin 2\pi(t-1)}{\pi(t-1)}\right]$

(g) $x(t)$ as shown in Figure P4.21(a)

(h) $x(t)$ as shown in Figure P4.21(b)

(i) $x(t) = \begin{cases} 1-t^2, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

(j) $\sum_{n=-\infty}^{+\infty} e^{-|t-2n|}$

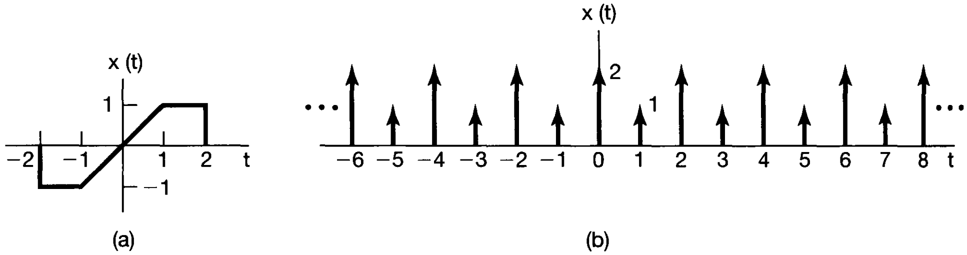


Figure P4.21

4.22. Determine the continuous-time signal corresponding to each of the following transforms.

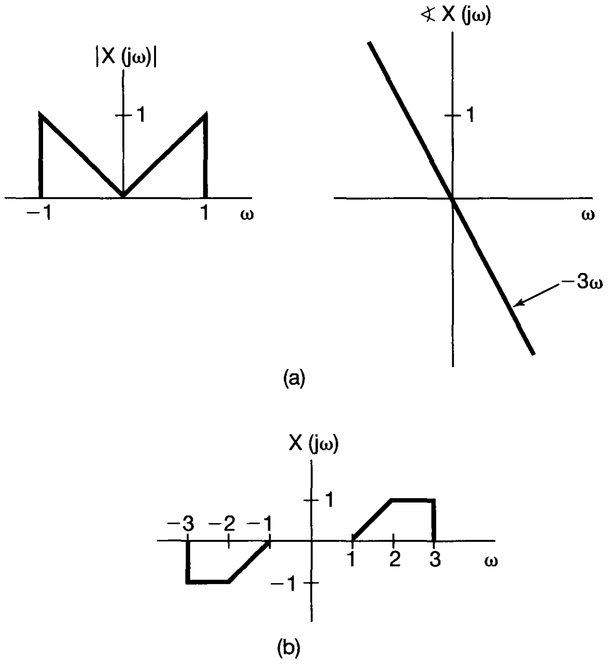


Figure P4.22

(a) $X(j\omega) = \frac{2\sin[3(\omega-2\pi)]}{(\omega-2\pi)}$
(b) $X(j\omega) = \cos(4\omega + \pi/3)$
(c) $X(j\omega)$ as given by the magnitude and phase plots of Figure P4.22(a)
(d) $X(j\omega) = 2[\delta(\omega - 1) - \delta(\omega + 1)] + 3[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$
(e) $X(j\omega)$ as in Figure P4.22(b)

4.23. Consider the signal

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the Fourier transform of each of the signals shown in Figure P4.23. You should be able to do this by explicitly evaluating *only* the transform of $x_0(t)$ and then using properties of the Fourier transform.

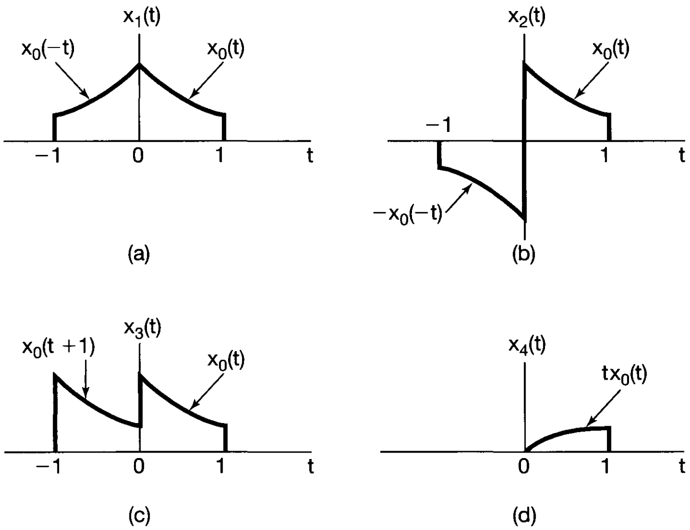


Figure P4.23

4.24. **(a)** Determine which, if any, of the real signals depicted in Figure P4.24 have Fourier transforms that satisfy each of the following conditions:
(1) $\Re e\{X(j\omega)\} = 0$
(2) $\Im m\{X(j\omega)\} = 0$
(3) There exists a real α such that $e^{j\alpha\omega}X(j\omega)$ is real
(4) $\int_{-\infty}^{\infty} X(j\omega)\,d\omega = 0$
(5) $\int_{-\infty}^{\infty} \omega X(j\omega)\,d\omega = 0$
(6) $X(j\omega)$ is periodic

(b) Construct a signal that has properties (1), (4), and (5) and does *not* have the others.

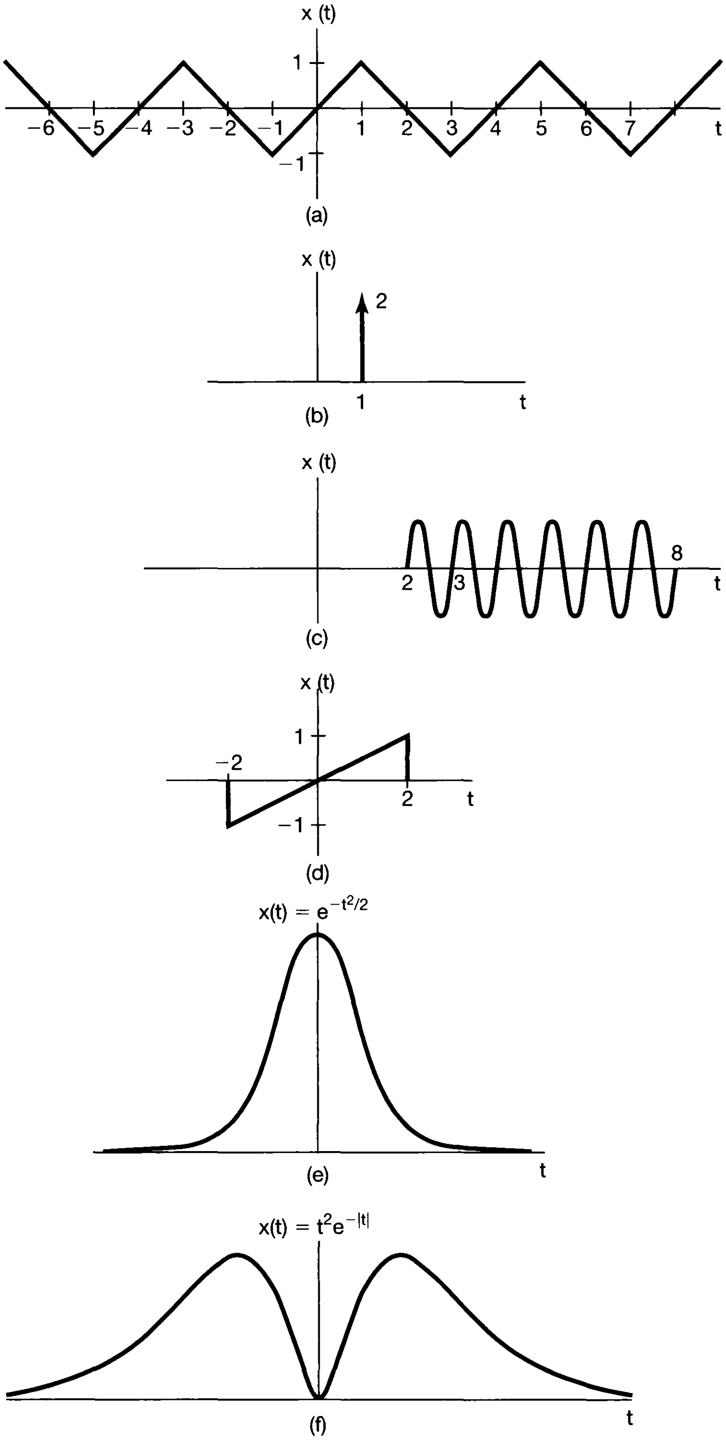

x (t)
1
−1
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 t
(a)
x (t)
2
1 t
(b)
x (t)
8
2 3 t
(c)
x (t)
1
−2
2 t
−1
(d)
x(t) = e^{−t²/2}
(e)
t
x(t) = t²e^{−|t|}
(f)
t

Figure P4.24

4.25. Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ depicted in Figure P4.25.

(a) Find $\sphericalangle X(j\omega)$.

(b) Find $X(j0)$.

(c) Find $\int_{-\infty}^{\infty} X(j\omega)\, d\omega$.

(d) Evaluate $\int_{-\infty}^{\infty} X(j\omega) \frac{2 \sin \omega}{\omega} e^{j2\omega}\, d\omega$.

(e) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$.

(f) Sketch the inverse Fourier transform of $\mathcal{R}e\{X(j\omega)\}$.

Note: You should perform all these calculations without explicitly evaluating $X(j\omega)$.

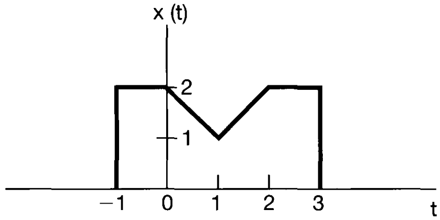


Figure P4.25

4.26. **(a)** Compute the convolution of each of the following pairs of signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$, using the convolution property, and inverse transforming.

(i) $x(t) = te^{-2t}u(t),\ h(t) = e^{-4t}u(t)$

(ii) $x(t) = te^{-2t}u(t),\ h(t) = te^{-4t}u(t)$

(iii) $x(t) = e^{-t}u(t),\ h(t) = e^{t}u(-t)$

(b) Suppose that $x(t) = e^{-(t-2)}u(t-2)$ and $h(t)$ is as depicted in Figure P4.26. Verify the convolution property for this pair of signals by showing that the Fourier transform of $y(t) = x(t) * h(t)$ equals $H(j\omega)X(j\omega)$.

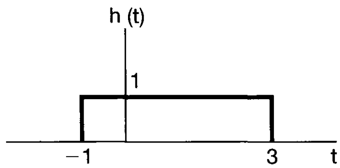


Figure P4.26

4.27. Consider the signals

$$x(t) = u(t-1) - 2u(t-2) + u(t-3)$$

and

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kT),$$

where $T > 0$. Let a_k denote the Fourier series coefficients of $\tilde{x}(t)$, and let $X(j\omega)$ denote the Fourier transform of $x(t)$.

(a) Determine a closed-form expression for $X(j\omega)$.

(b) Determine an expression for the Fourier coefficients a_k and verify that $a_k = \frac{1}{T}X\left(j\frac{2\pi k}{T}\right)$.

4.28. **(a)** Let $x(t)$ have the Fourier transform $X(j\omega)$, and let $p(t)$ be periodic with fundamental frequency ω_0 and Fourier series representation

$$p(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jn\omega_0 t}.$$

Determine an expression for the Fourier transform of

$$y(t) = x(t)p(t). \tag{P4.28–1}$$

(b) Suppose that $X(j\omega)$ is as depicted in Figure P4.28(a). Sketch the spectrum of $y(t)$ in eq. (P4.28–1) for each of the following choices of $p(t)$:

(i) $p(t) = \cos(t/2)$

(ii) $p(t) = \cos t$

(iii) $p(t) = \cos 2t$

(iv) $p(t) = (\sin t)(\sin 2t)$

(v) $p(t) = \cos 2t - \cos t$

(vi) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - \pi n)$

(vii) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 2\pi n)$

(viii) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 4\pi n)$

(ix) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 2\pi n) - \frac{1}{2}\sum_{n=-\infty}^{+\infty} \delta(t - \pi n)$

(x) $p(t) =$ the periodic square wave shown in Figure P4.28(b).

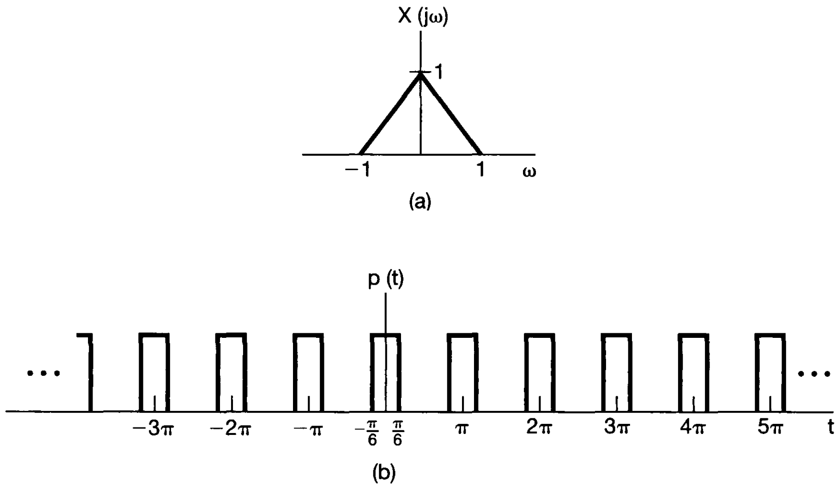
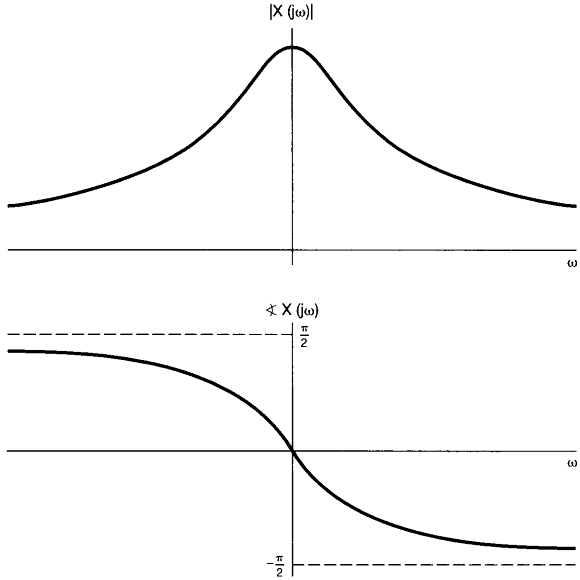


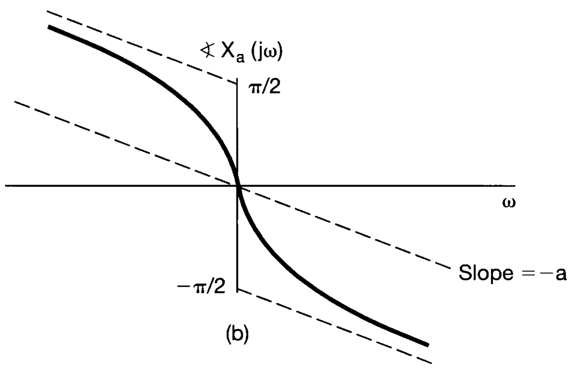
Figure P4.28

4.29. A real-valued continuous-time function $x(t)$ has a Fourier transform $X(j\omega)$ whose magnitude and phase are as illustrated in Figure P4.29(a).

The functions $x_a(t)$, $x_b(t)$, $x_c(t)$, and $x_d(t)$ have Fourier transforms whose magnitudes are identical to $X(j\omega)$, but whose phase functions differ, as shown in Figures P4.29(b)–(e). The phase functions $\sphericalangle X_a(j\omega)$ and $\sphericalangle X_b(j\omega)$ are formed by adding a linear phase to $\sphericalangle X(j\omega)$. The function $\sphericalangle X_c(j\omega)$ is formed by reflecting $\sphericalangle X(j\omega)$ about $\omega = 0$, and $\sphericalangle X_d(j\omega)$ is obtained by a combination of a reflection and an addition of a linear phase. Using the properties of Fourier transforms, determine the expressions for $x_a(t)$, $x_b(t)$, $x_c(t)$, and $x_d(t)$ in terms of $x(t)$.



(a)



(b)

Figure P4.29

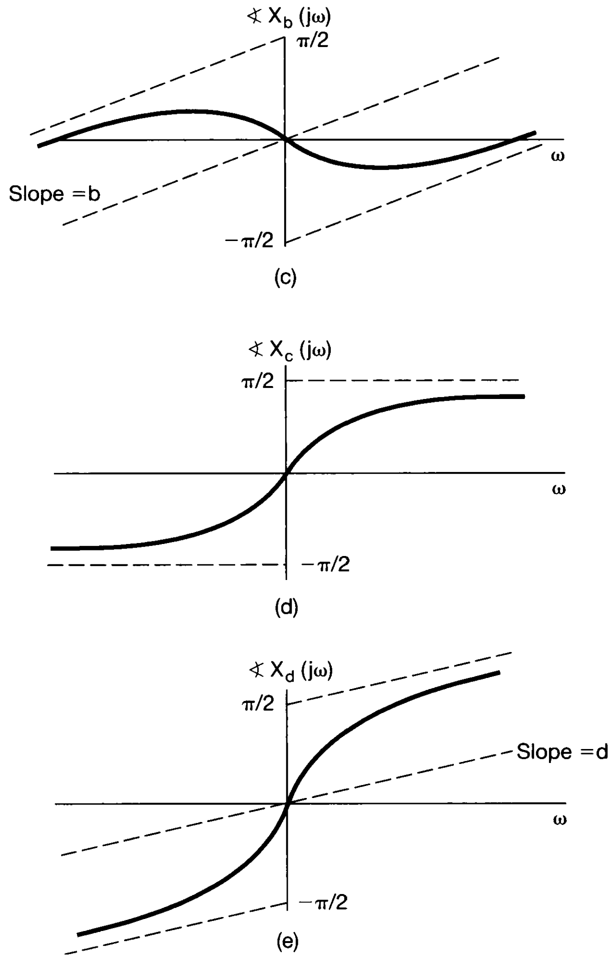


Figure P4.29 *Continued*

4.30. Suppose $g(t) = x(t)\cos t$ and the Fourier transform of the $g(t)$ is

$$G(j\omega) = \begin{cases} 1, & |\omega| \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Determine $x(t)$.

(b) Specify the Fourier transform $X_1(j\omega)$ of a signal $x_1(t)$ such that

$$g(t) = x_1(t)\cos\left(\frac{2}{3}t\right).$$

4.31. **(a)** Show that the three LTI systems with impulse responses

$$h_1(t) = u(t),$$

$$h_2(t) = -2\delta(t) + 5e^{-2t}u(t),$$

and

$$h_3(t) = 2te^{-t}u(t)$$

all have the same response to $x(t) = \cos t$.

(b) Find the impulse response of another LTI system with the same response to $\cos t$.

This problem illustrates the fact that the response to $\cos t$ cannot be used to specify an LTI system uniquely.

4.32. Consider an LTI system S with impulse response

$$h(t) = \frac{\sin(4(t-1))}{\pi(t-1)}.$$

Determine the output of S for each of the following inputs:

(a) $x_1(t) = \cos(6t + \frac{\pi}{2})$

(b) $x_2(t) = \sum_{k=0}^{\infty}(\frac{1}{2})^k \sin(3kt)$

(c) $x_3(t) = \frac{\sin(4(t+1))}{\pi(t+1)}$

(d) $x_4(t) = (\frac{\sin 2t}{\pi t})^2$

4.33. The input and the output of a stable and causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(a) Find the impulse response of this system.

(b) What is the response of this system if $x(t) = te^{-2t}u(t)$?

(c) Repeat part (a) for the stable and causal LTI system described by the equation

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{dy(t)}{dt} + y(t) = 2\frac{d^2x(t)}{dt^2} - 2x(t)$$

4.34. A causal and stable LTI system S has the frequency response

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}.$$

(a) Determine a differential equation relating the input $x(t)$ and output $y(t)$ of S.
(b) Determine the impulse response $h(t)$ of S.
(c) What is the output of S when the input is

$$x(t) = e^{-4t}u(t) - te^{-4t}u(t)?$$

4.35. In this problem, we provide examples of the effects of nonlinear changes in phase.
(a) Consider the continuous-time LTI system with frequency response

$$H(j\omega) = \frac{a - j\omega}{a + j\omega},$$

where $a > 0$. What is the magnitude of $H(j\omega)$? What is $\measuredangle H(j\omega)$? What is the impulse response of this system?
(b) Determine the output of the system of part (a) with $a = 1$ when the input is

$$\cos(t/\sqrt{3}) + \cos t + \cos \sqrt{3}t.$$

Roughly sketch both the input and the output.

4.36. Consider an LTI system whose response to the input

$$x(t) = [e^{-t} + e^{-3t}]u(t)$$

is

$$y(t) = [2e^{-t} - 2e^{-4t}]u(t).$$

(a) Find the frequency response of this system.
(b) Determine the system's impulse response.
(c) Find the differential equation relating the input and the output of this system.

ADVANCED PROBLEMS

4.37. Consider the signal $x(t)$ in Figure P4.37.
(a) Find the Fourier transform $X(j\omega)$ of $x(t)$.
(b) Sketch the signal

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k).$$

(c) Find another signal $g(t)$ such that $g(t)$ is not the same as $x(t)$ and

$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k).$$

(d) Argue that, although $G(j\omega)$ is different from $X(j\omega)$, $G(j\frac{\pi k}{2}) = X(j\frac{\pi k}{2})$ for all integers k. You should not explicitly evaluate $G(j\omega)$ to answer this question.

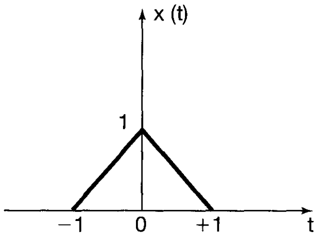


Figure P4.37

4.38. Let $x(t)$ be any signal with Fourier transform $X(j\omega)$. The frequency-shift property of the Fourier transform may be stated as

$$e^{j\omega_0 t}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0)).$$

(a) Prove the frequency-shift property by applying the frequency shift to the analysis equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

(b) Prove the frequency-shift property by utilizing the Fourier transform of $e^{j\omega_0 t}$ in conjunction with the multiplication property of the Fourier transform.

4.39. Suppose that a signal $x(t)$ has Fourier transform $X(j\omega)$. Now consider another signal $g(t)$ whose shape is the same as the shape of $X(j\omega)$; that is,

$$g(t) = X(jt).$$

(a) Show that the Fourier transform $G(j\omega)$ of $g(t)$ has the same shape as $2\pi x(-t)$; that is, show that

$$G(j\omega) = 2\pi x(-\omega).$$

(b) Using the fact that

$$\mathcal{F}\{\delta(t + B)\} = e^{jB\omega}$$

in conjunction with the result from part (a), show that

$$\mathcal{F}\{e^{jBt}\} = 2\pi\,\delta(\omega - B).$$

4.40. Use properties of the Fourier transform to show by induction that the Fourier transform of

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t), \ a > 0,$$

is

$$\frac{1}{(a + j\omega)^n}.$$

4.41. In this problem, we derive the multiplication property of the continuous-time Fourier transform. Let $x(t)$ and $y(t)$ be two continuous-time signals with Fourier transforms $X(j\omega)$ and $Y(j\omega)$, respectively. Also, let $g(t)$ denote the inverse Fourier transform of $\frac{1}{2\pi}\{X(j\omega) * Y(j\omega)\}$.

(a) Show that

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j(\omega - \theta))e^{j\omega t}\,d\omega\right]d\theta.$$

(b) Show that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j(\omega - \theta))e^{j\omega t}\,d\omega = e^{j\theta t}y(t).$$

(c) Combine the results of parts (a) and (b) to conclude that

$$g(t) = x(t)y(t).$$

4.42. Let

$$g_1(t) = \{[\cos(\omega_0 t)]x(t)\} * h(t) \qquad \text{and} \qquad g_2(t) = \{[\sin(\omega_0 t)]x(t)\} * h(t),$$

where

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk100t}$$

is a real-valued periodic signal and $h(t)$ is the impulse response of a stable LTI system.

(a) Specify a value for ω_0 and any necessary constraints on $H(j\omega)$ to ensure that

$$g_1(t) = \mathcal{Re}\{a_5\} \qquad \text{and} \qquad g_2(t) = \mathcal{Im}\{a_5\}.$$

(b) Give an example of $h(t)$ such that $H(j\omega)$ satisfies the constraints you specified in part (a).

4.43. Let

$$g(t) = x(t)\cos^2 t * \frac{\sin t}{\pi t}.$$

Assuming that $x(t)$ is real and $X(j\omega) = 0$ for $|\omega| \geq 1$, show that there exists an LTI system S such that

$$x(t) \xrightarrow{S} g(t).$$

4.44. The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$\frac{dy(t)}{dt} + 10y(t) = \int_{-\infty}^{+\infty} x(\tau)z(t-\tau)\,d\tau - x(t),$$

where $z(t) = e^{-t}u(t) + 3\,\delta(t)$.

(a) Find the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$ of this system.

(b) Determine the impulse response of the system.

4.45. In the discussion in Section 4.3.7 of Parseval's relation for continuous-time signals, we saw that

$$\int_{-\infty}^{+\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\,d\omega.$$

This says that the total energy of the signal can be obtained by integrating $|X(j\omega)|^2$ over all frequencies. Now consider a real-valued signal $x(t)$ processed by the ideal bandpass filter $H(j\omega)$ shown in Figure P4.45. Express the energy in the output signal $y(t)$ as an integration over frequency of $|X(j\omega)|^2$. For Δ sufficiently small so that $|X(j\omega)|$ is approximately constant over a frequency interval of width Δ, show that the energy in the output $y(t)$ of the bandpass filter is approximately proportional to $\Delta|X(j\omega_0)|^2$.

On the basis of the foregoing result, $\Delta|X(j\omega_0)|^2$ is proportional to the energy in the signal in a bandwidth Δ around the frequency ω_0. For this reason, $|X(j\omega)|^2$ is often referred to as the *energy-density spectrum* of the signal $x(t)$.

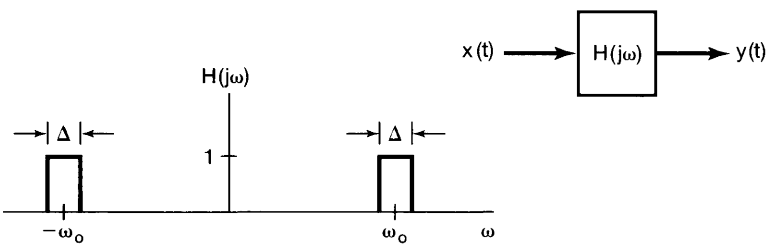


Figure P4.45

4.46. In Section 4.5.1, we discussed the use of amplitude modulation with a complex exponential carrier to implement a bandpass filter. The specific system was shown in Figure 4.26, and if only the real part of $f(t)$ is retained, the equivalent bandpass filter is that shown in Figure 4.30.

In Figure P4.46, we indicate an implementation of a bandpass filter using sinusoidal modulation and lowpass filters. Show that the output $y(t)$ of the system is identical to that which would be obtained by retaining only $\mathcal{R}e\{f(t)\}$ in Figure 4.26.

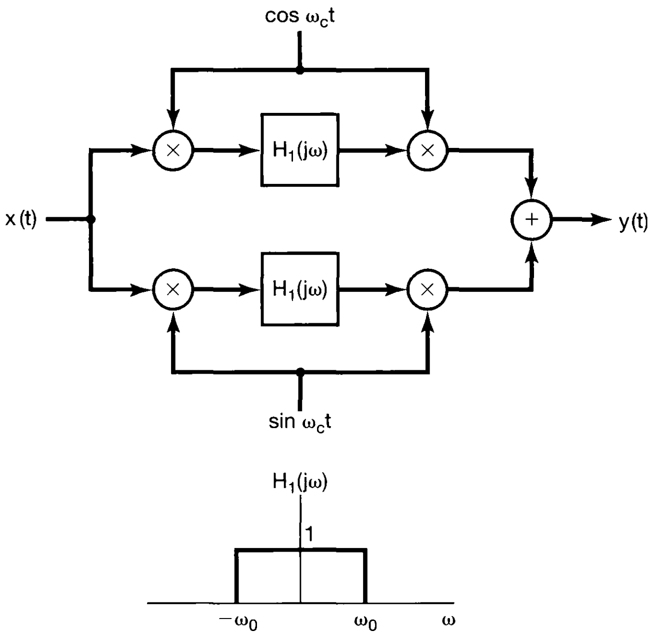


Figure P4.46

4.47. An important property of the frequency response $H(j\omega)$ of a continuous-time LTI system with a real, causal impulse response $h(t)$ is that $H(j\omega)$ is completely specified by its real part, $\mathcal{R}e\{H(j\omega)\}$. The current problem is concerned with deriving and examining some of the implications of this property, which is generally referred to as *real-part sufficiency*.

(a) Prove the property of real-part sufficiency by examining the signal $h_e(t)$, which is the even part of $h(t)$. What is the Fourier transform of $h_e(t)$? Indicate how $h(t)$ can be recovered from $h_e(t)$.

(b) If the real part of the frequency response of a causal system is

$$\mathcal{R}e\{H(j\omega)\} = \cos\omega,$$

what is $h(t)$?

(c) Show that $h(t)$ can be recovered from $h_o(t)$, the odd part of $h(t)$, for every value of t except $t = 0$. Note that if $h(t)$ does not contain any singularities [$\delta(t)$, $u_1(t)$, $u_2(t)$, etc.] at $t = 0$, then the frequency response

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}\,dt$$

will not change if $h(t)$ is set to some arbitrary finite value at the single point $t = 0$. Thus, in this case, show that $H(j\omega)$ is also completely specified by its imaginary part.

EXTENSION PROBLEMS

4.48. Let us consider a system with a real and causal impulse response $h(t)$ that does not have any singularities at $t = 0$. In Problem 4.47, we saw that either the real or the imaginary part of $H(j\omega)$ completely determines $H(j\omega)$. In this problem we derive an explicit relationship between $H_R(j\omega)$ and $H_I(j\omega)$, the real and imaginary parts of $H(j\omega)$.

(a) To begin, note that since $h(t)$ is causal,

$$h(t) = h(t)u(t), \qquad \text{(P4.48–1)}$$

except perhaps at $t = 0$. Now, since $h(t)$ contains no singularities at $t = 0$, the Fourier transforms of both sides of eq. (P4.48–1) must be identical. Use this fact, together with the multiplication property, to show that

$$H(j\omega) = \frac{1}{j\pi}\int_{-\infty}^{+\infty}\frac{H(j\eta)}{\omega-\eta}d\eta. \qquad \text{(P4.48–2)}$$

Use eq. (P4.48–2) to determine an expression for $H_R(j\omega)$ in terms of $H_I(j\omega)$ and one for $H_I(j\omega)$ in terms of $H_R(j\omega)$.

(b) The operation

$$y(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x(\tau)}{t-\tau}\,d\tau \qquad \text{(P4.48–3)}$$

is called the *Hilbert transform*. We have just seen that the real and imaginary parts of the transform of a real, causal impulse response $h(t)$ can be determined from one another using the Hilbert transform.

Now consider eq. (P4.48–3), and regard $y(t)$ as the output of an LTI system with input $x(t)$. Show that the frequency response of this system is

$$H(j\omega) = \begin{cases} -j, & \omega > 0 \\ j, & \omega < 0 \end{cases}.$$

(c) What is the Hilbert transform of the signal $x(t) = \cos 3t$?

4.49. Let $H(j\omega)$ be the frequency response of a continuous-time LTI system, and suppose that $H(j\omega)$ is real, even, and positive. Also, assume that

$$\max_{\omega}\{H(j\omega)\} = H(0).$$

(a) Show that:

(i) The impulse response, $h(t)$, is real.

(ii) $\max\{|h(t)|\} = h(0)$.

Hint: If $f(t,\omega)$ is a complex function of two variables, then

$$\left|\int_{-\infty}^{+\infty} f(t,\omega)\,d\omega\right| \le \int_{-\infty}^{+\infty} |f(t,\omega)|\,d\omega.$$

(b) One important concept in system analysis is the *bandwidth* of an LTI system. There are many different mathematical ways in which to define bandwidth, but they are related to the qualitative and intuitive idea that a system with frequency response $G(j\omega)$ essentially "stops" signals of the form $e^{j\omega t}$ for values of ω where $G(j\omega)$ vanishes or is small and "passes" those complex exponentials in the band of frequency where $G(j\omega)$ is not small. The width of this band is the bandwidth. These ideas will be made much clearer in Chapter 6, but for now we will consider a special definition of bandwidth for those systems with frequency responses that have the properties specified previously for $H(j\omega)$. Specifically, one definition of the bandwidth B_w of such a system is the width of the rectangle of height $H(j0)$ that has an area equal to the area under $H(j\omega)$. This is illustrated in Figure P4.49(a). Note that since $H(j0) = \max_\omega H(j\omega)$, the frequencies within the band indicated in the figure are those for which $H(j\omega)$ is largest. The exact choice of the width in the figure is, of course, a bit arbitrary, but we have chosen one definition that allows us to compare different systems and to make precise a very important relationship between time and frequency.

What is the bandwidth of the system with frequency response

$$H(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}?$$

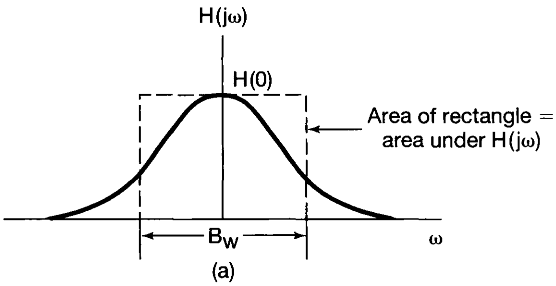


Figure P4.49a

(c) Find an expression for the bandwidth B_w in terms of $H(j\omega)$.

(d) Let $s(t)$ denote the step response of the system set out in part (a). An important measure of the speed of response of a system is the *rise time*, which, like the bandwidth, has a qualitative definition, leading to many possible mathematical definitions, one of which we will use. Intuitively, the rise time of a system is a measure of how fast the step response rises from zero to its final value,

$$s(\infty) = \lim_{t \to \infty} s(t).$$

Thus, the smaller the rise time, the faster is the response of the system. For the system under consideration in this problem, we will define the rise time as

$$t_r = \frac{s(\infty)}{h(0)}.$$

Since

$$s'(t) = h(t),$$

and also because of the property that $h(0) = \max_t h(t)$, t_r is the time it would take to go from zero to $s(\infty)$ while maintaining the maximum rate of change of $s(t)$. This is illustrated in Figure P4.49(b).

Find an expression for t_r in terms of $H(j\omega)$.

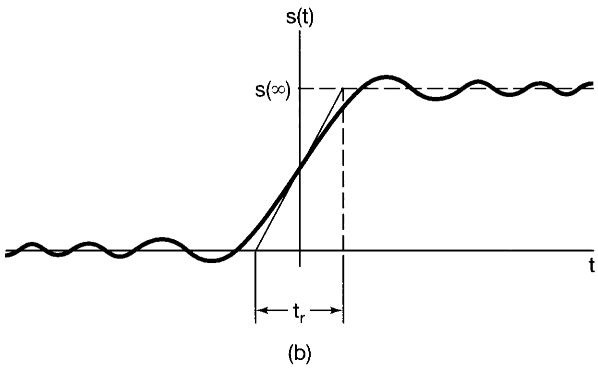


Figure P4.49b

(e) Combine the results of parts (c) and (d) to show that

$$B_w t_r = 2\pi. \qquad \text{(P4.49–1)}$$

Thus, we *cannot* independently specify both the rise time and the bandwidth of our system. For example, eq. (P4.49–1) implies that, if we want a fast system (t_r small), the system must have a large bandwidth. This is a fundamental trade-off that is of central importance in many problems of system design.

4.50. In Problems 1.45 and 2.67, we defined and examined several of the properties and uses of correlation functions. In the current problem, we examine the properties of such functions in the frequency domain. Let $x(t)$ and $y(t)$ be two real signals. Then the cross-correlation function of $x(t)$ and $y(t)$ is defined as

$$\phi_{xy}(t) = \int_{-\infty}^{+\infty} x(t + \tau)y(\tau)\,d\tau.$$

Similarly, we can define $\phi_{yx}(t)$, $\phi_{xx}(t)$, and $\phi_{yy}(t)$. [The last two of these are called the autocorrelation functions of the signals $x(t)$ and $y(t)$, respectively.] Let $\Phi_{xy}(j\omega)$, $\Phi_{yx}(j\omega)$, $\Phi_{xx}(j\omega)$, and $\Phi_{yy}(j\omega)$ denote the Fourier transforms of $\phi_{xy}(t)$, $\phi_{yx}(t)$, $\phi_{xx}(t)$, and $\phi_{yy}(t)$, respectively.

(a) What is the relationship between $\Phi_{xy}(j\omega)$ and $\Phi_{yx}(j\omega)$?

(b) Find an expression for $\Phi_{xy}(j\omega)$ in terms of $X(j\omega)$ and $Y(j\omega)$.

(c) Show that $\Phi_{xx}(j\omega)$ is real and nonnegative for every ω.

(d) Suppose now that $x(t)$ is the input to an LTI system with a real-valued impulse response and with frequency response $H(j\omega)$ and that $y(t)$ is the output. Find expressions for $\Phi_{xy}(j\omega)$ and $\Phi_{yy}(j\omega)$ in terms of $\Phi_{xx}(j\omega)$ and $H(j\omega)$.

(e) Let $x(t)$ be as is illustrated in Figure P4.50, and let the LTI system impulse response be $h(t) = e^{-at}u(t)$, $a > 0$. Compute $\Phi_{xx}(j\omega)$, $\Phi_{xy}(j\omega)$, and $\Phi_{yy}(j\omega)$ using the results of parts (a)–(d).

(f) Suppose that we are given the following Fourier transform of a function $\phi(t)$:

$$\Phi(j\omega) = \frac{\omega^2 + 100}{\omega^2 + 25}.$$

Find the impulse responses of *two* causal, stable LTI systems that have autocorrelation functions equal to $\phi(t)$. Which one of these has a causal, stable inverse?

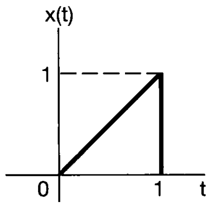


Figure P4.50

4.51. (a) Consider two LTI systems with impulse responses $h(t)$ and $g(t)$, respectively, and suppose that these systems are inverses of one another. Suppose also that the systems have frequency responses denoted by $H(j\omega)$ and $G(j\omega)$, respectively. What is the relationship between $H(j\omega)$ and $G(j\omega)$?

(b) Consider the continuous-time LTI system with frequency response

$$H(j\omega) = \begin{cases} 1, & 2 < |\omega| < 3 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Is it possible to find an input $x(t)$ to this system such that the output is as depicted in Figure P4.50? If so, find $x(t)$. If not, explain why not.

(ii) Is this system invertible? Explain your answer.

(c) Consider an auditorium with an echo problem. As discussed in Problem 2.64, we can model the acoustics of the auditorium as an LTI system with an impulse response consisting of an impulse train, with the kth impulse in the train corresponding to the kth echo. Suppose that in this particular case the impulse response is

$$h(t) = \sum_{k=0}^{\infty} e^{-kT}\,\delta(t - kT),$$

where the factor e^{-kT} represents the attenuation of the kth echo.

In order to make a high-quality recording from the stage, the effect of the echoes must be removed by performing some processing of the sounds sensed by the recording equipment. In Problem 2.64, we used convolutional techniques to consider one example of the design of such a processor (for a different acoustic model). In the current problem, we will use frequency-domain techniques. Specifically, let $G(j\omega)$ denote the frequency response of the LTI system to be used to process the sensed acoustic signal. Choose $G(j\omega)$ so that the echoes are completely removed and the resulting signal is a faithful reproduction of the original stage sounds.

(d) Find the differential equation for the inverse of the system with impulse response

$$h(t) = 2\,\delta(t) + u_1(t).$$

(e) Consider the LTI system initially at rest and described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = \frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t).$$

The inverse of this system is also initially at rest and described by a differential equation. Find the differential equation describing the inverse, and find the impulse responses $h(t)$ and $g(t)$ of the original system and its inverse.

4.52. Inverse systems frequently find application in problems involving imperfect measuring devices. For example, consider a device for measuring the temperature of a liquid. It is often reasonable to model such a device as an LTI system that, because of the response characteristics of the measuring element (e.g., the mercury in a thermometer), does not respond instantaneously to temperature changes. In particular, assume that the response of this device to a unit step in temperature is

$$s(t) = (1 - e^{-t/2})u(t). \qquad \text{(P4.52–1)}$$

(a) Design a compensatory system that, when provided with the output of the measuring device, produces an output equal to the instantaneous temperature of the liquid.

(b) One of the problems that often arises in using inverse systems as compensators for measuring devices is that gross inaccuracies in the indicated temperature may occur if the actual output of the measuring device produces errors due to small, erratic phenomena in the device. Since there always are such sources of error in real systems, one must take them into account. To illustrate this, consider a measuring device whose overall output can be modeled as the sum of the response of the measuring device characterized by eq. (P4.52–1) and an interfering "noise" signal $n(t)$. Such a model is depicted in Figure P4.52(a), where we have also included the inverse system of part (a), which now has as its input the *overall* output of the measuring device. Suppose that $n(t) = \sin\omega t$. What is the contribution of $n(t)$ to the output of the inverse system, and how does this output change as ω is increased?

(c) The issue raised in part (b) is an important one in many applications of LTI system analysis. Specifically, we are confronted with the fundamental tradeoff between the speed of response of the system and the ability of the system to attenuate high-frequency interference. In part (b) we saw that this tradeoff implied that, by attempting to speed up the response of a measuring device (by means of an inverse system), we produced a system that would also amplify

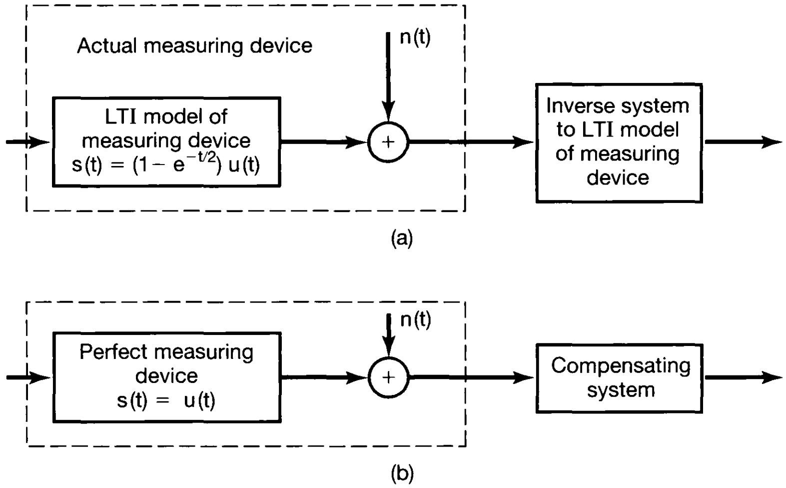


Figure P4.52

corrupting sinusoidal signals. To illustrate this concept further, consider a measuring device that responds instantaneously to changes in temperature, but that also is corrupted by noise. The response of such a system can be modeled, as depicted in Figure P4.52(b), as the sum of the response of a perfect measuring device and a corrupting signal $n(t)$. Suppose that we wish to design a compensatory system that will *slow down* the response to actual temperature variations, but also will attenuate the noise $n(t)$. Let the impulse response of this system be

$$h(t) = ae^{-at}u(t).$$

Choose a so that the overall system of Figure P4.52(b) responds as quickly as possible to a step change in temperature, subject to the constraint that the amplitude of the portion of the output due to the noise $n(t) = \sin 6t$ is no larger than 1/4.

4.53. As mentioned in the text, the techniques of Fourier analysis can be extended to signals having two independent variables. As their one-dimensional counterparts do in some applications, these techniques play an important role in other applications, such as image processing. In this problem, we introduce some of the elementary ideas of two-dimensional Fourier analysis.

Let $x(t_1, t_2)$ be a signal that depends upon two independent variables t_1 and t_2. The *two-dimensional Fourier transform* of $x(t_1, t_2)$ is defined as

$$X(j\omega_1, j\omega_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t_1, t_2)e^{-j(\omega_1 t_1 + \omega_2 t_2)}\,dt_1\,dt_2.$$

(a) Show that this double integral can be performed as two successive one-dimensional Fourier transforms, first in t_1 with t_2 regarded as fixed and then in t_2.

(b) Use the result of part (a) to determine the inverse transform—that is, an expression for $x(t_1, t_2)$ in terms of $X(j\omega_1, j\omega_2)$.

(c) Determine the two-dimensional Fourier transforms of the following signals:

(i) $x(t_1, t_2) = e^{-t_1+2t_2}u(t_1 - 1)u(2 - t_2)$

(ii) $x(t_1, t_2) = \begin{cases} e^{-|t_1|-|t_2|}, & \text{if } -1 < t_1 \leq 1 \text{ and } -1 \leq t_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(iii) $x(t_1, t_2) = \begin{cases} e^{-|t_1|-|t_2|}, & \text{if } 0 \leq t_1 \leq 1 \text{ or } 0 \leq t_2 \leq 1 \text{ (or both)} \\ 0, & \text{otherwise} \end{cases}$

(iv) $x(t_1, t_2)$ as depicted in Figure P4.53.

(v) $e^{-|t_1+t_2|-|t_1-t_2|}$

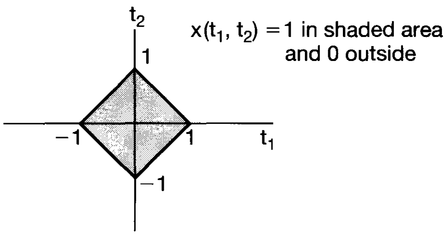


Figure P4.53

(d) Determine the signal $x(t_1, t_2)$ whose two-dimensional Fourier transform is

$$X(j\omega_1, j\omega_2) = \frac{2\pi}{4 + j\omega_1}\delta(\omega_2 - 2\omega_1).$$

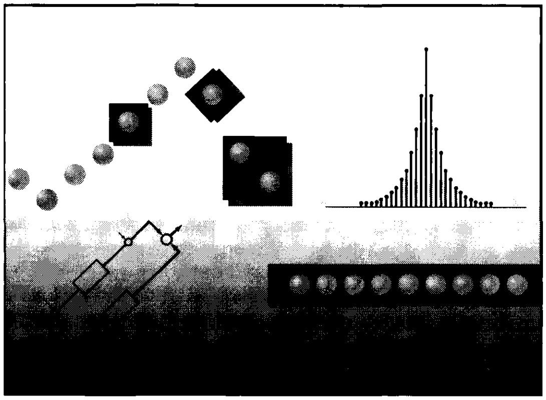
(e) Let $x(t_1, t_2)$ and $h(t_1, t_2)$ be two signals with two-dimensional Fourier transforms $X(j\omega_1, j\omega_2)$ and $H(j\omega_1, j\omega_2)$, respectively. Determine the transforms of the following signals in terms of $X(j\omega_1, j\omega_2)$ and $H(j\omega_1, j\omega_2)$:

(i) $x(t_1 - T_1, t_2 - T_2)$

(ii) $x(at_1, bt_2)$

(iii) $y(t_1, t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau_1, \tau_2)h(t_1 - \tau_1, t_2 - \tau_2)\,d\tau_1\,d\tau_2$

5
THE DISCRETE-TIME FOURIER TRANSFORM



5.0 INTRODUCTION

In Chapter 4, we introduced the continuous-time Fourier transform and developed the many characteristics of that transform which make the methods of Fourier analysis of such great value in analyzing and understanding the properties of continuous-time signals and systems. In the current chapter, we complete our development of the basic tools of Fourier analysis by introducing and examining the discrete-time Fourier transform.

In our discussion of Fourier series in Chapter 3, we saw that there are many similarities and strong parallels in analyzing continuous-time and discrete-time signals. However, there are also important differences. For example, as we saw in Section 3.6, the Fourier series representation of a discrete-time periodic signal is a *finite* series, as opposed to the infinite series representation required for continuous-time periodic signals. As we will see in this chapter, there are corresponding differences between continuous-time and discrete-time Fourier transforms.

In the remainder of the chapter, we take advantage of the similarities between continuous-time and discrete-time Fourier analysis by following a strategy essentially identical to that used in Chapter 4. In particular, we begin by extending the Fourier series description of periodic signals in order to develop a Fourier transform representation for discrete-time aperiodic signals, and we follow with an analysis of the properties and characteristics of the discrete-time Fourier transform that parallels that given in Chapter 4. By doing this, we not only will enhance our understanding of the basic concepts of Fourier analysis that are common to both continuous and discrete time, but also will contrast their differences in order to deepen our understanding of the distinct characteristics of each.

5.1 REPRESENTATION OF APERIODIC SIGNALS: THE DISCRETE-TIME FOURIER TRANSFORM

5.1.1 Development of the Discrete-Time Fourier Transform

In Section 4.1 [eq. (4.2) and Figure 4.2], we saw that the Fourier series coefficients for a continuous-time periodic square wave can be viewed as samples of an envelope function and that, as the period of the square wave increases, these samples become more and more finely spaced. This property suggested representing an aperiodic signal $x(t)$ by first constructing a periodic signal $\tilde{x}(t)$ that equaled $x(t)$ over one period. Then, as this period approached infinity, $\tilde{x}(t)$ was equal to $x(t)$ over larger and larger intervals of time, and the Fourier series representation for $\tilde{x}(t)$ converged to the Fourier transform representation for $x(t)$. In this section, we apply an analogous procedure to discrete-time signals in order to develop the Fourier transform representation for discrete-time aperiodic sequences.

Consider a general sequence $x[n]$ that is of finite duration. That is, for some integers N_1 and N_2, $x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$. A signal of this type is illustrated in Figure 5.1(a). From this aperiodic signal, we can construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is one period, as illustrated in Figure 5.1(b). As we choose the period N to be larger, $\tilde{x}[n]$ is identical to $x[n]$ over a longer interval, and as $N \to \infty$, $\tilde{x}[n] = x[n]$ for any finite value of n.

Let us now examine the Fourier series representation of $\tilde{x}[n]$. Specifically, from eqs. (3.94) and (3.95), we have

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}, \tag{5.1}$$

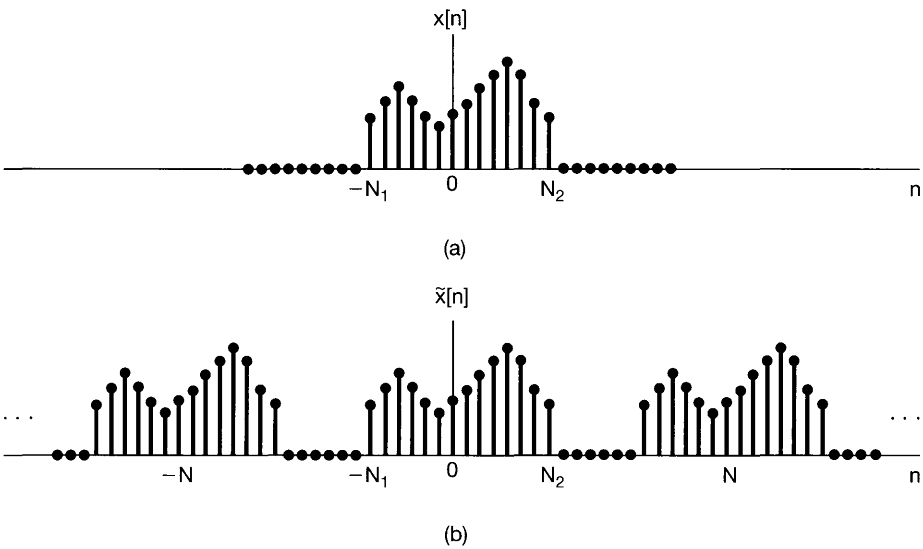


Figure 5.1 (a) Finite-duration signal $x[n]$; (b) periodic signal $\tilde{x}[n]$ constructed to be equal to $x[n]$ over one period.

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}. \tag{5.2}$$

Since $x[n] = \tilde{x}[n]$ over a period that includes the interval $-N_1 \leq n \leq N_2$, it is convenient to choose the interval of summation in eq. (5.2) to include this interval, so that $\tilde{x}[n]$ can be replaced by $x[n]$ in the summation. Therefore,

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(2\pi/N)n}, \tag{5.3}$$

where in the second equality in eq. (5.3) we have used the fact that $x[n]$ is zero outside the interval $-N_1 \leq n \leq N_2$. Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}, \tag{5.4}$$

we see that the coefficients a_k are proportional to samples of $X(e^{j\omega})$, i.e.,

$$a_k = \frac{1}{N} X(e^{jk\omega_0}), \tag{5.5}$$

where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain. Combining eqs. (5.1) and (5.5) yields

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}. \tag{5.6}$$

Since $\omega_0 = 2\pi/N$, or equivalently, $1/N = \omega_0/2\pi$, eq. (5.6) can be rewritten as

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0. \tag{5.7}$$

As with eq. (4.7), as N increases ω_0 decreases, and as $N \rightarrow \infty$ eq. (5.7) passes to an integral. To see this more clearly, consider $X(e^{j\omega})e^{j\omega n}$ as sketched in Figure 5.2. From

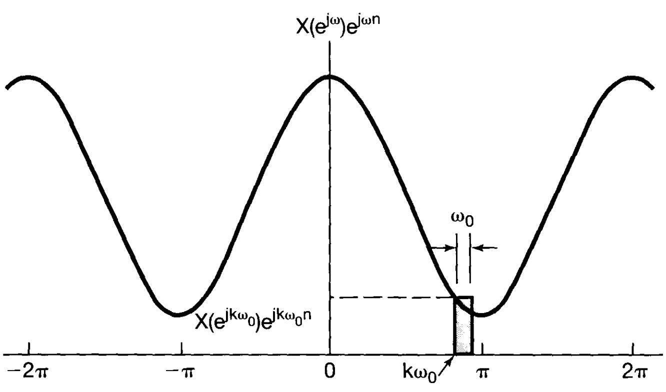


Figure 5.2 Graphical interpretation of eq. (5.7).

eq. (5.4), $X(e^{j\omega})$ is seen to be periodic in ω with period 2π, and so is $e^{j\omega n}$. Thus, the product $X(e^{j\omega})e^{j\omega n}$ will also be periodic. As depicted in the figure, each term in the summation in eq. (5.7) represents the area of a rectangle of height $X(e^{jk\omega_0})e^{j\omega_0 n}$ and width ω_0. As $\omega_0 \to 0$, the summation becomes an integral. Furthermore, since the summation is carried out over N consecutive intervals of width $\omega_0 = 2\pi/N$, the total interval of integration will always have a width of 2π. Therefore, as $N \to \infty$, $\tilde{x}[n] = x[n]$, and eq. (5.7) becomes

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega,$$

where, since $X(e^{j\omega})e^{j\omega n}$ is periodic with period 2π, the interval of integration can be taken as *any* interval of length 2π. Thus, we have the following pair of equations:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \tag{5.8}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \tag{5.9}$$

Equations (5.8) and (5.9) are the discrete-time counterparts of eqs. (4.8) and (4.9). The function $X(e^{j\omega})$ is referred to as the *discrete-time Fourier transform* and the pair of equations as the *discrete-time Fourier transform pair.* Equation (5.8) is the *synthesis equation,* eq. (5.9) the *analysis equation.* Our derivation of these equations indicates how an aperiodic sequence can be thought of as a linear combination of complex exponentials. In particular, the synthesis equation is in effect a representation of $x[n]$ as a linear combination of complex exponentials infinitesimally close in frequency and with amplitudes $X(e^{j\omega})(d\omega/2\pi)$. For this reason, as in continuous time, the Fourier transform $X(e^{j\omega})$ will often be referred to as the *spectrum* of $x[n]$, because it provides us with the information on how $x[n]$ is composed of complex exponentials at different frequencies.

Note also that, as in continuous time, our derivation of the discrete-time Fourier transform provides us with an important relationship between discrete-time Fourier series and transforms. In particular, the Fourier coefficients a_k of a periodic signal $\tilde{x}[n]$ can be expressed in terms of equally spaced *samples* of the Fourier transform of a finite-duration, aperiodic signal $x[n]$ that is equal to $\tilde{x}[n]$ over one period and is zero otherwise. This fact is of considerable importance in practical signal processing and Fourier analysis, and we look at it further in Problem 5.41.

As our derivation indicates, the discrete-time Fourier transform shares many similarities with the continuous-time case. The major differences between the two are the periodicity of the discrete-time transform $X(e^{j\omega})$ and the finite interval of integration in the synthesis equation. Both of these stem from a fact that we have noted several times before: Discrete-time complex exponentials that differ in frequency by a multiple of 2π are identical. In Section 3.6 we saw that, for periodic discrete-time signals, the implications of this statement are that the Fourier series coefficients are periodic and that the Fourier series representation is a finite sum. For aperiodic signals, the analogous implications are that $X(e^{j\omega})$ is periodic (with period 2π) and that the synthesis equation involves an integration only over a frequency interval that produces distinct complex exponentials (i.e., any interval of length 2π). In Section 1.3.3, we noted one further consequence of the periodicity of $e^{j\omega n}$ as a function of ω: $\omega = 0$ and $\omega = 2\pi$ yield the same signal. Signals at frequencies near these values or any other even multiple of π are slowly varying and therefore are all appropriately thought of as low-frequency signals. Similarly, the high frequencies in discrete time are the values of ω near odd multiples of π. Thus, the signal $x_1[n]$ shown in Figure 5.3(a) with Fourier transform depicted in Figure 5.3(b) varies more slowly than the signal $x_2[n]$ in Figure 5.3(c) whose transform is shown in Figure 5.3(d).

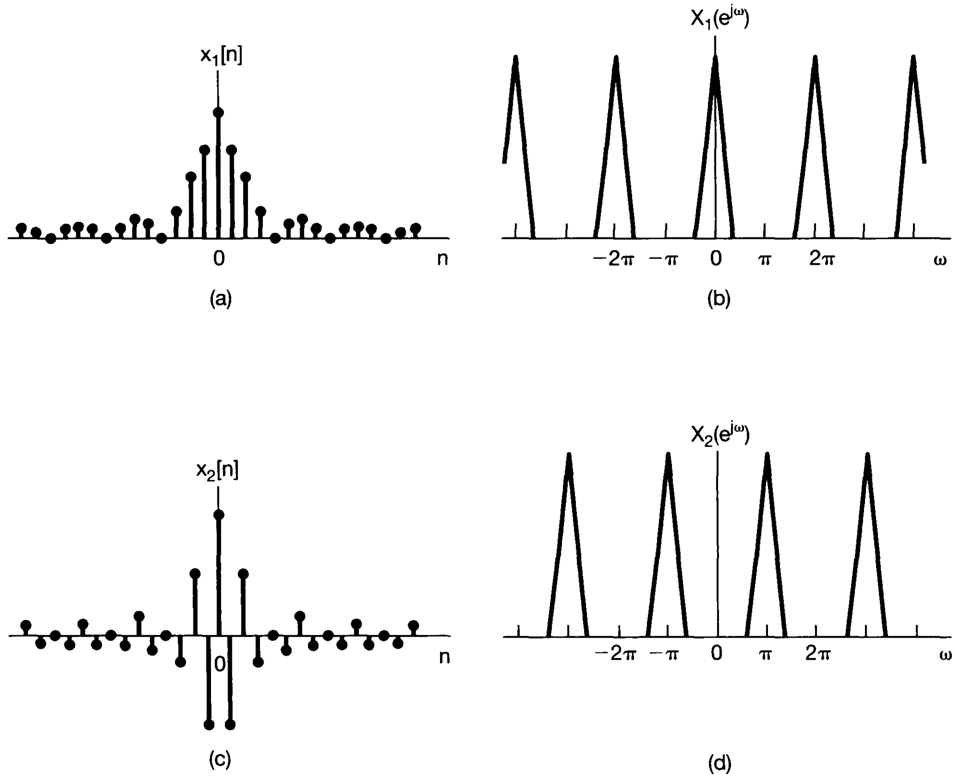


Figure 5.3 (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$.

5.1.2 Examples of Discrete-Time Fourier Transforms

To illustrate the discrete-time Fourier transform, let us consider several examples.

Example 5.1

Consider the signal

$$x[n] = a^n u[n], \qquad |a| < 1.$$

In this case,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

The magnitude and phase of $X(e^{j\omega})$ are shown in Figure 5.4(a) for $a > 0$ and in Figure 5.4(b) for $a < 0$. Note that all of these functions are periodic in ω with period 2π.

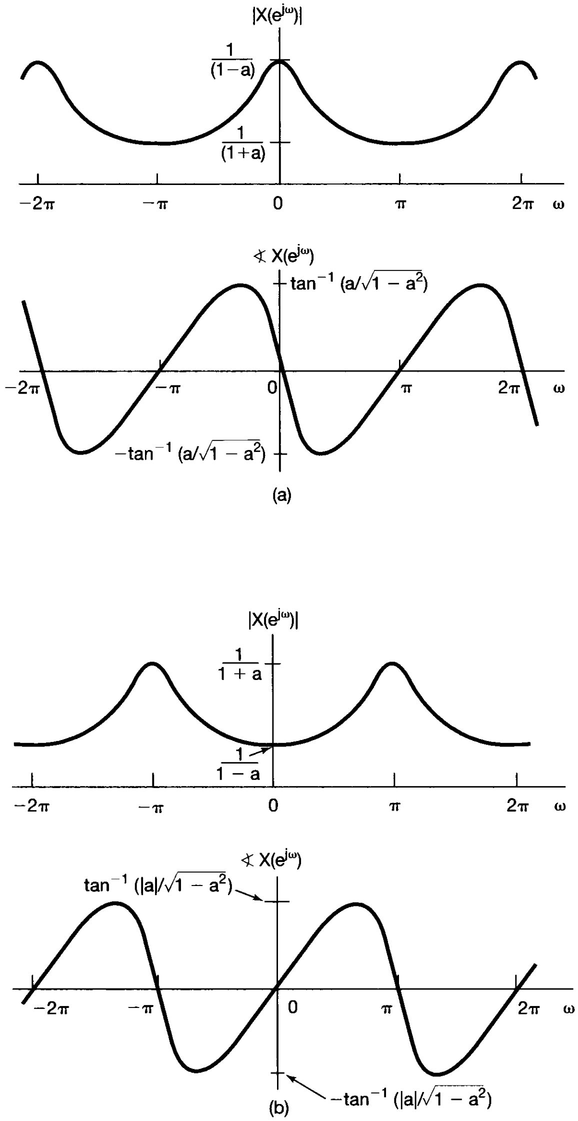


Figure 5.4 Magnitude and phase of the Fourier transform of Example 5.1 for (a) $a > 0$ and (b) $a < 0$.

Example 5.2

Let

$$x[n] = a^{|n|}, \qquad |a| < 1.$$

This signal is sketched for $0 < a < 1$ in Figure 5.5(a). Its Fourier transform is obtained from eq. (5.9):

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}. \end{aligned}$$

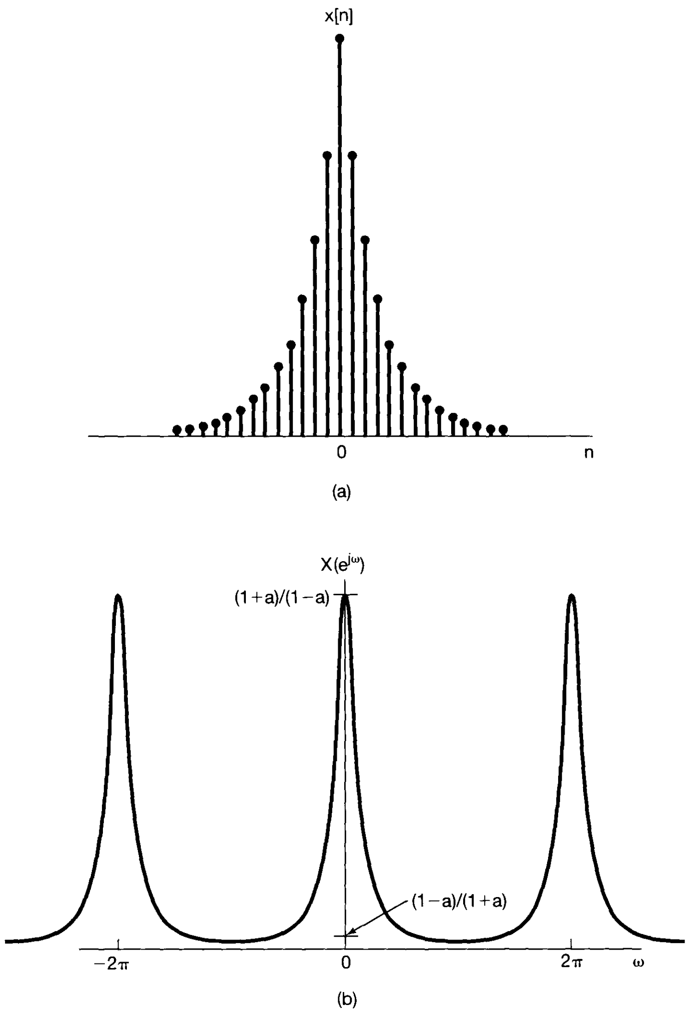


Figure 5.5 (a) Signal $x[n] = a^{|n|}$ of Example 5.2 and (b) its Fourier transform ($0 < a < 1$).

Making the substitution of variables $m = -n$ in the second summation, we obtain

$$X(e^{j\omega}) = \sum_{n=0}^{\infty}(ae^{-j\omega})^n + \sum_{m=1}^{\infty}(ae^{j\omega})^m.$$

Both of these summations are infinite geometric series that we can evaluate in closed form, yielding

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} \\ &= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}. \end{aligned}$$

In this case, $X(e^{j\omega})$ is real and is illustrated in Figure 5.5(b), again for $0 < a < 1$.

Example 5.3

Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}, \tag{5.10}$$

which is illustrated in Figure 5.6(a) for $N_1 = 2$. In this case,

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}. \tag{5.11}$$

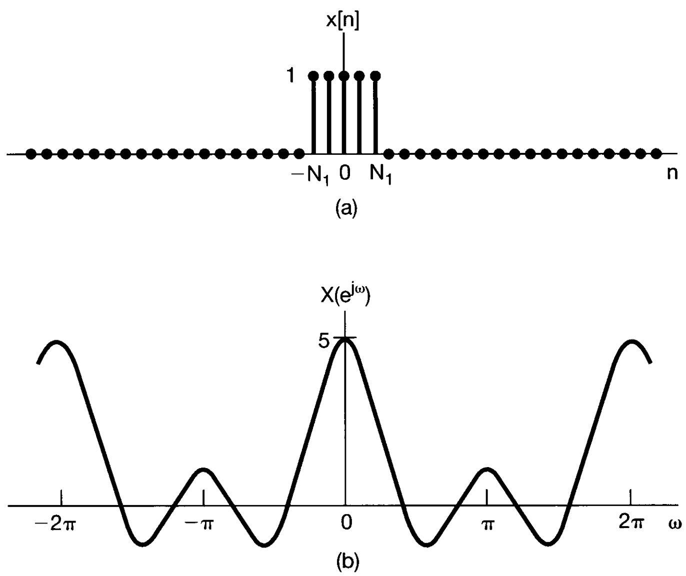


Figure 5.6 (a) Rectangular pulse signal of Example 5.3 for $N_1 = 2$ and (b) its Fourier transform.

Using calculations similar to those employed in obtaining eq. (3.104) in Example 3.12, we can write

$$X(e^{j\omega}) = \frac{\sin\omega\left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}. \tag{5.12}$$

This Fourier transform is sketched in Figure 5.6(b) for $N_1 = 2$. The function in eq. (5.12) is the discrete-time counterpart of the sinc function, which appears in the Fourier transform of the continuous-time rectangular pulse (see Example 4.4). An important difference between these two functions is that the function in eq. (5.12) is periodic with period 2π, whereas the sinc function is aperiodic.

5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

Although the argument we used to derive the discrete-time Fourier transform in Section 5.1.1 was constructed assuming that $x[n]$ was of arbitrary but finite duration, eqs. (5.8) and (5.9) remain valid for an extremely broad class of signals with infinite duration (such as the signals in Examples 5.1 and 5.2). In this case, however, we again must consider the question of convergence of the infinite summation in the analysis equation (5.9). The conditions on $x[n]$ that guarantee the convergence of this sum are direct counterparts of the convergence conditions for the continuous-time Fourier transform.[1] Specifically, eq. (5.9) will converge either if $x[n]$ is absolutely summable, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty, \tag{5.13}$$

or if the sequence has finite energy, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty. \tag{5.14}$$

In contrast to the situation for the analysis equation (5.9), there are generally no convergence issues associated with the synthesis equation (5.8), since the integral in this equation is over a finite interval of integration. This is very much the same situation as for the discrete-time Fourier series synthesis equation (3.94), which involves a finite sum and consequently has no issues of convergence associated with it either. In particular, if we approximate an aperiodic signal $x[n]$ by an integral of complex exponentials with frequencies taken over the interval $|\omega| \leq W$, i.e.,

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega})e^{j\omega n}\, d\omega, \tag{5.15}$$

[1]For discussions of the convergence issues associated with the discrete-time Fourier transform, see A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1989), and L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975).

then $\hat{x}[n] = x[n]$ for $W = \pi$. Thus, much as in Figure 3.18, we would expect not to see any behavior like the Gibbs phenomenon in evaluating the discrete-time Fourier transform synthesis equation. This is illustrated in the following example.

Example 5.4

Let $x[n]$ be the unit impulse; that is,

$$x[n] = \delta[n].$$

In this case the analysis equation (5.9) is easily evaluated, yielding

$$X(e^{j\omega}) = 1.$$

In other words, just as in continuous time, the unit impulse has a Fourier transform representation consisting of equal contributions at all frequencies. If we then apply eq. (5.15) to this example, we obtain

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\,d\omega = \frac{\sin Wn}{\pi n}. \tag{5.16}$$

This is plotted in Figure 5.7 for several values of W. As can be seen, the frequency of the oscillations in the approximation increases as W is increased, which is similar to what we observed in the continuous-time case. On the other hand, in contrast to the continuous-time case, the amplitude of these oscillations decreases relative to the magnitude of $\hat{x}[0]$ as W is increased, and the oscillations disappear entirely for $W = \pi$.

5.2 THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

As in the continuous-time case, discrete-time periodic signals can be incorporated within the framework of the discrete-time Fourier transform by interpreting the transform of a periodic signal as an impulse train in the frequency domain. To derive the form of this representation, consider the signal

$$x[n] = e^{j\omega_0 n}. \tag{5.17}$$

In continuous time, we saw that the Fourier transform of $e^{j\omega_0 t}$ can be interpreted as an impulse at $\omega = \omega_0$. Therefore, we might expect the same type of transform to result for the discrete-time signal of eq. (5.17). However, the discrete-time Fourier transform must be periodic in ω with period 2π. This then suggests that the Fourier transform of $x[n]$ in eq. (5.17) should have impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. In fact, the Fourier transform of $x[n]$ is the impulse train

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l), \tag{5.18}$$

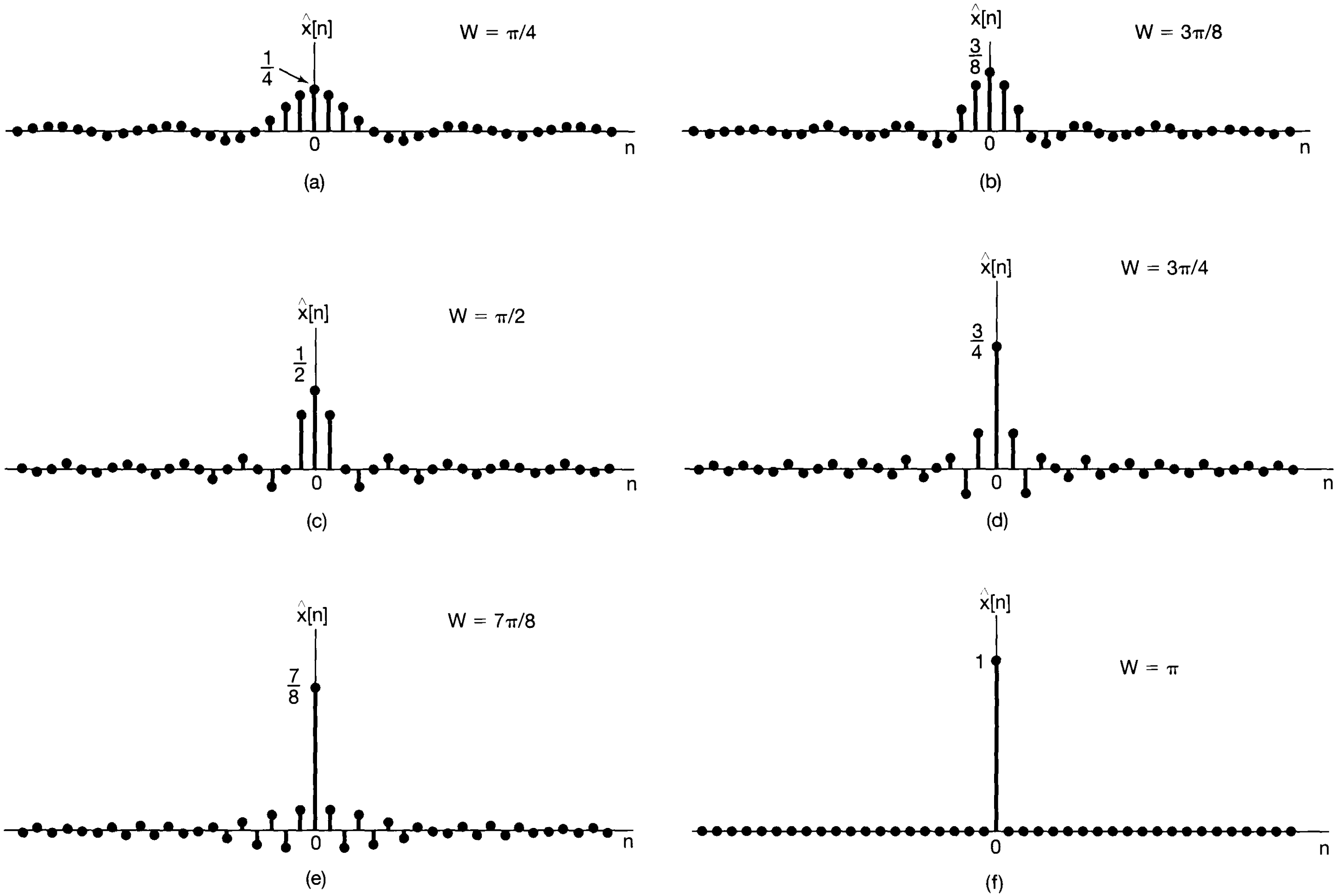


Figure 5.7 Approximation to the unit sample obtained as in eq. (5.16) using complex exponentials with frequencies $|\omega| \leq W$: (a) $W = \pi/4$; (b) $W = 3\pi/8$; (c) $W = \pi/2$; (d) $W = 3\pi/4$; (e) $W = 7\pi/8$; (f) $W = \pi$. Note that for $W = \pi$, $\hat{x}[n] = \delta[n]$.

which is illustrated in Figure 5.8. In order to check the validity of this expression, we must evaluate its inverse transform. Substituting eq. (5.18) into the synthesis equation (5.8), we find that

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega-\omega_0-2\pi l)e^{j\omega n}\,d\omega.$$

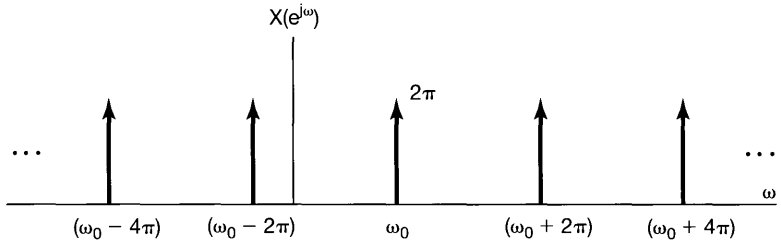


Figure 5.8 Fourier transform of $x[n] = e^{j\omega_0 n}$.

Note that any interval of length 2π includes exactly one impulse in the summation given in eq. (5.18). Therefore, if the interval of integration chosen includes the impulse located at $\omega_0 + 2\pi r$, then

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}.$$

Now consider a periodic sequence $x[n]$ with period N and with the Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}. \tag{5.19}$$

In this case, the Fourier transform is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k\,\delta\left(\omega - \frac{2\pi k}{N}\right), \tag{5.20}$$

so that the Fourier transform of a periodic signal can be directly constructed from its Fourier coefficients.

To verify that eq. (5.20) is in fact correct, note that $x[n]$ in eq. (5.19) is a linear combination of signals of the form in eq. (5.17), and thus the Fourier transform of $x[n]$ must be a linear combination of transforms of the form of eq. (5.18). In particular, suppose that we choose the interval of summation in eq. (5.19) as $k = 0, 1, \ldots, N-1$, so that

$$\begin{aligned} x[n] = {} & a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} \\ & + \cdots + a_{N-1}e^{j(N-1)(2\pi/N)n}. \end{aligned} \tag{5.21}$$

Thus, $x[n]$ is a linear combination of signals, as in eq. (5.17), with $\omega_0 = 0, 2\pi/N, 4\pi/N, \ldots, (N-1)2\pi/N$. The resulting Fourier transform is illustrated in Figure 5.9. In Figure 5.9(a), we have depicted the Fourier transform of the first term on the right-hand side of eq. (5.21): The Fourier transform of the constant signal $a_0 = a_0 e^{j0 \cdot n}$ is a periodic impulse train, as in eq. (5.18), with $\omega_0 = 0$ and a scaling of $2\pi a_0$ on each of the impulses. Moreover, from Chapter 4 we know that the Fourier series coefficients a_k are periodic with period N, so that $2\pi a_0 = 2\pi a_N = 2\pi a_{-N}$. In Figure 5.9(b) we have illustrated the Fourier transform of the second term in eq. (5.21), where we have again used eq. (5.18),

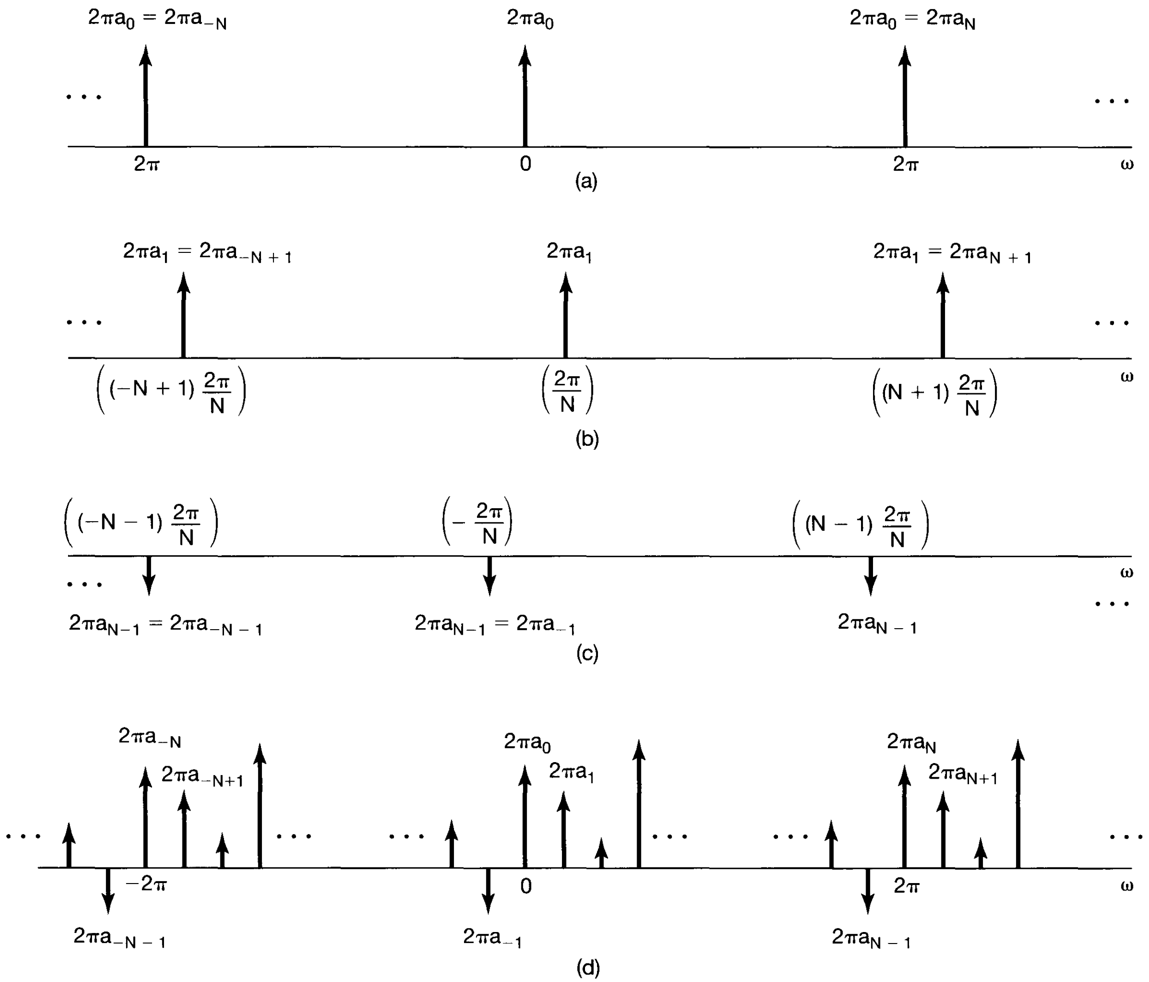


Figure 5.9 Fourier transform of a discrete-time periodic signal: (a) Fourier transform of the first term on the right-hand side of eq. (5.21); (b) Fourier transform of the second term in eq. (5.21); (c) Fourier transform of the last term in eq. (5.21); (d) Fourier transform of $x[n]$ in eq (5.21).

in this case for $a_1 e^{j(2\pi/N)n}$, and the fact that $2\pi a_1 = 2\pi a_{N+1} = 2\pi a_{-N+1}$. Similarly, Figure 5.9(c) depicts the final term. Finally, Figure 5.9(d) depicts the entire expression for $X(e^{j\omega})$. Note that because of the periodicity of the a_k, $X(e^{j\omega})$ can be interpreted as a train of impulses occurring at multiples of the fundamental frequency $2\pi/N$, with the area of the impulse located at $\omega = 2\pi k/N$ being $2\pi a_k$, which is exactly what is stated in eq. (5.20).

Example 5.5

Consider the periodic signal

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \qquad \text{with} \quad \omega_0 = \frac{2\pi}{5}. \tag{5.22}$$

From eq. (5.18), we can immediately write

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right). \tag{5.23}$$

That is,

$$X(e^{j\omega}) = \pi\,\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\,\delta\left(\omega + \frac{2\pi}{5}\right), \qquad -\pi \le \omega < \pi, \tag{5.24}$$

and $X(e^{j\omega})$ repeats periodically with a period of 2π, as illustrated in Figure 5.10.

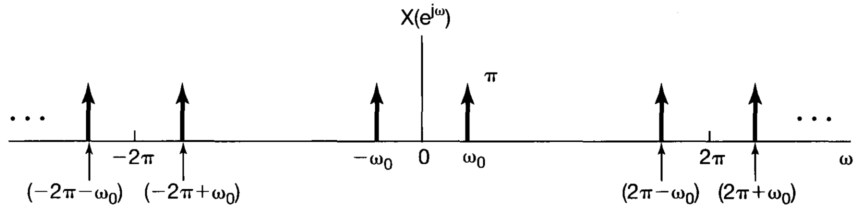


Figure 5.10 Discrete-time Fourier transform of $x[n] = \cos\omega_0 n$.

Example 5.6

The discrete-time counterpart of the periodic impulse train of Example 4.8 is the sequence

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN], \tag{5.25}$$

as sketched in Figure 5.11(a). The Fourier series coefficients for this signal can be calculated directly from eq. (3.95):

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}.$$

Choosing the interval of summation as $0 \le n \le N-1$, we have

$$a_k = \frac{1}{N}. \tag{5.26}$$

Using eqs. (5.26) and (5.20), we can then represent the Fourier transform of the signal as

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right), \tag{5.27}$$

which is illustrated in Figure 5.11(b).

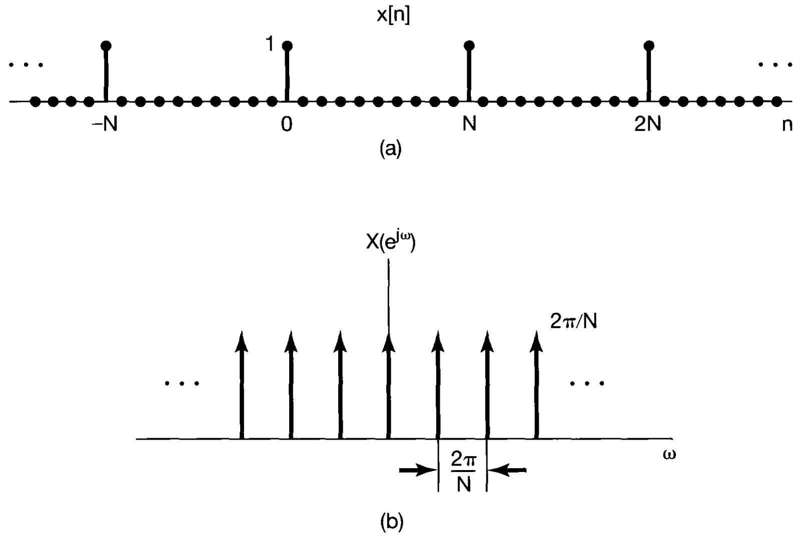


Figure 5.11 (a) Discrete-time periodic impulse train; (b) its Fourier transform.

5.3 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

As with the continuous-time Fourier transform, a variety of properties of the discrete-time Fourier transform provide further insight into the transform and, in addition, are often useful in reducing the complexity in the evaluation of transforms and inverse transforms. In this and the following two sections we consider these properties, and in Table 5.1 we present a concise summary of them. By comparing this table with Table 4.1, we can get a clear picture of some of the similarities and differences between continuous-time and discrete-time Fourier transform properties. When the derivation or interpretation of a discrete-time Fourier transform property is essentially identical to its continuous-time counterpart, we will simply state the property. Also, because of the close relationship between the Fourier series and the Fourier transform, many of the transform properties translate directly into corresponding properties for the discrete-time Fourier series, which we summarized in Table 3.2 and briefly discussed in Section 3.7.

In the following discussions, it will be convenient to adopt notation similar to that used in Section 4.3 to indicate the pairing of a signal and its transform. That is,

$$X(e^{j\omega}) = \mathfrak{F}\{x[n]\},$$
$$x[n] = \mathfrak{F}^{-1}\{X(e^{j\omega})\},$$
$$x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}).$$

5.3.1 Periodicity of the Discrete-Time Fourier Transform

As we discussed in Section 5.1, the discrete-time Fourier transform is *always* periodic in ω with period 2π; i.e.,

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega}).} \tag{5.28}$$

This is in contrast to the continuous-time Fourier transform, which in general is not periodic.

5.3.2 Linearity of the Fourier Transform

If

$$x_1[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

and

$$x_2[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X_2(e^{j\omega}),$$

then

$$\boxed{ax_1[n] + bx_2[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega}).} \tag{5.29}$$

5.3.3 Time Shifting and Frequency Shifting

If

$$x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$\boxed{x[n-n_0] \stackrel{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega n_0}X(e^{j\omega})} \tag{5.30}$$

and

$$e^{j\omega_0 n}x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}). \tag{5.31}$$

Equation (5.30) can be obtained by direct substitution of $x[n-n_0]$ into the analysis equation (5.9), while eq. (5.31) is derived by substituting $X(e^{j(\omega-\omega_0)})$ into the synthesis equation (5.8).

As a consequence of the periodicity and frequency-shifting properties of the discrete-time Fourier transform, there exists a special relationship between ideal lowpass and ideal highpass discrete-time filters. This is illustrated in the next example.

Example 5.7

In Figure 5.12(a) we have depicted the frequency response $H_{\text{lp}}(e^{j\omega})$ of a lowpass filter with cutoff frequency ω_c, while in Figure 5.12(b) we have displayed $H_{\text{lp}}(e^{j(\omega-\pi)})$—that is, the frequency response $H_{\text{lp}}(e^{j\omega})$ shifted by one-half period, i.e., by π. Since high frequencies in discrete time are concentrated near π (and other odd multiples of π), the filter in Figure 5.12(b) is an ideal highpass filter with cutoff frequency $\pi-\omega_c$. That is,

$$H_{\text{hp}}(e^{j\omega}) = H_{\text{lp}}(e^{j(\omega-\pi)}). \tag{5.32}$$

As we can see from eq. (3.122), and as we will discuss again in Section 5.4, the frequency response of an LTI system is the Fourier transform of the impulse response of the system. Thus, if $h_{\text{lp}}[n]$ and $h_{\text{hp}}[n]$ respectively denote the impulse responses of

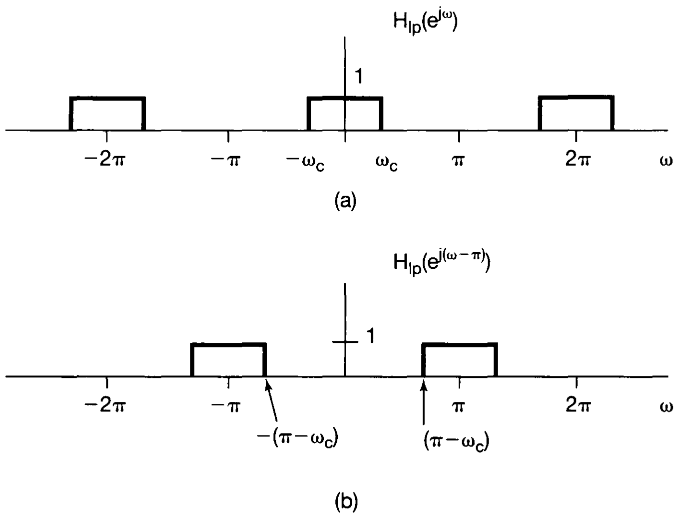


Figure 5.12 (a) Frequency response of a lowpass filter; (b) frequency response of a highpass filter obtained by shifting the frequency response in (a) by $\omega = \pi$ corresponding to one-half period.

Figure 5.12, eq. (5.32) and the frequency-shifting property imply that the lowpass and highpass filters in

$$h_{\mathrm{hp}}[n] = e^{j\pi n} h_{\mathrm{lp}}[n] \tag{5.33}$$

$$= (-1)^n h_{\mathrm{lp}}[n]. \tag{5.34}$$

5.3.4 Conjugation and Conjugate Symmetry

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$\boxed{x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega}).} \tag{5.35}$$

Also, if $x[n]$ is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is,

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n]\text{real}].} \tag{5.36}$$

From this, it follows that $\mathcal{R}e\{X(e^{j\omega})\}$ is an even function of ω and $\mathcal{I}m\{X(e^{j\omega})\}$ is an odd function of ω. Similarly, the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function. Furthermore,

$$\mathcal{E}v\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(e^{j\omega})\}$$

and

$$\mathcal{O}d\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} j\mathcal{I}m\{X(e^{j\omega})\},$$

where $\mathcal{E}v$ and $\mathcal{O}d$ denote the even and odd parts, respectively, of $x[n]$. For example, if $x[n]$ is real and even, its Fourier transform is also real and even. Example 5.2 illustrates this symmetry for $x[n] = a^{|n|}$.

5.3.5 Differencing and Accumulation

In this subsection, we consider the discrete-time counterpart of integration—that is, accumulation—and its inverse, first differencing. Let $x[n]$ be a signal with Fourier transform $X(e^{j\omega})$. Then, from the linearity and time-shifting properties, the Fourier transform pair for the first-difference signal $x[n] - x[n-1]$ is given by

$$\boxed{x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega}).} \tag{5.37}$$

Next, consider the signal

$$y[n] = \sum_{m=-\infty}^{n} x[m]. \tag{5.38}$$

Since $y[n] - y[n-1] = x[n]$, we might conclude that the transform of $y[n]$ should be related to the transform of $x[n]$ by division by $(1 - e^{-j\omega})$. This is partly correct, but as with the continuous-time integration property given by eq. (4.32), there is more involved. The precise relationship is

$$\boxed{\sum_{m=-\infty}^{n} x[m] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k).} \tag{5.39}$$

The impulse train on the right-hand side of eq. (5.39) reflects the dc or average value that can result from summation.

Example 5.8

Let us derive the Fourier transform $X(e^{j\omega})$ of the unit step $x[n] = u[n]$ by making use of the accumulation property and the knowledge that

$$g[n] = \delta[n] \stackrel{\mathcal{F}}{\longleftrightarrow} G(e^{j\omega}) = 1.$$

From Section 1.4.1 we know that the unit step is the running sum of the unit impulse. That is,

$$x[n] = \sum_{m=-\infty}^{n} g[m].$$

Taking the Fourier transform of both sides and using accumulation yields

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{(1-e^{-j\omega})}G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \\ &= \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k). \end{aligned}$$

5.3.6 Time Reversal

Let $x[n]$ be a signal with spectrum $X(e^{j\omega})$, and consider the transform $Y(e^{j\omega})$ of $y[n] = x[-n]$. From eq. (5.9),

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[-n]e^{-j\omega n}. \tag{5.40}$$

Substituting $m = -n$ into eq. (5.40), we obtain

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[m]e^{-j(-\omega)m} = X(e^{-j\omega}). \tag{5.41}$$

That is,

$$\boxed{x[-n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{-j\omega}).} \quad (5.42)$$

5.3.7 Time Expansion

Because of the discrete nature of the time index for discrete-time signals, the relation between time and frequency scaling in discrete time takes on a somewhat different form from its continuous-time counterpart. Specifically, in Section 4.3.5 we derived the continuous-time property

$$x(at) \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right). \quad (5.43)$$

However, if we try to define the signal $x[an]$, we run into difficulties if a is not an integer. Therefore, we cannot slow down the signal by choosing $a < 1$. On the other hand, if we let a be an integer other than ± 1—for example, if we consider $x[2n]$—we do not merely speed up the original signal. That is, since n can take on only integer values, the signal $x[2n]$ consists of the even samples of $x[n]$ alone.

There is a result that does closely parallel eq. (5.43), however. Let k be a positive integer, and define the signal

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases} \quad (5.44)$$

As illustrated in Figure 5.13 for $k = 3$, $x_{(k)}[n]$ is obtained from $x[n]$ by placing $k-1$ zeros between successive values of the original signal. Intuitively, we can think of $x_{(k)}[n]$ as a slowed-down version of $x[n]$. Since $x_{(k)}[n]$ equals 0 unless n is a multiple of k, i.e., unless $n = rk$, we see that the Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}.$$

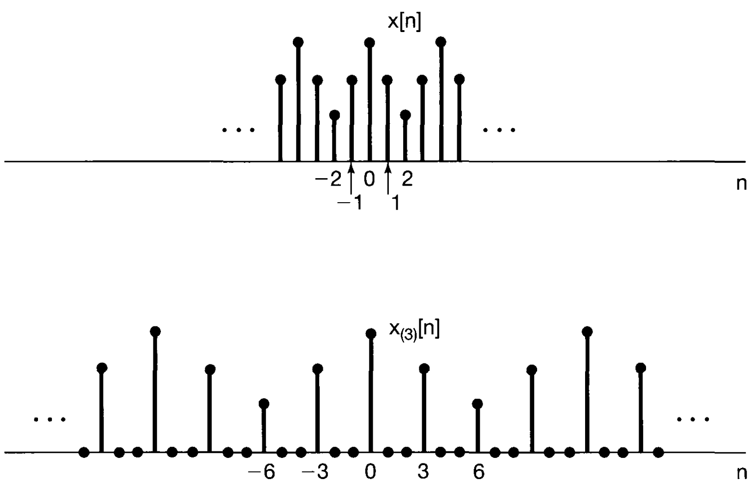


Figure 5.13 The signal $x_{(3)}[n]$ obtained from $x[n]$ by inserting two zeros between successive values of the original signal.

Furthermore, since $x_{(k)}[rk] = x[r]$, we find that

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}).$$

That is,

$$\boxed{x_{(k)}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{jk\omega}).} \tag{5.45}$$

Note that as the signal is spread out and slowed down in time by taking $k > 1$, its Fourier transform is compressed. For example, since $X(e^{j\omega})$ is periodic with period 2π, $X(e^{jk\omega})$ is periodic with period $2\pi/k$. This property is illustrated in Figure 5.14 for a rectangular pulse.

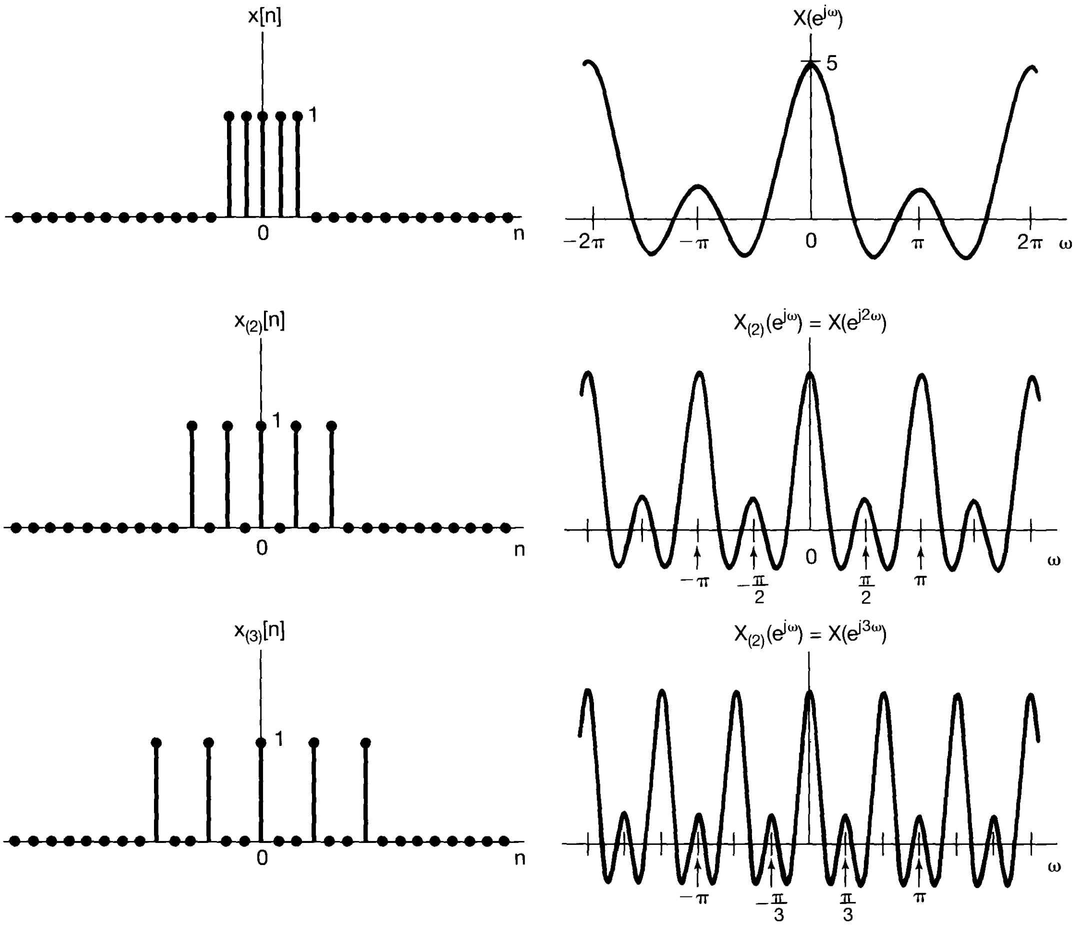


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

Example 5.9

As an illustration of the usefulness of the time-expansion property in determining Fourier transforms, let us consider the sequence $x[n]$ displayed in Figure 5.15(a). This sequence can be related to the simpler sequence $y[n]$ depicted in Figure 5.15(b). In particular

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

where

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

and $y_{(2)}[n-1]$ represents $y_{(2)}[n]$ shifted one unit to the right. The signals $y_{(2)}[n]$ and $2y_{(2)}[n-1]$ are depicted in Figures 5.15(c) and (d), respectively.

Next, note that $y[n] = g[n-2]$, where $g[n]$ is a rectangular pulse as considered in Example 5.3 (with $N_1 = 2$) and as depicted in Figure 5.6(a). Consequently, from Example 5.3 and the time-shifting property, we see that

$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

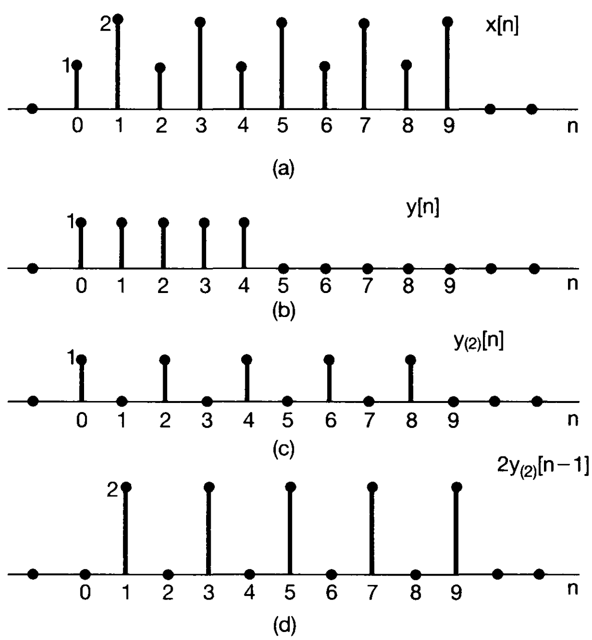


Figure 5.15 (a) The signal $x[n]$ in Example 5.9; (b) the signal $y[n]$; (c) the signal $y_{(2)}[n]$ obtained by inserting one zero between successive values of $y[n]$; and (d) the signal $2y_{(2)}[n-1]$.

Using the time-expansion property, we then obtain

$$y_{(2)}[n] \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)},$$

and using the linearity and time-shifting properties, we get

$$2y_{(2)}[n-1] \overset{\mathfrak{F}}{\longleftrightarrow} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}.$$

Combining these two results, we have

$$X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right).$$

5.3.8 Differentiation in Frequency

Again, let

$$x[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}).$$

If we use the definition of $X(e^{j\omega})$ in the analysis equation (5.9) and differentiate both sides, we obtain

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}.$$

The right-hand side of this equation is the Fourier transform of $-jnx[n]$. Therefore, multiplying both sides by j, we see that

$$\boxed{nx[n] \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.} \tag{5.46}$$

The usefulness of this property will be illustrated in Example 5.13 in Section 5.4.

5.3.9 Parseval's Relation

If $x[n]$ and $X(e^{j\omega})$ are a Fourier transform pair, then

$$\boxed{\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega.} \tag{5.47}$$

We note that this is similar to eq. (4.43), and the derivation proceeds in a similar manner. The quantity on the left-hand side of eq. (5.47) is the total energy in the signal $x[n]$, and

Parseval's relation states that this energy can also be determined by integrating the energy per unit frequency, $|X(e^{j\omega})|^2/2\pi$, over a full 2π interval of distinct discrete-time frequencies. In analogy with the continuous-time case, $|X(e^{j\omega})|^2$ is referred to as the *energy-density spectrum* of the signal $x[n]$. Note also that eq. (5.47) is the counterpart for aperiodic signals of Parseval's relation, eq. (3.110), for periodic signals, which equates the average power in a periodic signal with the sum of the average powers of its individual harmonic components.

Given the Fourier transform of a sequence, it is possible to use Fourier transform properties to determine whether a particular sequence has a number of different properties. To illustrate this idea, we present the following example.

Example 5.10

Consider the sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ is depicted for $-\pi \leq \omega \leq \pi$ in Figure 5.16. We wish to determine whether or not, in the time domain, $x[n]$ is periodic, real, even, and/or of finite energy.

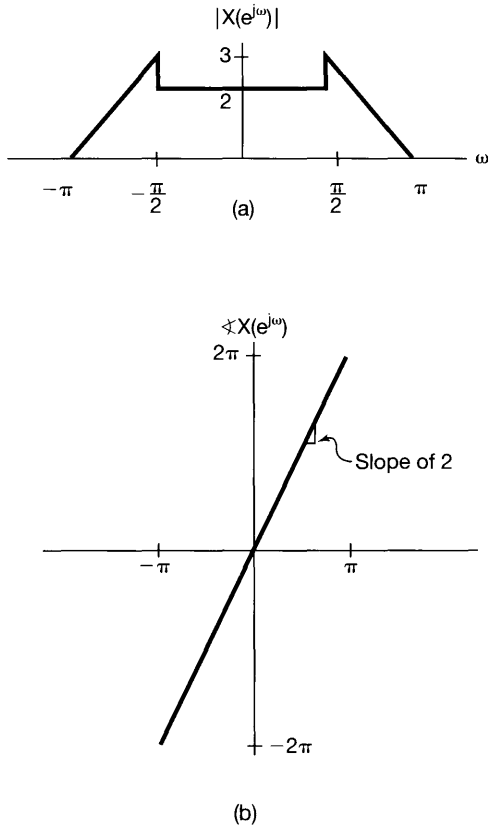


Figure 5.16 Magnitude and phase of the Fourier transform for Example 5.10.

Accordingly, we note first that periodicity in the time domain implies that the Fourier transform is zero, except possibly for impulses located at various integer multiples of the fundamental frequency. This is not true for $X(e^{j\omega})$. We conclude, then, that $x[n]$ is *not* periodic.

Next, from the symmetry properties for Fourier transforms, we know that a real-valued sequence must have a Fourier transform of even magnitude and a phase function that is odd. This is true for the given $|X(e^{j\omega})|$ and $\sphericalangle X(e^{j\omega})$. We thus conclude that $x[n]$ is real.

Third, if $x[n]$ is an even function, then, by the symmetry properties for real signals, $X(e^{j\omega})$ must be real and even. However, since $X(e^{j\omega}) = |X(e^{j\omega})|e^{-j2\omega}$, $X(e^{j\omega})$ is not a real-valued function. Consequently, $x[n]$ is not even.

Finally, to test for the finite-energy property, we may use Parseval's relation,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega.$$

It is clear from Figure 5.16 that integrating $|X(e^{j\omega})|^2$ from $-\pi$ to π will yield a finite quantity. We conclude that $x[n]$ has finite energy.

In the next few sections, we consider several additional properties. The first two of these are the convolution and multiplication properties, similar to those discussed in Sections 4.4 and 4.5. The third is the property of duality, which is examined in Section 5.7, where we consider not only duality in the discrete-time domain, but also the duality that exists *between* the continuous-time and discrete-time domains.

5.4 THE CONVOLUTION PROPERTY

In Section 4.4, we discussed the importance of the continuous-time Fourier transform with regard to its effect on the operation of convolution and its use in dealing with continuous-time LTI systems. An identical relation applies in discrete time, and this is one of the principal reasons that the discrete-time Fourier transform is of such great value in representing and analyzing discrete-time LTI systems. Specifically, if $x[n]$, $h[n]$, and $y[n]$ are the input, impulse response, and output, respectively, of an LTI system, so that

$$y[n] = x[n] * h[n],$$

then

$$\boxed{Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}),} \tag{5.48}$$

where $X(e^{j\omega})$, $H(e^{j\omega})$, and $Y(e^{j\omega})$ are the Fourier transforms of $x[n]$, $h[n]$, and $y[n]$, respectively. Furthermore, comparing eqs. (3.122) and (5.9), we see that the frequency response of a discrete-time LTI system, as first defined in Section 3.8, is the Fourier transform of the impulse response of the system.

The derivation of eq. (5.48) exactly parallels that carried out in Section 4.4. In particular, as in continuous time, the Fourier synthesis equation (5.8) for $x[n]$ can be interpreted as a decomposition of $x[n]$ into a linear combination of complex exponentials with infinitesimal amplitudes proportional to $X(e^{j\omega})$. Each of these exponentials is an eigenfunction of the system. In Chapter 3, we used this fact to show that the Fourier series coefficients of the response of an LTI system to a periodic input are simply the Fourier coefficients of the input multiplied by the system's frequency response evaluated at the corresponding harmonic frequencies. The convolution property (5.48) represents the extension of this result to aperiodic inputs and outputs by using the Fourier transform rather than the Fourier series.

As in continuous time, eq. (5.48) maps the convolution of two signals to the simple algebraic operation of multiplying their Fourier transforms, a fact that both facilitates the analysis of signals and systems and adds significantly to our understanding of the way in which an LTI system responds to the input signals that are applied to it. In particular, from eq. (5.48), we see that the frequency response $H(e^{j\omega})$ captures the change in complex amplitude of the Fourier transform of the input at each frequency ω. Thus, in frequency-selective filtering, for example, we want $H(e^{j\omega}) \approx 1$ over the range of frequencies corresponding to the desired passband and $H(e^{j\omega}) \approx 0$ over the band of frequencies to be eliminated or significantly attenuated.

5.4.1 Examples

To illustrate the convolution property, along with a number of other properties, we consider several examples in this section.

Example 5.11

Consider an LTI system with impulse response

$$h[n] = \delta[n - n_0].$$

The frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0]e^{-j\omega n} = e^{-j\omega n_0}.$$

Thus, for any input $x[n]$ with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is

$$Y(e^{j\omega}) = e^{-j\omega n_0}X(e^{j\omega}). \tag{5.49}$$

We note that, for this example, $y[n] = x[n - n_0]$ and eq. (5.49) is consistent with the time-shifting property. Note also that the frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ of a pure time shift has unity magnitude at all frequencies and a phase characteristic $-\omega n_0$ that is linear with frequency.

Example 5.12

Consider the discrete-time ideal lowpass filter introduced in Section 3.9.2. This system has the frequency response $H(e^{j\omega})$ illustrated in Figure 5.17(a). Since the impulse response and frequency response of an LTI system are a Fourier transform pair, we can determine the impulse response of the ideal lowpass filter from the frequency response using the Fourier transform synthesis equation (5.8). In particular, using $-\pi \leq \omega \leq \pi$ as the interval of integration in that equation, we see from Figure 5.17(a) that

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \frac{\sin \omega_c n}{\pi n}, \end{aligned} \tag{5.50}$$

which is shown in Figure 5.17(b).

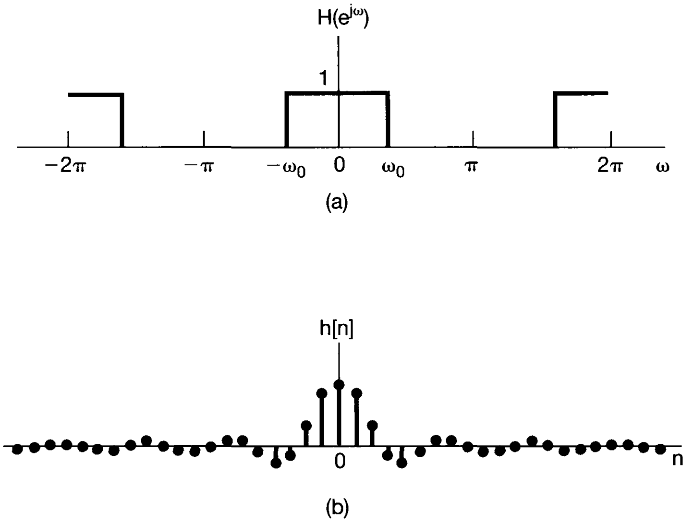


Figure 5.17 (a) Frequency response of a discrete-time ideal lowpass filter; (b) impulse response of the ideal lowpass filter.

In Figure 5.17, we come across many of the same issues that surfaced with the continuous-time ideal lowpass filter in Example 4.18. First, since $h[n]$ is not zero for $n < 0$, the ideal lowpass filter is not causal. Second, even if causality is not an important issue, there are other reasons, including ease of implementation and preferable time domain characteristics, that nonideal filters are generally used to perform frequency-selective filtering. In particular, the impulse response of the ideal lowpass filter in Figure 5.17(b) is oscillatory, a characteristic that is undesirable in some applications. In such cases, a tradeoff between frequency-domain objectives such as frequency selectivity and time-domain properties such as nonoscillatory behavior must be made. In Chapter 6, we will discuss these and related ideas in more detail.

As the following example illustrates, the convolution property can also be of value in facilitating the calculation of convolution sums.

Example 5.13

Consider an LTI system with impulse response

$$h[n] = \alpha^n u[n],$$

with $|\alpha| < 1$, and suppose that the input to this system is

$$x[n] = \beta^n u[n],$$

with $|\beta| < 1$. Evaluating the Fourier transforms of $h[n]$ and $x[n]$, we have

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \tag{5.51}$$

and

$$X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}, \tag{5.52}$$

so that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}. \tag{5.53}$$

As with Example 4.19 , determining the inverse transform of $Y(e^{j\omega})$ is most easily done by expanding $Y(e^{j\omega})$ by the method of partial fractions. Specifically, $Y(e^{j\omega})$ is a ratio of polynomials in powers of $e^{-j\omega}$, and we would like to express this as a sum of simpler terms of this type so that we can find the inverse transform of each term by inspection (together, perhaps, with the use of the frequency differentiation property of Section 5.3.8). The general algebraic procedure for rational transforms is described in the appendix. For this example, if $\alpha \neq \beta$, the partial fraction expansion of $Y(e^{j\omega})$ is of the form

$$Y(e^{j\omega}) = \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}}. \tag{5.54}$$

Equating the right-hand sides of eqs (5.53) and (5.54), we find that

$$A = \frac{\alpha}{\alpha - \beta}, \quad B = -\frac{\beta}{\alpha - \beta}.$$

Therefore, from Example 5.1 and the linearity property, we can obtain the inverse transform of eq. (5.54) by inspection:

$$\begin{aligned} y[n] &= \frac{\alpha}{\alpha - \beta}\alpha^n u[n] - \frac{\beta}{\alpha - \beta}\beta^n u[n] \\ &= \frac{1}{\alpha - \beta}[\alpha^{n+1}u[n] - \beta^{n+1}u[n]]. \end{aligned} \tag{5.55}$$

For $\alpha = \beta$, the partial-fraction expansion in eq. (5.54) is not valid. However, in this case,

$$Y(e^{j\omega}) = \left(\frac{1}{1 - \alpha e^{-j\omega}}\right)^2,$$

which can be expressed as

$$Y(e^{j\omega}) = \frac{j}{\alpha}e^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right). \tag{5.56}$$

As in Example 4.19, we can use the frequency differentiation property, eq. (5.46), together with the Fourier transform pair

$$\alpha^n u[n] \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\omega}},$$

to conclude that

$$n\alpha^n u[n] \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right).$$

To account for the factor $e^{j\omega}$, we use the time-shifting property to obtain

$$(n+1)\alpha^{n+1}u[n+1] \overset{\mathfrak{F}}{\longleftrightarrow} je^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right),$$

and finally, accounting for the factor $1/\alpha$, in eq. (5.56), we obtain

$$y[n] = (n+1)\alpha^n u[n+1]. \tag{5.57}$$

It is worth noting that, although the right-hand side is multiplied by a step that begins at $n = -1$, the sequence $(n+1)\alpha^n u[n+1]$ is still zero prior to $n = 0$, since the factor $n+1$ is zero at $n = -1$. Thus, we can alternatively express $y[n]$ as

$$y[n] = (n+1)\alpha^n u[n]. \tag{5.58}$$

As illustrated in the next example, the convolution property, along with other Fourier transform properties, is often useful in analyzing system interconnections.

Example 5.14

Consider the system shown in Figure 5.18(a) with input $x[n]$ and output $y[n]$. The LTI systems with frequency response $H_{lp}(e^{j\omega})$ are ideal lowpass filters with cutoff frequency $\pi/4$ and unity gain in the passband.

Let us first consider the top path in Figure 5.18(a). The Fourier transform of the signal $w_1[n]$ can be obtained by noting that $(-1)^n = e^{j\pi n}$ so that $w_1[n] = e^{j\pi n}x[n]$. Using the frequency-shifting property, we then obtain

$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)}).$$

The convolution property yields

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)}).$$

Since $w_3[n] = e^{j\pi n}w_2[n]$, we can again apply the frequency-shifting property to obtain

$$\begin{aligned} W_3(e^{j\omega}) &= W_2(e^{j(\omega-\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}). \end{aligned}$$

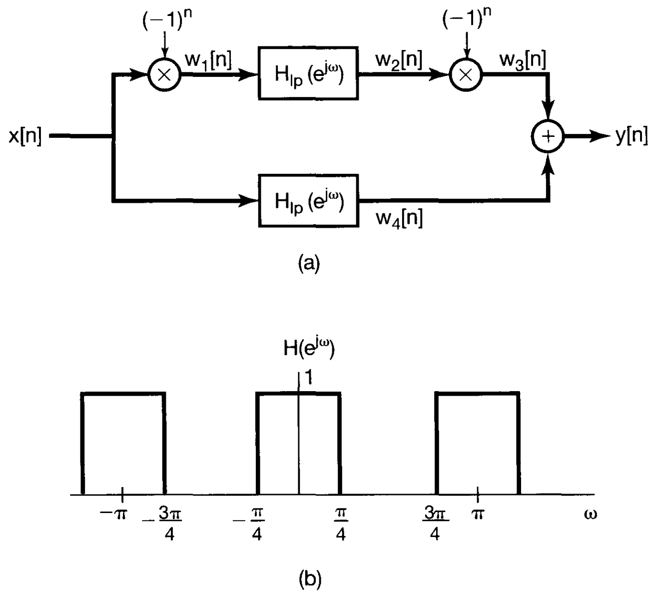


Figure 5.18 (a) System interconnection for Example 5.14; (b) the overall frequency response for this system.

Since discrete-time Fourier transforms are always periodic with period 2π,

$$W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega}).$$

Applying the convolution property to the lower path, we get

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega}).$$

From the linearity property of the Fourier transform, we obtain

$$\begin{aligned} Y(e^{j\omega}) &= W_3(e^{j\omega}) + W_4(e^{j\omega}) \\ &= [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]X(e^{j\omega}). \end{aligned}$$

Consequently, the overall system in Figure 5.18(a) has the frequency response

$$H(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]$$

which is shown in Figure 5.18(b).

As we saw in Example 5.7, $H_{lp}(e^{j(\omega-\pi)})$ is the frequency response of an ideal highpass filter. Thus, the overall system passes both low and high frequencies and stops frequencies between these two passbands. That is, the filter has what is often referred to as an *ideal bandstop characteristic,* where the stopband is the region $\pi/4 < |\omega| < 3\pi/4$.

It is important to note that, as in continuous time, not every discrete-time LTI system has a frequency response. For example, the LTI system with impulse response $h[n] = 2^n u[n]$ does not have a finite response to sinusoidal inputs, which is reflected in the fact that the Fourier transform analysis equation for $h[n]$ diverges. However, if an LTI system is stable, then, from Section 2.3.7, its impulse response is absolutely summable; that is,

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty. \tag{5.59}$$

Therefore, the frequency response always converges for stable systems. In using Fourier methods, we will be restricting ourselves to systems with impulse responses that have well-defined Fourier transforms. In Chapter 10, we will introduce an extension of the Fourier transform referred to as the z-transform that will allow us to use transform techniques for LTI systems for which the frequency response does not converge.

5.5 THE MULTIPLICATION PROPERTY

In Section 4.5, we introduced the multiplication property for continuous-time signals and indicated some of its applications through several examples. An analogous property exists for discrete-time signals and plays a similar role in applications. In this section, we derive this result directly and give an example of its use. In Chapters 7 and 8, we will use the multiplication property in the context of our discussions of sampling and communications.

Consider $y[n]$ equal to the product of $x_1[n]$ and $x_2[n]$, with $Y(e^{j\omega})$, $X_1(e^{j\omega})$, and $X_2(e^{j\omega})$ denoting the corresponding Fourier transforms. Then

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n]e^{-j\omega n},$$

or since

$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta, \tag{5.60}$$

it follows that

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\} e^{-j\omega n}. \tag{5.61}$$

Interchanging the order of summation and integration, we obtain

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\left[\sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n}\right]d\theta. \tag{5.62}$$

The bracketed summation is $X_2(e^{j(\omega-\theta)})$, and consequently, eq. (5.62) becomes

$$\boxed{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.} \tag{5.63}$$

Equation (5.63) corresponds to a *periodic* convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, and the integral in this equation can be evaluated over any interval of length 2π. The usual form of convolution (in which the integral ranges from $-\infty$ to $+\infty$) is often referred to as aperiodic convolution to distinguish it from periodic convolution. The mechanics of periodic convolution are most easily illustrated through an example.

Example 5.15

Consider the problem of finding the Fourier transform $X(e^{j\omega})$ of a signal $x[n]$ which is the product of two other signals; that is,

$$x[n] = x_1[n]x_2[n],$$

where

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

and

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

From the multiplication property given in eq. (5.63), we know that $X(e^{j\omega})$ is the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, where the integral in eq. (5.63) can be taken over any interval of length 2π. Choosing the interval $-\pi < \theta \leq \pi$, we obtain

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta. \tag{5.64}$$

Equation (5.64) resembles aperiodic convolution, except for the fact that the integration is limited to the interval $-\pi < \theta \leq \pi$. However, we can convert the equation into an ordinary convolution by defining

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & \text{for } -\pi < \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}.$$

Then, replacing $X_1(e^{j\theta})$ in eq. (5.64) by $\hat{X}_1(e^{j\theta})$, and using the fact that $\hat{X}_1(e^{j\theta})$ is zero for $|\theta| > \pi$, we see that

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta. \end{aligned}$$

Thus, $X(e^{j\omega})$ is $1/2\pi$ times the aperiodic convolution of the rectangular pulse $\hat{X}_1(e^{j\omega})$ and the periodic square wave $X_2(e^{j\omega})$, both of which are shown in Figure 5.19. The result of this convolution is the Fourier transform $X(e^{j\omega})$ shown in Figure 5.20.

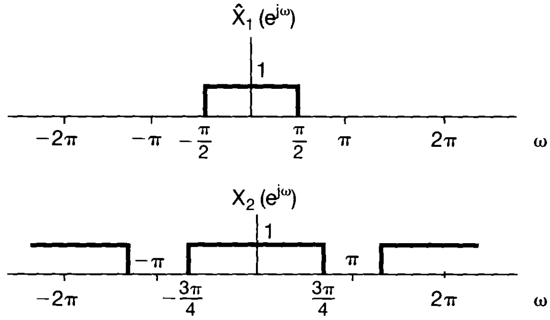


Figure 5.19 $\hat{X}_1(e^{j\omega})$ representing one period of $X_1(e^{j\omega})$, and $X_2(e^{j\omega})$. The linear convolution of $\hat{X}_1(e^{j\omega})$ and $X_2(e^{j\omega})$ corresponds to the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$.

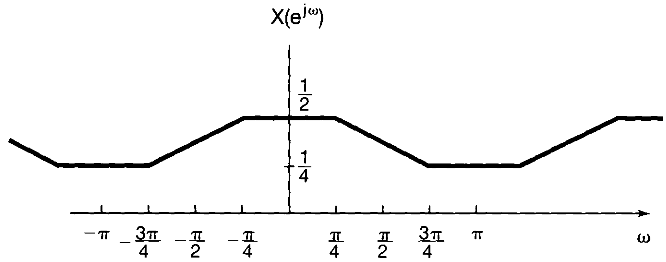


Figure 5.20 Result of the periodic convolution in Example 5.15.

5.6 TABLES OF FOURIER TRANSFORM PROPERTIES AND BASIC FOURIER TRANSFORM PAIRS

In Table 5.1, we summarize a number of important properties of the discrete-time Fourier transform and indicate the section of the text in which each is discussed. In Table 5.2, we summarize some of the basic and most important discrete-time Fourier transform pairs. Many of these have been derived in examples in the chapter.

5.7 DUALITY

In considering the continuous-time Fourier transform, we observed a symmetry or duality between the analysis equation (4.9) and the synthesis equation (4.8). No corresponding duality exists between the analysis equation (5.9) and the synthesis equation (5.8) for the discrete-time Fourier transform. However, there *is* a duality in the discrete-time Fourier *series* equations (3.94) and (3.95), which we develop in Section 5.7.1. In addition, there is

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & N_1 < \|n\| \le N/2 \end{cases}$ and $x[n + N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \|a\| < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \|n\| \le N_1 \\ 0, & \|n\| > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \|\omega\| \le W \\ 0, & W < \|\omega\| \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n + 1)a^n u[n], \quad \|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r - 1)!} a^n u[n], \quad \|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

signal, reversed in time). To see this in more detail, consider two periodic sequences with period N, related through the summation

$$f[m] = \frac{1}{N} \sum_{r=\langle N \rangle} g[r] e^{-jr(2\pi/N)m}. \tag{5.65}$$

If we let $m = k$ and $r = n$, eq. (5.65) becomes

$$f[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk(2\pi/N)n}.$$

Comparing this with eq. (3.95), we see that the sequence $f[k]$ corresponds to the Fourier series coefficients of the signal $g[n]$. That is, if we adopt the notation

$$x[n] \overset{\mathfrak{FS}}{\longleftrightarrow} a_k$$

introduced in Chapter 3 for a periodic discrete-time signal and its set of Fourier coefficients, the two periodic sequences related through eq. (5.65) satisfy

$$g[n] \overset{\mathfrak{FS}}{\longleftrightarrow} f[k]. \tag{5.66}$$

Alternatively, if we let $m = n$ and $r = -k$, eq. (5.65) becomes

$$f[n] = \sum_{k=\langle N \rangle} \frac{1}{N} g[-k] e^{jk(2\pi/N)n}.$$

Comparing this with eq. (3.94), we find that $(1/N)g[-k]$ corresponds to the sequence of Fourier series coefficients of $f[n]$. That is,

$$f[n] \overset{\mathfrak{FS}}{\longleftrightarrow} \frac{1}{N} g[-k]. \tag{5.67}$$

As in continuous time, this duality implies that every property of the discrete-time Fourier series has a dual. For example, referring to Table 3.2, we see that the pair of properties

$$x[n - n_0] \overset{\mathfrak{FS}}{\longleftrightarrow} a_k e^{-jk(2\pi/N)n_0} \tag{5.68}$$

and

$$e^{jm(2\pi/N)n} x[n] \overset{\mathfrak{FS}}{\longleftrightarrow} a_{k-m} \tag{5.69}$$

are dual. Similarly, from the same table, we can extract another pair of dual properties:

$$\sum_{r=\langle N \rangle} x[r] y[n-r] \overset{\mathfrak{FS}}{\longleftrightarrow} N a_k b_k \tag{5.70}$$

and

$$x[n] y[n] \overset{\mathfrak{FS}}{\longleftrightarrow} \sum_{l=\langle N \rangle} a_l b_{k-l}. \tag{5.71}$$

In addition to its consequences for the properties of discrete-time Fourier series, duality can often be useful in reducing the complexity of the calculations involved in determining Fourier series representations. This is illustrated in the following example.

Example 5.16

Consider the following periodic signal with a period of $N = 9$:

$$x[n] = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ \dfrac{5}{9}, & n = \text{multiple of } 9 \end{cases} \tag{5.72}$$

In Chapter 3, we found that a rectangular square wave has Fourier coefficients in a form much as in eq. (5.72). Duality, then, suggests that the coefficients for $x[n]$ must be in the form of a rectangular square wave. To see this more precisely, let $g[n]$ be a rectangular square wave with period $N = 9$ such that

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4. \end{cases}$$

The Fourier series coefficients b_k for $g[n]$ can be determined from Example 3.12 as

$$b_k = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ \dfrac{5}{9}, & k = \text{multiple of } 9 \end{cases}$$

The Fourier series analysis equation (3.95) for $g[n]$ can now be written as

$$b_k = \frac{1}{9}\sum_{n=-2}^{2}(1)e^{-j2\pi nk/9}.$$

Interchanging the names of the variables k and n and noting that $x[n] = b_n$, we find that

$$x[n] = \frac{1}{9}\sum_{k=-2}^{2}(1)e^{-j2\pi nk/9}.$$

Letting $k' = -k$ in the sum on the right side, we obtain

$$x[n] = \frac{1}{9}\sum_{k'=-2}^{2} e^{+j2\pi nk'/9}.$$

Finally, moving the factor 1/9 inside the summation, we see that the right side of this equation has the form of the synthesis equation (3.94) for $x[n]$. We thus conclude that the Fourier coefficients of $x[n]$ are given by

$$a_k = \begin{cases} 1/9, & |k| \leq 2 \\ 0, & 2 < |k| \leq 4, \end{cases}$$

and, of course, are periodic with period $N = 9$.

5.7.2 Duality between the Discrete-Time Fourier Transform and the Continuous-Time Fourier Series

In addition to the duality for the discrete Fourier series, there is a duality between the *discrete-time* Fourier *transform* and the *continuous-time* Fourier *series*. Specifically, let us compare the continuous-time Fourier series equations (3.38) and (3.39) with the discrete-time Fourier transform equations (5.8) and (5.9). We repeat these equations here for convenience:

[eq. (5.8)] $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega, \tag{5.73}$$

[eq. (5.9)] $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \tag{5.74}$$

[eq. (3.38)] $$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \tag{5.75}$$

[eq. (3.39)] $$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt. \tag{5.76}$$

Note that eqs. (5.73) and (5.76) are very similar, as are eqs. (5.74) and (5.75), and in fact, we can interpret eqs. (5.73) and (5.74) as a *Fourier series* representation of the periodic frequency response $X(e^{j\omega})$. In particular, since $X(e^{j\omega})$ is a periodic function of ω with period 2π, it has a Fourier series representation as a weighted sum of harmonically related periodic exponential functions of ω, all of which have the common period of 2π. That is, $X(e^{j\omega})$ can be represented in a Fourier series as a weighted sum of the signals $e^{j\omega n}$, $n = 0, \pm 1, \pm 2, \ldots$. From eq. (5.74), we see that the nth Fourier coefficient in this expansion—i.e., the coefficient multiplying $e^{j\omega n}$—is $x[-n]$. Furthermore, since the period of $X(e^{j\omega})$ is 2π, eq. (5.73) can be interpreted as the Fourier series analysis equation for the Fourier series coefficient $x[n]$—i.e., for the coefficient multiplying $e^{-j\omega n}$ in the expression for $X(e^{j\omega})$ in eq. (5.74). The use of this duality relationship is best illustrated with an example.

Example 5.17

The duality between the discrete-time Fourier transform synthesis equation and the continuous-time Fourier series analysis equation may be exploited to determine the discrete-time Fourier transform of the sequence

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

To use duality, we first must identify a continuous-time signal $g(t)$ with period $T = 2\pi$ and Fourier coefficients $a_k = x[k]$. From Example 3.5, we know that if $g(t)$ is a periodic square wave with period 2π (or, equivalently, with fundamental frequency $\omega_0 = 1$) and with

$$g(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & T_1 < |t| \le \pi \end{cases},$$

then the Fourier series coefficients of $g(t)$ are

$$a_k = \frac{\sin(kT_1)}{k\pi}$$

Consequently, if we take $T_1 = \pi/2$, we will have $a_k = x[k]$. In this case the analysis equation for $g(t)$ is

$$\frac{\sin(\pi k/2)}{\pi k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-jkt}dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{-jkt}dt.$$

Renaming k as n and t as ω, we have

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{-jn\omega}d\omega. \tag{5.77}$$

Replacing n by $-n$ on both sides of eq. (5.77) and noting that the sinc function is even, we obtain

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{jn\omega}d\omega.$$

The right-hand side of this equation has the form of the Fourier transform synthesis equation for $x[n]$, where

$$X(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \pi/2 \\ 0 & \pi/2 < |\omega| \le \pi \end{cases}.$$

In Table 5.3, we present a compact summary of the Fourier series and Fourier transform expressions for both continuous-time and discrete-time signals, and we also indicate the duality relationships that apply in each case.

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

(duality: Fourier Series discrete time ⟷ discrete frequency; Fourier Transform continuous time ⟷ continuous frequency; continuous-time Fourier series frequency domain ⟷ discrete-time Fourier transform time domain)

5.8 SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

A general linear constant-coefficient difference equation for an LTI system with input $x[n]$ and output $y[n]$ is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{5.78}$$

The class of systems described by such difference equations is quite an important and useful one. In this section, we take advantage of several of the properties of the discrete-time Fourier transform to determine the frequency response $H(e^{j\omega})$ for an LTI system described by such an equation. The approach we follow closely parallels the discussion in Section 4.7 for continuous-time LTI systems described by linear constant-coefficient differential equations.

There are two related ways in which to determine $H(e^{j\omega})$. The first of these, which we illustrated in Section 3.11 for several simple difference equations, explicitly uses the fact that complex exponentials are eigenfunctions of LTI systems. Specifically, if $x[n] = e^{j\omega n}$ is the input to an LTI system, then the output must be of the form $H(e^{j\omega})e^{j\omega n}$. Substituting these expressions into eq. (5.78) and performing some algebra allows us to solve for $H(e^{j\omega})$. In this section, we follow a second approach making use of the convolution, linearity, and time-shifting properties of the discrete-time Fourier transform. Let $X(e^{j\omega})$, $Y(e^{j\omega})$, and $H(e^{j\omega})$ denote the Fourier transforms of the input $x[n]$, output $y[n]$, and impulse response $h[n]$, respectively. The convolution property, eq. (5.48), of the discrete-time Fourier transform then implies that

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}. \tag{5.79}$$

Applying the Fourier transform to both sides of eq. (5.78) and using the linearity and time-shifting properties, we obtain the expression

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}),$$

or equivalently,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}. \tag{5.80}$$

Comparing eq. (5.80) with eq. (4.76), we see that, as in the case of continuous time, $H(e^{j\omega})$ is a ratio of polynomials, but in discrete time the polynomials are in the variable $e^{-j\omega}$. The coefficients of the numerator polynomial are the same coefficients as appear on the right side of eq. (5.78), and the coefficients of the denominator polynomial are the same as appear on the left side of that equation. Therefore, the frequency response of the LTI system specified by eq. (5.78) can be written down by inspection.

The difference equation (5.78) is generally referred to as an Nth-order difference equation, as it involves delays in the output $y[n]$ of up to N time steps. Also, the denominator of $H(e^{j\omega})$ in eq. (5.80) is an Nth-order polynomial in $e^{-j\omega}$.

Example 5.18

Consider the causal LTI system that is characterized by the difference equation

$$y[n] - ay[n-1] = x[n], \tag{5.81}$$

with $|a| < 1$. From eq. (5.80), the frequency response of this system is

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{5.82}$$

Comparing this with Example 5.1, we recognize it as the Fourier transform of the sequence $a^n u[n]$. Thus, the impulse response of the system is

$$h[n] = a^n u[n]. \tag{5.83}$$

Example 5.19

Consider a causal LTI system that is characterized by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]. \tag{5.84}$$

From eq. (5.80), the frequency response is

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}. \tag{5.85}$$

As a first step in obtaining the impulse response, we factor the denominator of eq. (5.85):

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}. \tag{5.86}$$

$H(e^{j\omega})$ can be expanded by the method of partial fractions, as in Example A.3 in the appendix. The result of this expansion is

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}. \tag{5.87}$$

The inverse transform of each term can be recognized by inspection, with the result that

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]. \tag{5.88}$$

The procedure followed in Example 5.19 is identical in style to that used in continuous time. Specifically, after expanding $H(e^{j\omega})$ by the method of partial fractions, we can find the inverse transform of each term by inspection. The same approach can be applied to the frequency response of any LTI system described by a linear constant-coefficient difference equation in order to determine the system impulse response. Also, as illustrated in the next example, if the Fourier transform $X(e^{j\omega})$ of the input to such a system is a ratio of polynomials in $e^{-j\omega}$, then $Y(e^{j\omega})$ is as well. In this case, we can use the same technique to find the response $y[n]$ to the input $x[n]$.

Example 5.20

Consider the LTI system of Example 5.19, and let the input to this system be

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

Then, using eq. (5.80) and Example 5.1 or 5.18, we obtain

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left[\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$
$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-e\omega})^2}. \quad (5.89)$$

As described in the appendix, the form of the partial-fraction expansion in this case is

$$Y(e^{j\omega}) = \frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}} + \frac{B_{12}}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}, \quad (5.90)$$

where the constants B_{11}, B_{12}, and B_{21} can be determined using the techniques described in the appendix. This particular expansion is worked out in detail in Example A.4, and the values obtained are

$$B_{11} = -4, \quad B_{12} = -2, \quad B_{21} = 8,$$

so that

$$Y(e^{j\omega}) = -\frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}. \quad (5.91)$$

The first and third terms are of the same type as those encountered in Example 5.19, while the second term is of the same form as one seen in Example 5.13. Either from these examples or from Table 5.2, we can invert each of the terms in eq. (5.91) to obtain the inverse transform

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\} u[n]. \quad (5.92)$$

5.9 SUMMARY

In this chapter, we have paralleled Chapter 4 as we developed the Fourier transform for discrete-time signals and examined many of its important properties. Throughout the chapter, we have seen a great many similarities between continuous-time and discrete-time Fourier analysis, and we have also seen some important differences. For example, the relationship between Fourier series and Fourier transforms in discrete time is exactly analogous to that in continuous time. In particular, our derivation of the discrete-time Fourier transform for aperiodic signals from the discrete-time Fourier series representations is very much the same as the corresponding continuous-time derivation. Furthermore, many of the properties of continuous-time transforms have exact discrete-time counterparts. On the other hand, in contrast to the continuous-time case, the discrete-time Fourier transform of an aperiodic signal is always periodic with period 2π. In addition to similarities and differences such as these, we have described the duality relationships among the Fourier representations of continuous-time and discrete-time signals.

The most important similiarities between continuous- and discrete-time Fourier analysis are in their uses in analyzing and representing signals and LTI systems. Specifically, the convolution property provides us with the basis for the frequency-domain analysis of LTI systems. We have already seen some of the utility of this approach in our discussion of filtering in Chapters 3–5 and in our examination of systems described by linear constant-coefficient differential or difference equations, and we will gain a further appreciation for its utility in Chapter 6, in which we examine filtering and time-versus-frequency issues in more detail. In addition, the multiplication properties in continuous and discrete time are essential to our development of sampling in Chapter 7 and communications in Chapter 8.

Chapter 5 Problems

The first section of problems belongs to the basic category and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

5.1. Use the Fourier transform analysis equation (5.9) to calculate the Fourier transforms of:
(a) $(\frac{1}{2})^{n-1}u[n-1]$ **(b)** $(\frac{1}{2})^{|n-1|}$
Sketch and label one period of the magnitude of each Fourier transform.

5.2. Use the Fourier transform analysis equation (5.9) to calculate the Fourier transforms of:
(a) $\delta[n-1]+\delta[n+1]$ **(b)** $\delta[n+2]-\delta[n-2]$
Sketch and label one period of the magnitude of each Fourier transform.

5.3. Determine the Fourier transform for $-\pi \le \omega < \pi$ in the case of each of the following periodic signals:
(a) $\sin(\frac{\pi}{3}n+\frac{\pi}{4})$ **(b)** $2+\cos(\frac{\pi}{6}n+\frac{\pi}{8})$

5.4. Use the Fourier transform synthesis equation (5.8) to determine the inverse Fourier transforms of:
(a) $X_1(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\{2\pi\delta(\omega-2\pi k)+\pi\delta(\omega-\frac{\pi}{2}-2\pi k)+\pi\delta(\omega+\frac{\pi}{2}-2\pi k)\}$
(b) $X_2(e^{j\omega}) = \begin{cases} 2j, & 0<\omega\le\pi \\ -2j, & -\pi<\omega\le 0 \end{cases}$

5.5. Use the Fourier transform synthesis equation (5.8) to determine the inverse Fourier transform of $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}$, where

$$|X(e^{j\omega})| = \begin{cases} 1, & 0\le|\omega|<\frac{\pi}{4} \\ 0, & \frac{\pi}{4}\le|\omega|\le\pi \end{cases} \quad \text{and } \sphericalangle X(e^{j\omega}) = -\frac{3\omega}{2}.$$

Use your answer to determine the values of n for which $x[n]=0$.

5.6. Given that $x[n]$ has Fourier transform $X(e^{j\omega})$, express the Fourier transforms of the following signals in terms of $X(e^{j\omega})$. You may use the Fourier transform properties listed in Table 5.1.
(a) $x_1[n] = x[1-n]+x[-1-n]$
(b) $x_2[n] = \frac{x^*[-n]+x[n]}{2}$
(c) $x_3[n] = (n-1)^2x[n]$

5.7. For each of the following Fourier transforms, use Fourier transform properties (Table 5.1) to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

(a) $X_1(e^{j\omega}) = e^{-j\omega}\sum_{k=1}^{10}(\sin k\omega)$

(b) $X_2(e^{j\omega}) = j\sin(\omega)\cos(5\omega)$

(c) $X_3(e^{j\omega}) = A(\omega) + e^{jB(\omega)}$ where

$$A(\omega) = \begin{cases} 1, & 0 \le |\omega| \le \frac{\pi}{8} \\ 0, & \frac{\pi}{8} < |\omega| \le \pi \end{cases} \quad \text{and } B(\omega) = -\frac{3\omega}{2} + \pi.$$

5.8. Use Tables 5.1 and 5.2 to help determine $x[n]$ when its Fourier transform is

$$X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}}\left(\frac{\sin\frac{3}{2}\omega}{\sin\frac{\omega}{2}}\right) + 5\pi\delta(\omega), \qquad -\pi < \omega \le \pi$$

5.9. The following four facts are given about a real signal $x[n]$ with Fourier transform $X(e^{jw})$:

1. $x[n] = 0$ for $n > 0$.
2. $x[0] > 0$.
3. $\mathcal{I}m\{X(e^{j\omega})\} = \sin\omega - \sin 2\omega$.
4. $\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2\,d\omega = 3$.

Determine $x[n]$.

5.10. Use Tables 5.1 and 5.2 in conjunction with the fact that

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$$

to determine the numerical value of

$$A = \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n.$$

5.11. Consider a signal $g[n]$ with Fourier transform $G(e^{j\omega})$. Suppose

$$g[n] = x_{(2)}[n],$$

where the signal $x[n]$ has a Fourier transform $X(e^{j\omega})$. Determine a real number α such that $0 < \alpha < 2\pi$ and $G(e^{j\omega}) = G(e^{j(\omega-\alpha)})$.

5.12. Let

$$y[n] = \left(\frac{\sin\frac{\pi}{4}n}{\pi n}\right)^2 * \left(\frac{\sin\omega_c n}{\pi n}\right),$$

where $*$ denotes convolution and $|\omega_c| \le \pi$. Determine a stricter constraint on ω_c

which ensures that

$$y[n] = \left(\frac{\sin \frac{\pi}{4} n}{\pi n}\right)^2.$$

5.13. An LTI system with impulse response $h_1[n] = (\frac{1}{3})^n u[n]$ is connected in parallel with another causal LTI system with impulse response $h_2[n]$. The resulting parallel interconnection has the frequency response

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}}.$$

Determine $h_2[n]$.

5.14. Suppose we are given the following facts about an LTI system S with impulse response $h[n]$ and frequency response $H(e^{j\omega})$:

1. $(\frac{1}{4})^n u[n] \longrightarrow g[n]$, where $g[n] = 0$ for $n \geq 2$ and $n < 0$.
2. $H(e^{j\pi/2}) = 1$.
3. $H(e^{j\omega}) = H(e^{j(\omega - \pi)})$.

Determine $h[n]$.

5.15. Let the inverse Fourier transform of $Y(e^{j\omega})$ be

$$y[n] = \left(\frac{\sin \omega_c n}{\pi n}\right)^2,$$

where $0 < \omega_c < \pi$. Determine the value of ω_c which ensures that

$$Y(e^{j\pi}) = \frac{1}{2}.$$

5.16. The Fourier transform of a particular signal is

$$X(e^{j\omega}) = \sum_{k=0}^{3} \frac{(1/2)^k}{1 - \frac{1}{4}e^{-j(\omega - \pi/2k)}}.$$

It can be shown that

$$x[n] = g[n]q[n],$$

where $g[n]$ is of the form $\alpha^n u[n]$ and $q[n]$ is a periodic signal with period N.

(a) Determine the value of α.
(b) Determine the value of N.
(c) Is $x[n]$ real?

5.17. The signal $x[n] = (-1)^n$ has a fundamental period of 2 and corresponding Fourier series coefficients a_k. Use duality to determine the Fourier series coefficients b_k of the signal $g[n] = a_n$ with a fundamental period of 2.

5.18. Given the fact that

$$a^{|n|} \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1 - a^2}{1 - 2a\cos\omega + a^2}, \quad |a| < 1,$$

use duality to determine the Fourier series coefficients of the following continuous-time signal with period $T = 1$:

$$x(t) = \frac{1}{5 - 4\cos(2\pi t)}.$$

5.19. Consider a causal and stable LTI system S whose input $x[n]$ and output $y[n]$ are related through the second-order difference equation

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n].$$

(a) Determine the frequency response $H(e^{j\omega})$ for the system S.
(b) Determine the impulse response $h[n]$ for the system S.

5.20. A causal and stable LTI system S has the property that

$$\left(\frac{4}{5}\right)^n u[n] \longrightarrow n\left(\frac{4}{5}\right)^n u[n].$$

(a) Determine the frequency response $H(e^{j\omega})$ for the system S.
(b) Determine a difference equation relating any input $x[n]$ and the corresponding output $y[n]$.

BASIC PROBLEMS

5.21. Compute the Fourier transform of each of the following signals:
(a) $x[n] = u[n-2] - u[n-6]$
(b) $x[n] = (\frac{1}{2})^{-n}u[-n-1]$
(c) $x[n] = (\frac{1}{3})^{|n|}u[-n-2]$
(d) $x[n] = 2^n \sin(\frac{\pi}{4}n)u[-n]$
(e) $x[n] = (\frac{1}{2})^{|n|}\cos(\frac{\pi}{8}(n-1))$
(f) $x[n] = \begin{cases} n, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$
(g) $x[n] = \sin(\frac{\pi}{2}n) + \cos(n)$
(h) $x[n] = \sin(\frac{5\pi}{3}n) + \cos(\frac{7\pi}{3}n)$
(i) $x[n] = x[n-6]$, and $x[n] = u[n] - u[n-5]$ for $0 \le n \le 5$
(j) $x[n] = (n-1)(\frac{1}{3})^{|n|}$
(k) $x[n] = (\frac{\sin(\pi n/5)}{\pi n})\cos(\frac{7\pi}{2}n)$

5.22. The following are the Fourier transforms of discrete-time signals. Determine the signal corresponding to each transform.
(a) $X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} \le |\omega| \le \pi, 0 \le |\omega| < \frac{\pi}{4} \end{cases}$
(b) $X(e^{j\omega}) = 1 + 3e^{-j\omega} + 2e^{-j2\omega} - 4e^{-j3\omega} + e^{-j10\omega}$
(c) $X(e^{j\omega}) = e^{-j\omega/2}$ for $-\pi \le \omega \le \pi$
(d) $X(e^{j\omega}) = \cos^2\omega + \sin^2 3\omega$

(e) $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty}(-1)^k\delta(\omega - \frac{\pi}{2}k)$

(f) $X(e^{j\omega}) = \frac{e^{-j\omega}-\frac{1}{5}}{1-\frac{1}{5}e^{-j\omega}}$

(g) $X(e^{j\omega}) = \frac{1-\frac{1}{3}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}}$

(h) $X(e^{j\omega}) = \frac{1-(\frac{1}{3})^6 e^{-j6\omega}}{1-\frac{1}{3}e^{-j\omega}}$

5.23. Let $X(e^{j\omega})$ denote the Fourier transform of the signal $x[n]$ depicted in Figure P5.23. Perform the following calculations without explicitly evaluating $X(e^{j\omega})$:

(a) Evaluate $X(e^{j0})$.

(b) Find $\sphericalangle X(e^{j\omega})$.

(c) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.

(d) Find $X(e^{j\pi})$.

(e) Determine and sketch the signal whose Fourier transform is $\mathcal{R}e\{x(\omega)\}$.

(f) Evaluate:

(i) $\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega$

(ii) $\int_{-\pi}^{\pi}\left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega$

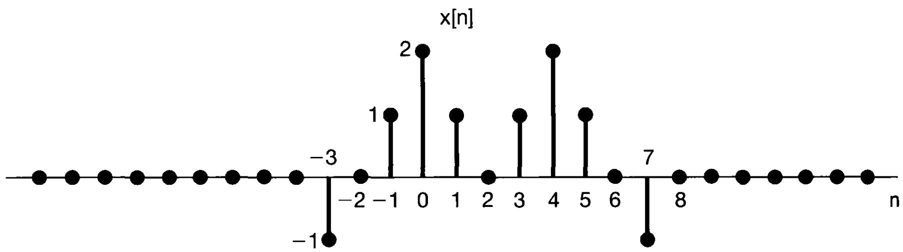


Fig P5.23

5.24. Determine which, if any, of the following signals have Fourier transforms that satisfy each of the following conditions:

1. $\mathcal{R}e\{X(e^{j\omega})\} = 0$.
2. $\mathcal{I}m\{X(e^{j\omega})\} = 0$.
3. There exists a real α such that $e^{j\alpha\omega}X(e^{j\omega})$ is real.
4. $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 0$.
5. $X(e^{j\omega})$ periodic.
6. $X(e^{j0}) = 0$.

(a) $x[n]$ as in Figure P5.24(a)

(b) $x[n]$ as in Figure P5.24(b)

(c) $x[n] = (\frac{1}{2})^n u[n]$

(d) $x[n] = (\frac{1}{2})^{|n|}$

(e) $x[n] = \delta[n-1] + \delta[n+2]$

(f) $x[n] = \delta[n-1] + \delta[n+3]$

(g) $x[n]$ as in Figure P5.24(c)

(h) $x[n]$ as in Figure P5.24(d)

(i) $x[n] = \delta[n-1] - \delta[n+1]$

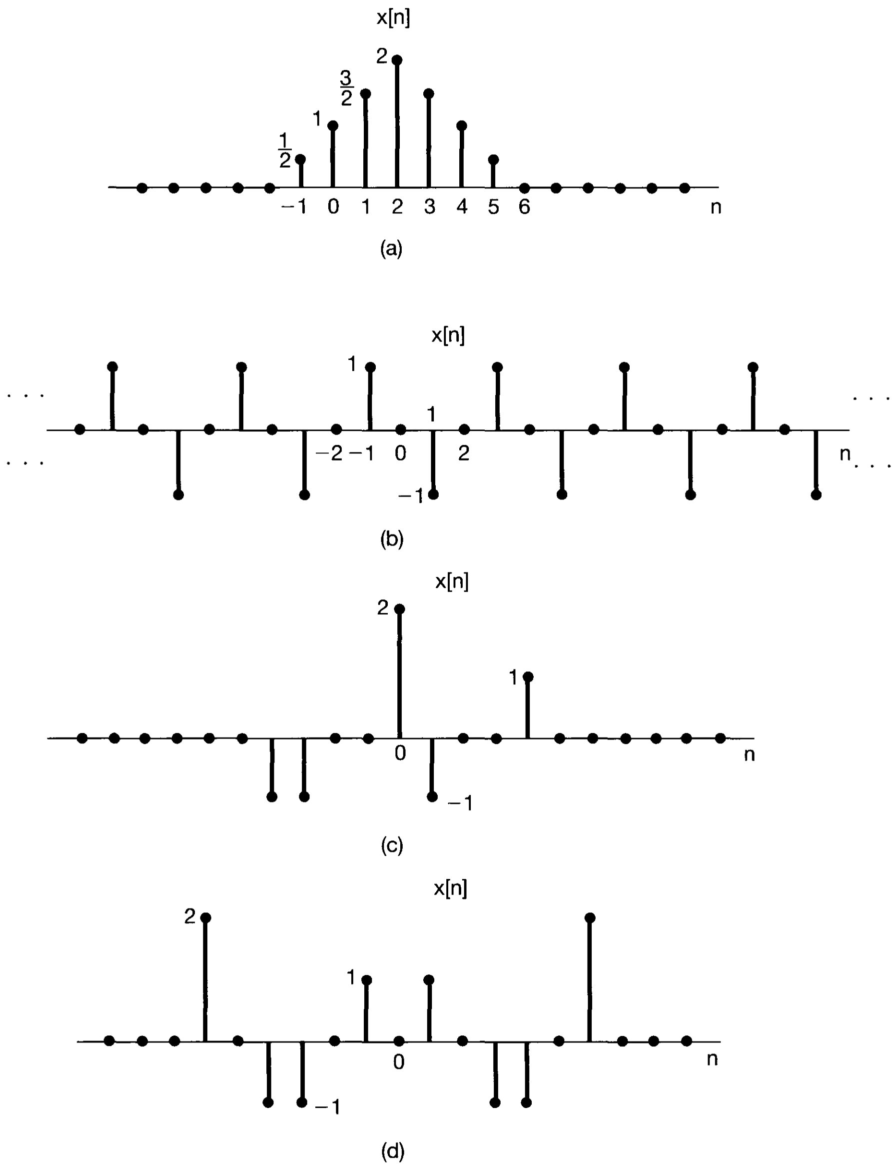


Fig P5.24

5.25. Consider the signal depicted in Figure P5.25. Let the Fourier transform of this signal be written in rectangular form as

$$X(e^{j\omega}) = A(\omega) + jB(\omega).$$

Sketch the function of time corresponding to the transform

$$Y(e^{j\omega}) = [B(\omega) + A(\omega)e^{j\omega}].$$

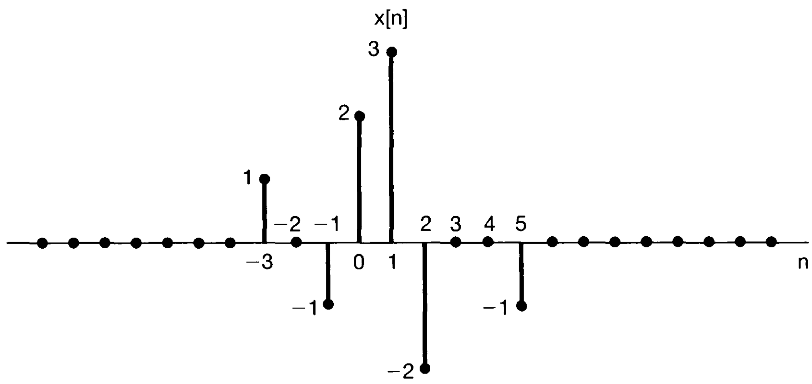


Fig P5.25

5.26. Let $x_1[n]$ be the discrete-time signal whose Fourier transform $X_1(e^{j\omega})$ is depicted in Figure P5.26(a).

(a) Consider the signal $x_2[n]$ with Fourier transform $X_2(e^{j\omega})$, as illustrated in Figure P5.26(b). Express $x_2[n]$ in terms of $x_1[n]$. [*Hint:* First express $X_2(e^{j\omega})$ in terms of $X_1(e^{j\omega})$, and then use properties of the Fourier transform.]

(b) Repeat part (a) for $x_3[n]$ with Fourier transform $X_3(e^{j\omega})$, as shown in Figure P5.26(c).

(c) Let

$$\alpha = \frac{\sum_{n=-\infty}^{\infty} n x_1[n]}{\sum_{n=-\infty}^{\infty} x_1[n]}.$$

This quantity, which is the center of gravity of the signal $x_1[n]$, is usually referred to as the *delay time* of $x_1[n]$. Find α. (You can do this without first determining $x_1[n]$ explicitly.)

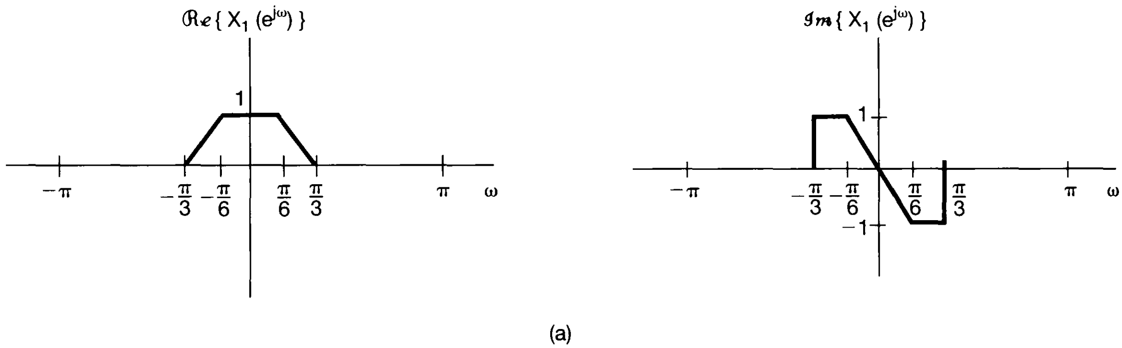
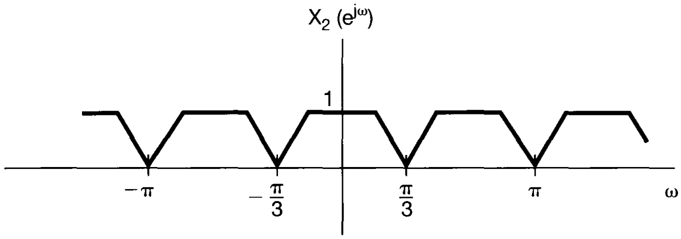
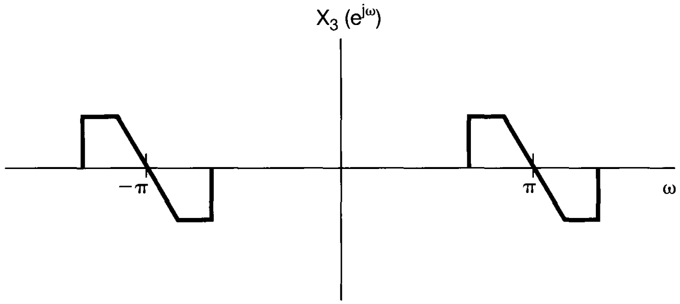


Fig P5.26a



(b)



(c)

Fig P5.26b,c

(d) Consider the signal $x_4[n] = x_1[n] * h[n]$, where

$$h[n] = \frac{\sin(\pi n/6)}{\pi n}.$$

Sketch $X_4(e^{j\omega})$.

5.27. (a) Let $x[n]$ be a discrete-time signal with Fourier transform $X(e^{j\omega})$, which is illustrated in Figure P5.27. Sketch the Fourier transform of

$$w[n] = x[n]p[n]$$

for each of the following signals $p[n]$:

(i) $p[n] = \cos \pi n$

(ii) $p[n] = \cos(\pi n/2)$

(iii) $p[n] = \sin(\pi n/2)$

(iv) $p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 2k]$

(v) $p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]$

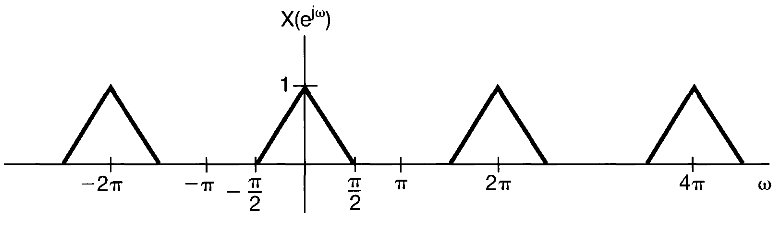


Fig P5.27

(b) Suppose that the signal $w[n]$ of part (a) is applied as the input to an LTI system with unit sample response

$$h[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

Determine the output $y[n]$ for each of the choices of $p[n]$ in part (a).

5.28. The signals $x[n]$ and $g[n]$ are known to have Fourier transforms $X(e^{j\omega})$ and $G(e^{j\omega})$, respectively. Furthermore, $X(e^{j\omega})$ and $G(e^{j\omega})$ are related as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\theta})G(e^{j(\omega-\theta)})d\theta = 1 + e^{-j\omega} \qquad \text{(P5.28–1)}$$

(a) If $x[n] = (-1)^n$, determine a sequence $g[n]$ such that its Fourier transform $G(e^{j\omega})$ satisfies eq. (P5.28–1). Are there other possible solutions for $g[n]$?
(b) Repeat the previous part for $x[n] = (\frac{1}{2})^n u[n]$.

5.29. **(a)** Consider a discrete-time LTI system with impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

Use Fourier transforms to determine the response to each of the following input signals:
(i) $x[n] = (\frac{3}{4})^n u[n]$
(ii) $x[n] = (n+1)(\frac{1}{4})^n u[n]$
(iii) $x[n] = (-1)^n$

(b) Suppose that

$$h[n] = \left[\left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{2}\right)\right] u[n].$$

Use Fourier transforms to determine the response to each of the following inputs:
(i) $x[n] = (\frac{1}{2})^n u[n]$
(ii) $x[n] = \cos(\pi n/2)$

(c) Let $x[n]$ and $h[n]$ be signals with the following Fourier transforms:

$$X(e^{j\omega}) = 3e^{j\omega} + 1 - e^{-j\omega} + 2e^{-j3\omega},$$
$$H(e^{j\omega}) = -e^{j\omega} + 2e^{-2j\omega} + e^{j4\omega}.$$

Determine $y[n] = x[n] * h[n]$.

5.30. In Chapter 4, we indicated that the continuous-time LTI system with impulse response

$$h(t) = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right) = \frac{\sin Wt}{\pi t}$$

plays a very important role in LTI system analysis. The same is true of the discrete-time LTI system with impulse response

$$h[n] = \frac{W}{\pi}\text{sinc}\left(\frac{Wn}{\pi}\right) = \frac{\sin Wn}{\pi n}.$$

(a) Determine and sketch the frequency response for the system with impulse response $h[n]$.

(b) Consider the signal

$$x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right).$$

Suppose that this signal is the input to LTI systems with the following impulse responses. Determine the output in each case.

(i) $h[n] = \frac{\sin(\pi n/6)}{\pi n}$

(ii) $h[n] = \frac{\sin(\pi n/6)}{\pi n} + \frac{\sin(\pi n/2)}{\pi n}$

(iii) $h[n] = \frac{\sin(\pi n/6)\sin(\pi n/3)}{\pi^2 n^2}$

(iv) $h[n] = \frac{\sin(\pi n/6)\sin(\pi n/3)}{\pi n}$

(c) Consider an LTI system with unit sample response

$$h[n] = \frac{\sin(\pi n/3)}{\pi n}.$$

Determine the output for each of the following inputs:

(i) $x[n] =$ the square wave depicted in Figure P5.30

(ii) $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-8k]$

(iii) $x[n] = (-1)^n$ times the square wave depicted in Figure P5.30

(iv) $x[n] = \delta[n+1] + \delta[n-1]$

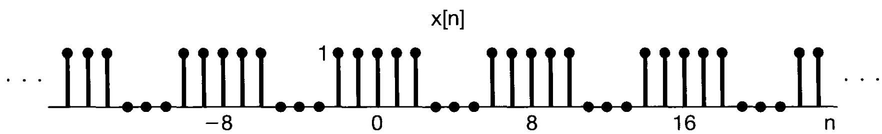


Fig P5.30

5.31. An LTI system S with impulse response $h[n]$ and frequency response $H(e^{j\omega})$ is known to have the property that, when $-\pi \leq \omega_0 \leq \pi$,

$$\cos \omega_0 n \longrightarrow \omega_0 \cos \omega_0 n.$$

(a) Determine $H(e^{j\omega})$.

(b) Determine $h[n]$.

5.32. Let $h_1[n]$ and $h_2[n]$ be the impulse responses of causal LTI systems, and let $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ be the corresponding frequency responses. Under these conditions, is the following equation true in general or not? Justify your answer.

$$\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(e^{j\omega})d\omega\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} H_2(e^{j\omega})d\omega\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(e^{j\omega})H_2(e^{j\omega})d\omega.$$

5.33. Consider a causal LTI system described by the difference equation

$$y[n] + \frac{1}{2}y[n-1] = x[n].$$

(a) Determine the frequency response $H(e^{j\omega})$ of this system.
(b) What is the response of the system to the following inputs?
  (i) $x[n] = (\frac{1}{2})^n u[n]$
  (ii) $x[n] = (-\frac{1}{2})^n u[n]$
  (iii) $x[n] = \delta[n] + \frac{1}{2}\delta[n-1]$
  (iv) $x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$
(c) Find the response to the inputs with the following Fourier transforms:
  (i) $X(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$
  (ii) $X(e^{j\omega}) = \frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}}$
  (iii) $X(e^{j\omega}) = \frac{1}{(1-\frac{1}{4}e^{-j\omega})(1+\frac{1}{2}e^{-j\omega})}$
  (iv) $X(e^{j\omega}) = 1 + 2e^{-3j\omega}$

5.34. Consider a system consisting of the cascade of two LTI systems with frequency responses

$$H_1(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}$$

and

$$H_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}.$$

(a) Find the difference equation describing the overall system.
(b) Determine the impulse response of the overall system.

5.35. A causal LTI system is described by the difference equation

$$y[n] - ay[n-1] = bx[n] + x[n-1],$$

where a is real and less than 1 in magnitude.
(a) Find a value of b such that the frequency response of the system satisfies

$$|H(e^{j\omega})| = 1, \text{ for all } \omega.$$

This kind of system is called an *all-pass system*, as it does not attenuate the input $e^{j\omega n}$ for *any* value of ω. Use the value of b that you have found in the rest of the problem.
(b) Roughly sketch $\sphericalangle H(e^{j\omega})$, $0 \le \omega \le \pi$, when $a = \frac{1}{2}$.
(c) Roughly sketch $\sphericalangle H(e^{j\omega})$, $0 \le \omega \le \pi$, when $a = -\frac{1}{2}$.

(d) Find and plot the output of this system with $a = -\frac{1}{2}$ when the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

From this example, we see that a nonlinear change in phase can have a significantly different effect on a signal than the time shift that results from a linear phase.

5.36. (a) Let $h[n]$ and $g[n]$ be the impulse responses of two stable discrete-time LTI systems that are inverses of each other. What is the relationship between the frequency responses of these two systems?

(b) Consider causal LTI systems described by the following difference equations. In each case, determine the impulse response of the inverse system and the difference equation that characterizes the inverse.

(i) $y[n] = x[n] - \frac{1}{4}x[n-1]$

(ii) $y[n] + \frac{1}{2}y[n-1] = x[n]$

(iii) $y[n] + \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$

(iv) $y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{1}{4}x[n-1] - \frac{1}{8}x[n-2]$

(v) $y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{1}{2}x[n-1]$

(vi) $y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$

(c) Consider the causal, discrete-time LTI system described by the difference equation

$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n-1] - \frac{1}{2}x[n-2]. \qquad \text{(P5.36–1)}$$

What is the inverse of this system? Show that the inverse is not causal. Find another causal LTI system that is an "inverse with delay" of the system described by eq. (P5.36–1). Specifically, find a causal LTI system such that the output $w[n]$ in Figure P5.36 equals $x[n-1]$.

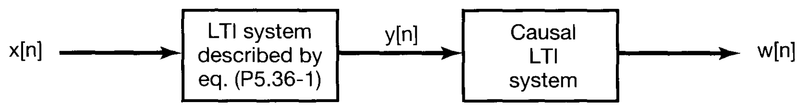


Fig P5.36

ADVANCED PROBLEMS

5.37. Let $X(e^{j\omega})$ be the Fourier transform of $x[n]$. Derive expressions in terms of $X(e^{j\omega})$ for the Fourier transforms of the following signals. (Do not assume that $x[n]$ is real.)

(a) $\mathcal{R}e\{x[n]\}$

(b) $x^*[-n]$

(c) $\mathcal{E}v\{x[n]\}$

5.38. Let $X(e^{j\omega})$ be the Fourier transform of a real signal $x[n]$. Show that $x[n]$ can be written as

$$x[n] = \int_0^{\pi} \{B(\omega)\cos\omega + C(\omega)\sin\omega\}d\omega$$

by finding expressions for $B(\omega)$ and $C(\omega)$ in terms of $X(e^{j\omega})$.

5.39. Derive the convolution property

$$x[n] * h[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega})H(e^{j\omega}).$$

5.40. Let $x[n]$ and $h[n]$ be two signals, and let $y[n] = x[n] * h[n]$. Write two expressions for $y[0]$, one (using the convolution sum directly) in terms of $x[n]$ and $h[n]$, and one (using the convolution property of Fourier transforms) in terms of $X(e^{j\omega})$ and $H(e^{j\omega})$. Then, by a judicious choice of $h[n]$, use these two expressions to derive Parseval's relation—that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

In a similar fashion, derive the following generalization of Parseval's relation:

$$\sum_{n-\infty}^{+\infty} x[n]z^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Z^*(e^{j\omega})d\omega.$$

5.41 Let $\tilde{x}[n]$ be a periodic signal with period N. A finite-duration signal $x[n]$ is related to $\tilde{x}[n]$ through

$$x[n] = \begin{cases} \tilde{x}[n], & n_0 \le n \le n_0 + N - 1 \\ 0, & \text{otherwise} \end{cases},$$

for some integer n_0. That is, $x[n]$ is equal to $\tilde{x}[n]$ over one period and zero elsewhere.

(a) If $\tilde{x}[n]$ has Fourier series coefficients a_k and $x[n]$ has Fourier transform $X(e^{j\omega})$, show that

$$a_k = \frac{1}{N}X(e^{j2\pi k/N})$$

regardless of the value of n_0.

(b) Consider the following two signals:

$$x[n] = u[n] - u[n-5]$$

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n-kN]$$

where N is a positive integer. Let a_k denote the Fourier coefficients of $\tilde{x}[n]$ and let $X(e^{j\omega})$ denote the Fourier transform of $x[n]$.

(i) Determine a closed-form expression for $X(e^{j\omega})$.

(ii) Using the result of part (i), determine an expression for the Fourier coefficients a_k.

5.42. In this problem, we derive the frequency-shift property of the discrete-time Fourier transform as a special case of the multiplication property. Let $x[n]$ be any discrete-time signal with Fourier transform $X(e^{j\omega})$, and let

$$g[n] = e^{j\omega_0 n}x[n].$$

(a) Determine and sketch the Fourier transform of

$$p[n] = e^{j\omega_0 n}.$$

(b) The multiplication property of the Fourier transform tells us that, since

$$g[n] = p[n]x[n],$$

$$G(e^{j\omega}) = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\theta})P(e^{j(\omega-\theta)})d\theta.$$

Evaluate this integral to show that

$$G(e^{j\omega}) = X(e^{j(\omega-\omega_0)}).$$

5.43. Let $x[n]$ be a signal with Fourier transform $X(e^{j\omega})$, and let

$$g[n] = x[2n]$$

be a signal whose Fourier transform is $G(e^{j\omega})$. In this problem, we derive the relationship between $G(e^{j\omega})$ and $X(e^{j\omega})$.

(a) Let

$$v[n] = \frac{(e^{-j\pi n}x[n]) + x[n]}{2}.$$

Express the Fourier transform $V(e^{j\omega})$ of $v[n]$ in terms of $X(e^{j\omega})$.

(b) Noting that $v[n] = 0$ for n odd, show that the Fourier transform of $v[2n]$ is equal to $V(e^{j\frac{\omega}{2}})$.

(c) Show that

$$x[2n] = v[2n].$$

It follows that

$$G(e^{j\omega}) = V(e^{j\omega/2}).$$

Now use the result of part (a) to express $G(e^{j\omega})$ in terms of $X(e^{j\omega})$.

5.44. (a) Let

$$x_1[n] = \cos\left(\frac{\pi n}{3}\right) + \sin\left(\frac{\pi n}{2}\right)$$

be a signal, and let $X_1(e^{j\omega})$ denote the Fourier transform of $x_1[n]$. Sketch $x_1[n]$, together with the signals with the following Fourier transforms:

(i) $X_2(e^{j\omega}) = X_1(e^{j\omega})e^{j\omega}, \ |\omega| < \pi$

(ii) $X_3(e^{j\omega}) = X_1(e^{j\omega})e^{-j3\omega/2}, \ |\omega| < \pi$

(b) Let

$$w(t) = \cos\left(\frac{\pi t}{3T}\right) + \sin\left(\frac{\pi t}{2T}\right)$$

be a continuous-time signal. Note that $x_1[n]$ can be regarded as a sequence of evenly spaced samples of $w(t)$; that is,

$$x_1[n] = w(nT).$$

Show that

$$x_2[n] = w(nT - \alpha)$$

and

$$x_3[n] = w(nT - \beta)$$

and specify the values of α and β. From this result we can conclude that $x_2[n]$ and $x_3[n]$ are also evenly spaced samples of $w(t)$.

5.45. Consider a discrete-time signal $x[n]$ with Fourier transform as illustrated in Figure P5.45. Provide dimensioned sketches of the following continuous-time signals:

(a) $x_1(t) = \sum_{n=-\infty}^{\infty} x[n]e^{j(2\pi/10)nt}$

(b) $x_2(t) = \sum_{n=-\infty}^{\infty} x[-n]e^{j(2\pi/10)nt}$

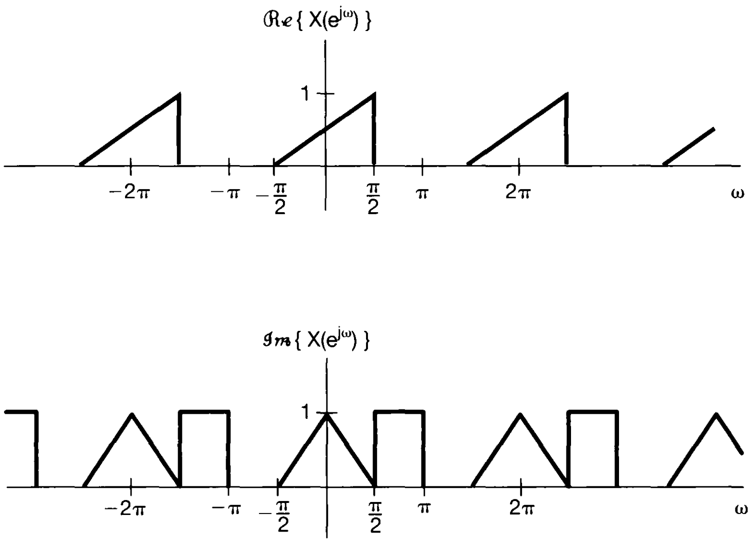


Fig P5.45

(c) $x_3(t) = \sum_{n=-\infty}^{\infty} \mathcal{O}d\{x[n]\}e^{j(2\pi/8)nt}$
(d) $x_4(t) = \sum_{n=-\infty}^{\infty} \mathcal{R}e\{x[n]\}e^{j(2\pi/6)nt}$

5.46. In Example 5.1, we showed that for $|\alpha| < 1$,

$$\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \alpha e^{-j\omega}}.$$

(a) Use properties of the Fourier transform to show that

$$(n + 1)\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(1 - \alpha e^{-j\omega})^2}.$$

(b) Show by induction that the inverse Fourier transform of

$$X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^r}$$

is

$$x[n] = \frac{(n + r - 1)!}{n!(r - 1)!}\alpha^n u[n].$$

5.47. Determine whether each of the following statements is true or false. Justify your answers. In each statement, the Fourier transform of $x[n]$ is denoted by $X(e^{j\omega})$.
(a) If $X(e^{j\omega}) = X(e^{j(\omega-1)})$, then $x[n] = 0$ for $|n| > 0$.
(b) If $X(e^{j\omega}) = X(e^{j(\omega-\pi)})$, then $x[n] = 0$ for $|n| > 0$.
(c) If $X(e^{j\omega}) = X(e^{j\omega/2})$, then $x[n] = 0$ for $|n| > 0$.
(d) If $X(e^{j\omega}) = X(e^{j2\omega})$, then $x[n] = 0$ for $|n| > 0$.

5.48. We are given a discrete-time, linear, time-invariant, causal system with input denoted by $x[n]$ and output denoted by $y[n]$. This system is specified by the following *pair* of difference equations, involving an intermediate signal $w[n]$:

$$y[n] + \frac{1}{4}y[n-1] + w[n] + \frac{1}{2}w[n-1] = \frac{2}{3}x[n],$$

$$y[n] - \frac{5}{4}y[n-1] + 2w[n] - 2w[n-1] = -\frac{5}{3}x[n].$$

(a) Find the frequency response and unit sample response of the system.
(b) Find a single difference equation relating $x[n]$ and $y[n]$ for the system.

5.49. **(a)** A particular discrete-time system has input $x[n]$ and output $y[n]$. The Fourier transforms of these signals are related by the equation

$$Y(e^{j\omega}) = 2X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) - \frac{dX(e^{j\omega})}{d\omega}.$$

(i) Is the system linear? Clearly justify your answer.
(ii) Is the system time invariant? Clearly justify your answer.
(iii) What is $y[n]$ if $x[n] = \delta[n]$?

(b) Consider a discrete-time system for which the transform $Y(e^{j\omega})$ of the output is related to the transform of the input through the relation

$$Y(e^{j\omega}) = \int_{\omega-\pi/4}^{\omega+\pi/4} X(e^{j\omega})d\omega.$$

Find an expression for $y[n]$ in terms of $x[n]$.

5.50. (a) Suppose we want to design a discrete-time LTI system which has the property that if the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1],$$

then the output is

$$y[n] = \left(\frac{1}{3}\right)^n u[n].$$

(i) Find the impulse response *and* frequency response of a discrete-time LTI system that has the foregoing property.
(ii) Find a difference equation relating $x[n]$ and $y[n]$ that characterizes the system.

(b) Suppose that a system has the response $(1/4)^n u[n]$ to the input $(n+2)(1/2)^n u[n]$. If the output of this system is $\delta[n] - (-1/2)^n u[n]$, what is the input?

5.51. (a) Consider a discrete-time system with unit sample response

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{4}\right)^n u[n].$$

Determine a linear constant-coefficient difference equation relating the input and output of the system.

(b) Figure P5.51 depicts a block diagram implementation of a causal LTI system.
(i) Find a difference equation relating $x[n]$ and $y[n]$ for this system.
(ii) What is the frequency response of the system?
(iii) Determine the system's impulse response.

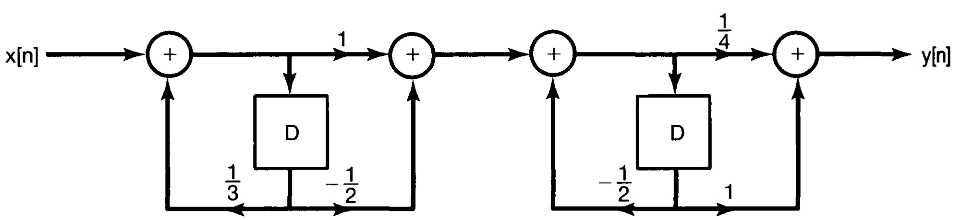


Fig P5.51

5.52. **(a)** Let $h[n]$ be the impulse response of a real, causal, discrete-time LTI system. Show that the system is completely specified by the real part of its frequency response. (*Hint:* Show how $h[n]$ can be recovered from $\mathcal{E}v\{h[n]\}$. What is the Fourier transform of $\mathcal{E}v\{h[n]\}$?) This is the discrete-time counterpart of the *real-part sufficiency* property of causal LTI systems considered in Problem 4.47 for continuous-time systems.

(b) Let $h[n]$ be real and causal. If

$$\mathcal{R}e\{H(e^{j\omega})\} = 1 + \alpha \cos 2\omega (\alpha \text{ real}),$$

determine $h[n]$ and $H(e^{j\omega})$.

(c) Show that $h[n]$ can be completely recovered from knowledge of $\mathcal{I}m\{H(e^{j\omega})\}$ and $h[0]$.

(d) Find two real, causal LTI systems whose frequency responses have imaginary parts equal to $\sin \omega$.

EXTENSION PROBLEMS

5.53. One of the reasons for the tremendous growth in the use of discrete-time methods for the analysis and synthesis of signals and systems was the development of exceedingly efficient tools for performing Fourier analysis of discrete-time sequences. At the heart of these methods is a technique that is very closely allied with discrete-time Fourier analysis and that is ideally suited for use on a digital computer or for implementation in digital hardware. This technique is the *discrete Fourier transform (DFT)* for finite-duration signals.

Let $x[n]$ be a signal of finite duration; that is, there is an integer N_1 so that

$$x[n] = 0, \qquad \text{outside the interval } 0 \le n \le N_1 - 1$$

Furthermore, let $X(e^{j\omega})$ denote the Fourier transform of $x[n]$. We can construct a periodic signal $\tilde{x}[n]$ that is equal to $x[n]$ over one period. Specifically, let $N \ge N_1$ be a given integer, and let $\tilde{x}[n]$ be periodic with period N and such that

$$\tilde{x}[n] = x[n], \qquad 0 \le n \le N - 1$$

The Fourier series coefficients for $\tilde{x}[n]$ are given by

$$a_k = \frac{1}{N} \sum_{\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

Choosing the interval of summation to be that over which $\tilde{x}[n] = x[n]$, we obtain

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n} \qquad \text{(P5.53–1)}$$

The set of coefficients defined by eq. (P5.53–1) comprise the DFT of $x[n]$. Specifically, the DFT of $x[n]$ is usually denoted by $\tilde{X}[k]$, and is defined as

$$\tilde{X}[k] = a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}, \qquad k = 0, 1, \ldots, N-1 \qquad \text{(P5.53–2)}$$

The importance of the DFT stems from several facts. First note that the original finite duration signal can be recovered from its DFT. Specifically, we have

$$x[n] = \sum_{k=0}^{N-1} \tilde{X}[k]e^{jk(2\pi/N)n}, \qquad n = 0, 1, \ldots, N-1 \qquad \text{(P5.53–3)}$$

Thus, the finite-duration signal can either be thought of as being specified by the finite set of nonzero values it assumes or by the finite set of values of $\tilde{X}[k]$ in its DFT. A second important feature of the DFT is that there is an extremely fast algorithm, called the *fast Fourier transform (FFT),* for its calculation (see Problem 5.54 for an introduction to this extremely important technique). Also, because of its close relationship to the discrete-time Fourier series and transform, the DFT inherits some of their important properties.

(a) Assume that $N \geq N_1$. Show that

$$\tilde{X}[k] = \frac{1}{N}X\left(e^{j(2\pi k/N)}\right)$$

where $\tilde{X}[k]$ is the DFT of $x[n]$. That is, the DFT corresponds to samples of $X(e^{j\omega})$ taken every $2\pi/N$. Equation (P5.53–3) leads us to conclude that $x[n]$ can be uniquely represented by these samples of $X(e^{j\omega})$.

(b) Let us consider samples of $X(e^{j\omega})$ taken every $2\pi/M$, where $M < N_1$. These samples correspond to more than one sequence of duration N_1. To illustrate this, consider the two signals $x_1[n]$ and $x_2[n]$ depicted in Figure P5.53. Show that if we choose $M = 4$, we have

$$X_1\left(e^{j(2\pi k/4)}\right) = X_2\left(e^{j(2\pi k/4)}\right)$$

for all values of k.

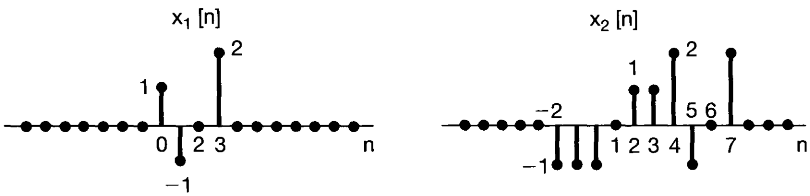


Fig P5.53

5.54. As indicated in Problem 5.53, there are many problems of practical importance in which one wishes to calculate the discrete Fourier transform (DFT) of discrete-time signals. Often, these signals are of quite long duration, and in such cases it is very important to use computationally efficient procedures. One of the reasons for the significant increase in the use of computerized techniques for the analysis of signals was the development of a very efficient technique known as the fast Fourier transform (FFT) algorithm for the calculation of the DFT of finite-duration sequences. In this problem, we develop the principle on which the FFT is based.

Let $x[n]$ be a signal that is 0 outside the interval $0 \leq n \leq N_1 - 1$. For $N \geq N_1$, the N-point DFT of $x[n]$ is given by

$$\tilde{X}[k] = \frac{1}{N}\sum_{k=0}^{N-1} x[n]e^{-jk(2\pi/N)n}, \quad k = 0, 1, \ldots, N-1. \qquad \text{(P5.54–1)}$$

It is convenient to write eq. (P5.54–1) as

$$\tilde{X}[k] = \frac{1}{N}\sum_{k=0}^{N-1} x[n]W_N^{nk}, \qquad \text{(P5.54–2)}$$

where

$$W_N = e^{-j2\pi/N}.$$

(a) One method for calculating $\tilde{X}[k]$ is by direct evaluation of eq. (P5.54–2). A useful measure of the complexity of such a computation is the total number of complex multiplications required. Show that the number of complex multiplications required to evaluate eq. (P5.54–2) directly, for $k = 0, 1, \ldots, N-1$, is N^2. Assume that $x[n]$ is complex and that the required values of W_N^{nk} have been precomputed and stored in a table. For simplicity, *do not* exploit the fact that, for certain values of n and k, W_N^{nk} is equal to ± 1 or $\pm j$ and hence does not, strictly speaking, require a full complex multiplication.

(b) Suppose that N is even. Let $f[n] = x[2n]$ represent the even-indexed samples of $x[n]$, and let $g[n] = x[2n+1]$ represent the odd-indexed samples.

(i) Show that $f[n]$ and $g[n]$ are zero outside the interval $0 \leq n \leq (N/2) - 1$.

(ii) Show that the N-point DFT $\tilde{X}[k]$ of $x[n]$ can be expressed as

$$\begin{aligned}\tilde{X}[k] &= \frac{1}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{nk} + \frac{1}{N}W_N^k \sum_{n=0}^{(N/2)-1} g[n]W_{N/2}^{nk} \\ &= \frac{1}{2}\tilde{F}[k] + \frac{1}{2}W_N^k\tilde{G}[k], \quad k = 0, 1, \ldots, N-1, \qquad \text{(P5.54–3)}\end{aligned}$$

where

$$\tilde{F}[k] = \frac{2}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{nk},$$

$$\tilde{G}[k] = \frac{2}{N}\sum_{n=0}^{(N/2)-1} g[n]W_{N/2}^{nk}.$$

(iii) Show that, for all k,

$$\tilde{F}\left[k + \frac{N}{2}\right] = \tilde{F}[k],$$

$$\tilde{G}\left[k + \frac{N}{2}\right] = \tilde{G}[k].$$

Note that $\tilde{F}[k]$, $k = 0, 1, \ldots, (N/2) - 1$, and $\tilde{G}[k]$, $k = 0, 1, \ldots, (N/2) - 1$, are the $(N/2)$-point DFTs of $f[n]$ and $g[n]$, respectively. Thus, eq. (P5.54–3) indicates that the length-N DFT of $x[n]$ can be calculated in terms of two DFTs of length $N/2$.

(iv) Determine the number of complex multiplications required to compute $\tilde{X}[k]$, $k = 0, 1, 2, \ldots, N - 1$, from eq. (P5.54–3) by first computing $\tilde{F}[k]$ and $\tilde{G}[k]$. [Make the same assumptions about multiplications as in part (a), and ignore the multiplications by the quantity 1/2 in eq. (P5.54–3).]

(c) If, like N, $N/2$ is even, then $f[n]$ and $g[n]$ can each be decomposed into sequences of even- and odd-indexed samples, and therefore, their DFTs can be computed using the same process as in eq. (P5.54–3). Furthermore, if N is an integer power of 2, we can continue to iterate the process, thus achieving significant savings in computation time. With this procedure, approximately how many complex multiplications are required for $N = 32, 256$, 1,024, and 4,096? Compare this to the direct method of calculation in part (a).

5.55. In this problem we introduce the concept of *windowing,* which is of great importance both in the design of LTI systems and in the spectral analysis of signals. Windowing is the operation of taking a signal $x[n]$ and multiplying it by a finite-duration *window signal* $w[n]$. That is,

$$p[n] = x[n]w[n].$$

Note that $p[n]$ is also of finite duration.

The importance of windowing in spectral analysis stems from the fact that in numerous applications one wishes to compute the Fourier transform of a signal that has been measured. Since in practice we can measure a signal $x[n]$ only over a finite time interval (the *time window*), the actual signal available for spectral analysis is

$$p[n] = \begin{cases} x[n], & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases},$$

where $-M \leq n \leq M$ is the time window. Thus,

$$p[n] = x[n]w[n],$$

where $w[n]$ is the *rectangular window;* that is,

$$w[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}. \qquad \text{(P5.55–1)}$$

Windowing also plays a role in LTI system design. Specifically, for a variety of reasons (such as the potential utility of the FFT algorithm; see Problem P5.54), it is often advantageous to design a system that has an impulse response of finite duration to achieve some desired signal-processing objective. That is, we often begin with a desired frequency response $H(e^{j\omega})$ whose inverse transform $h[n]$ is an impulse response of infinite (or at least excessively long) duration. What is required then is the construction of an impulse response $g[n]$ of finite duration whose transform $G(e^{j\omega})$ adequately approximates $H(e^{j\omega})$. One general approach to choosing $g[n]$ is to find a window function $w[n]$ such that the transform of $h[n]w[n]$ meets the desired specifications for $G(e^{j\omega})$.

Clearly, the windowing of a signal has an effect on the resulting spectrum. In this problem, we illustrate that effect.

(a) To gain some understanding of the effect of windowing, consider windowing the signal

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-k]$$

using the rectangular window signal given in eq. (P5.55–1).

(i) What is $X(e^{j\omega})$?

(ii) Sketch the transform of $p[n] = x[n]w[n]$ when $M = 1$.

(iii) Do the same for $M = 10$.

(b) Next, consider a signal $x[n]$ whose Fourier transform is specified by

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < \pi/4 \\ 0, & \pi/4 < |\omega| \leq \pi \end{cases}.$$

Let $p[n] = x[n]w[n]$, where $w[n]$ is the rectangular window of eq. (P5.55–1). Roughly sketch $P(e^{j\omega})$ for $M = 4$, 8, and 16.

(c) One of the problems with the use of a rectangular window is that it introduces ripples in the transform $P(e^{j\omega})$. (This is in fact directly related to the Gibbs phenomenon.) For that reason, a variety of other window signals have been developed. These signals are tapered; that is, they go from 0 to 1 more gradually than the abrupt transition of the rectangular window. The result is a reduction in the *amplitude* of the ripples in $P(e^{j\omega})$ at the expense of adding a bit of distortion in terms of further smoothing of $X(e^{j\omega})$.

To illustrate the points just made, consider the signal $x[n]$ described in part (b), and let $p[n] = x[n]w[n]$, where $w[n]$ is the *triangular* or *Bartlett window;* that is,

$$w[n] = \begin{cases} 1 - \frac{|n|}{M+1}, & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}.$$

Roughly sketch the Fourier transform of $p[n] = x[n]w[n]$ for $M = 4$, 8, and 16. [*Hint:* Note that the triangular signal can be obtained as a convolution of a rectangular signal with itself. This fact leads to a convenient expression for $W(e^{j\omega})$.]

(d) Let $p[n] = x[n]w[n]$, where $w[n]$ is a raised cosine signal known as the *Hanning window;* i.e.,

$$w[n] = \begin{cases} \frac{1}{2}[1 + \cos(\pi n/M)], & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}.$$

Roughly sketch $P(e^{j\omega})$ for $M = 4, 8$, and 16.

5.56. Let $x[m, n]$ be a signal that is a function of the two independent, discrete variables m and n. In analogy with one dimension and with the continuous-time case treated in Problem 4.53, we can define the two-dimensional Fourier transform of $x[m, n]$ as

$$X(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m, n] e^{-j(\omega_1 m + \omega_2 n)}. \qquad \text{(P5.56–1)}$$

(a) Show that eq. (P5.56–1) can be calculated as two successive one-dimensional Fourier transforms, first in m, with n regarded as fixed, and then in n. Use this result to determine an expression for $x[m, n]$ in terms of $X(e^{j\omega_1}, e^{j\omega_2})$.

(b) Suppose that

$$x[m, n] = a[m]b[n],$$

where $a[m]$ and $b[n]$ are each functions of only one independent variable. Let $A(e^{j\omega})$ and $B(e^{j\omega})$ denote the Fourier transforms of $a[m]$ and $b[n]$, respectively. Express $X(e^{j\omega_1}, e^{j\omega_2})$ in terms of $A(e^{j\omega})$ and $B(e^{j\omega})$.

(c) Determine the two-dimensional Fourier transforms of the following signals:

(i) $x[m, n] = \delta[m - 1]\delta[n + 4]$

(ii) $x[m, n] = (\frac{1}{2})^{n-m} u[n - 2]u[-m]$

(iii) $x[m, n] = (\frac{1}{2})^n \cos(2\pi m/3)u[n]$

(iv) $x[m, n] = \begin{cases} 1, & -2 < m < 2 \text{ and } -4 < n < 4 \\ 0, & \text{otherwise} \end{cases}$

(v) $x[m, n] = \begin{cases} 1, & -2 + n < m < 2 + n \text{ and } -4 < n < 4 \\ 0, & \text{otherwise} \end{cases}$

(vi) $x[m, n] = \sin\left(\frac{\pi n}{3} + \frac{2\pi m}{5}\right)$

(d) Determine the signal $x[m, n]$ whose Fourier transform is

$$X(e^{j\omega_1}, e^{j\omega_2}) = \begin{cases} 1, & 0 < |\omega_1| \leq \pi/4 \text{ and } 0 < |\omega_2| \leq \pi/2 \\ 0, & \pi/4 < |\omega_1| < \pi \text{ or } \pi/2 < |\omega_2| < \pi \end{cases}.$$

(e) Let $x[m, n]$ and $h[m, n]$ be two signals whose two-dimensional Fourier transforms are denoted by $X(e^{j\omega_1}, e^{j\omega_2})$ and $H(e^{j\omega_1}, e^{j\omega_2})$, respectively. Determine the transforms of the following signals in terms of $X(e^{j\omega_1}, e^{j\omega_2})$ and $H(e^{j\omega_1}, e^{j\omega_2})$:

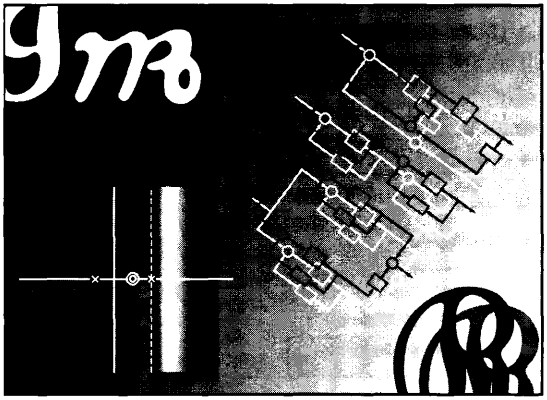
(i) $x[m, n]e^{jW_1 m}e^{jW_2 n}$

(ii) $y[m, n] = \begin{cases} x[k, r], & \text{if } m = 2k \text{ and } n = 2r \\ 0, & \text{if } m \text{ is not a multiple of 2 or } n \text{ is not a multiple of 3} \end{cases}$

(iii) $y[m, n] = x[m, n]h[m, n]$

9

THE LAPLACE TRANSFORM



9.0 INTRODUCTION

In the preceding chapters, we have seen that the tools of Fourier analysis are extremely useful in the study of many problems of practical importance involving signals and LTI systems. This is due in large part to the fact that broad classes of signals can be represented as linear combinations of periodic complex exponentials and that complex exponentials are eigenfunctions of LTI systems. The continuous-time Fourier transform provides us with a representation for signals as linear combinations of complex exponentials of the form e^{st} with $s = j\omega$. However the eigenfunction property introduced in Section 3.2 and many of its consequences apply as well for arbitrary values of s and not only those values that are purely imaginary. This observation leads to a generalization of the continuous-time Fourier transform, known as the Laplace transform, which we develop in this chapter. In the next chapter we develop the corresponding discrete-time generalization known as the z-transform.

As we will see, the Laplace and z-transforms have many of the properties that make Fourier analysis so useful. Moreover, not only do these transforms provide additional tools and insights for signals and systems that can be analyzed using the Fourier transform, but they also can be applied in some very important contexts in which Fourier transforms cannot. For example Laplace and z-transforms can be applied to the analysis of many unstable systems and consequently play an important role in the investigation of the stability or instability of systems. This fact, combined with the algebraic properties that Laplace and z-transforms share with Fourier transforms, leads to a very important set of tools for system analysis and in particular for the analysis of feedback systems, which we develop in Chapter 11.

9.1 THE LAPLACE TRANSFORM

In Chapter 3, we saw that the response of a linear time-invariant system with impulse response $h(t)$ to a complex exponential input of the form e^{st} is

$$y(t) = H(s)e^{st}, \tag{9.1}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt. \tag{9.2}$$

For s imaginary (i.e., $s = j\omega$), the integral in eq. (9.2) corresponds to the Fourier transform of $h(t)$. For general values of the complex variable s, it is referred to as the *Laplace transform* of the impulse response $h(t)$.

The Laplace transform of a general signal $x(t)$ is defined as[1]

$$\boxed{X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt,} \tag{9.3}$$

and we note in particular that it is a function of the independent variable s corresponding to the complex variable in the exponent of e^{-st}. The complex variable s can be written as $s = \sigma + j\omega$, with σ and ω the real and imaginary parts, respectively. For convenience, we will sometimes denote the Laplace transform in operator form as $\mathcal{L}\{x(t)\}$ and denote the transform relationship between $x(t)$ and $X(s)$ as

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s). \tag{9.4}$$

When $s = j\omega$, eq. (9.3) becomes

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt, \tag{9.5}$$

which corresponds to the *Fourier transform* of $x(t)$; that is,

$$X(s)|_{s=j\omega} = \mathfrak{F}\{x(t)\}. \tag{9.6}$$

The Laplace transform also bears a straightforward relationship to the Fourier transform when the complex variable s is not purely imaginary. To see this relationship, consider $X(s)$ as specified in eq. (9.3) with s expressed as $s = \sigma + j\omega$, so that

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t}\,dt, \tag{9.7}$$

[1]The transform defined by eq. (9.3) is often called the *bilateral Laplace transform,* to distinguish it from the unilateral Laplace transform, which we discuss in Section 9.9. The bilateral transform in eq. (9.3) involves an integration from $-\infty$ to $+\infty$, while the unilateral transform has a form similar to that in eq. (9.3), but with limits of integration from 0 to $+\infty$. As we are primarily concerned with the bilateral transform, we will omit the word "bilateral," except where it is needed in Section 9.9 to avoid ambiguity.

or

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt. \tag{9.8}$$

We recognize the right-hand side of eq. (9.8) as the Fourier transform of $x(t)e^{-\sigma t}$; that is, the Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)$ after multiplication by a real exponential signal. The real exponential $e^{-\sigma t}$ may be decaying or growing in time, depending on whether σ is positive or negative.

To illustrate the Laplace transform and its relationship to the Fourier transform, let us consider the following example:

Example 9.1

Let the signal $x(t) = e^{-at}u(t)$. From Example 4.1, the Fourier transform $X(j\omega)$ converges for $a > 0$ and is given by

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt = \frac{1}{j\omega + a}, \quad a > 0. \tag{9.9}$$

From eq. (9.3), the Laplace transform is

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt, \tag{9.10}$$

or, with $s = \sigma + j\omega$,

$$X(\sigma + j\omega) = \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}\,dt. \tag{9.11}$$

By comparison with eq. (9.9) we recognize eq. (9.11) as the Fourier transform of $e^{-(\sigma+a)t}u(t)$, and thus,

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \quad \sigma + a > 0, \tag{9.12}$$

or equivalently, since $s = \sigma + j\omega$ and $\sigma = \mathcal{R}e\{s\}$,

$$X(s) = \frac{1}{s + a}, \quad \mathcal{R}e\{s\} > -a. \tag{9.13}$$

That is,

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s + a}, \quad \mathcal{R}e\{s\} > -a. \tag{9.14}$$

For example, for $a = 0$, $x(t)$ is the unit step with Laplace transform $X(s) = 1/s$, $\mathcal{R}e\{s\} > 0$.

We note, in particular, that just as the Fourier transform does not converge for all signals, the Laplace transform may converge for some values of $\mathcal{R}e\{s\}$ and not for others. In eq. (9.13), the Laplace transform converges only for $\sigma = \mathcal{R}e\{s\} > -a$. If a is positive, then $X(s)$ can be evaluated at $\sigma = 0$ to obtain

$$X(0 + j\omega) = \frac{1}{j\omega + a}. \tag{9.15}$$

As indicated in eq. (9.6), for $\sigma = 0$ the Laplace transform is equal to the Fourier transform, as is evident in the preceding example by comparing eqs. (9.9) and (9.15). If a is negative or zero, the Laplace transform still exists, but the Fourier transform does not.

Example 9.2

For comparison with Example 9.1, let us consider as a second example the signal

$$x(t) = -e^{-at}u(-t). \tag{9.16}$$

Then

$$\begin{aligned} X(s) &= -\int_{-\infty}^{\infty} e^{-at}e^{-st}u(-t)\,dt \\ &= -\int_{-\infty}^{0} e^{-(s+a)t}\,dt, \end{aligned} \tag{9.17}$$

or

$$X(s) = \frac{1}{s + a}. \tag{9.18}$$

For convergence in this example, we require that $\mathcal{R}e\{s + a\} < 0$, or $\mathcal{R}e\{s\} < -a$; that is,

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s + a}, \qquad \mathcal{R}e\{s\} < -a. \tag{9.19}$$

Comparing eqs. (9.14) and (9.19), we see that the algebraic expression for the Laplace transform is identical for both of the signals considered in Examples 9.1 and 9.2. However, from the same equations, we also see that the set of values of s for which the expression is valid is very different in the two examples. This serves to illustrate the fact that, in specifying the Laplace transform of a signal, both the algebraic expression and the range of values of s for which this expression is valid are required. In general, the range of values of s for which the integral in eq.(9.3) converges is referred to as the *region of convergence* (which we abbreviate as ROC) of the Laplace transform. That is, the ROC consists of those values of $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-\sigma t}$ converges. We will have more to say about the ROC as we develop some insight into the properties of the Laplace transform.

A convenient way to display the ROC is shown in Figure 9.1. The variable s is a complex number, and in the figure we display the complex plane, generally referred to as the s-plane, associated with this complex variable. The coordinate axes are $\mathcal{R}e\{s\}$ along the horizontal axis and $\mathcal{I}m\{s\}$ along the vertical axis. The horizontal and vertical axes are sometimes referred to as the σ-axis and the $j\omega$-axis, respectively. The shaded region in Figure 9.1(a) represents the set of points in the s-plane corresponding to the region of convergence for Example 9.1. The shaded region in Figure 9.1(b) indicates the region of convergence for Example 9.2.

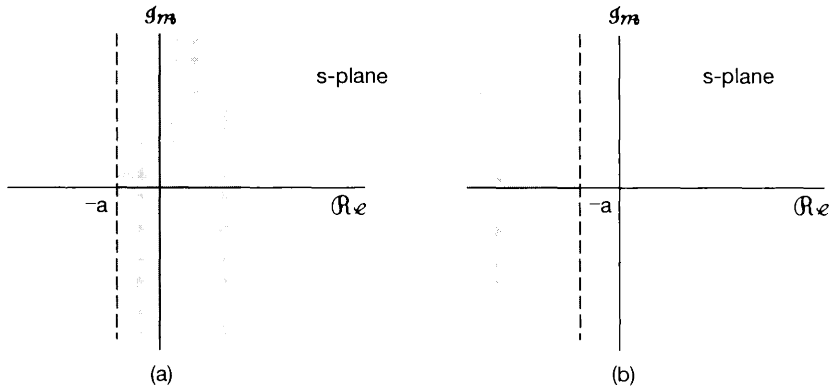


Figure 9.1 (a) ROC for Example 9.1; (b) ROC for Example 9.2.

Example 9.3

In this example, we consider a signal that is the sum of two real exponentials:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t). \tag{9.20}$$

The algebraic expression for the Laplace transform is then

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty}\left[3e^{-2t}u(t) - 2e^{-t}u(t)\right]e^{-st}\,dt \\ &= 3\int_{-\infty}^{\infty} e^{-2t}e^{-st}u(t)\,dt - 2\int_{-\infty}^{\infty} e^{-t}e^{-st}u(t)\,dt. \end{aligned} \tag{9.21}$$

Each of the integrals in eq. (9.21) is of the same form as the integral in eq. (9.10), and consequently, we can use the result in Example 9.1 to obtain

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}. \tag{9.22}$$

To determine the ROC we note that $x(t)$ is a sum of two real exponentials, and from eq. (9.21) we see that $X(s)$ is the sum of the Laplace transforms of each of the individual terms. The first term is the Laplace transform of $3e^{-2t}u(t)$ and the second term the Laplace transform of $-2e^{-t}u(t)$. From Example 9.1, we know that

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \qquad \mathcal{R}e\{s\} > -1,$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2.$$

The set of values of $\mathcal{R}e\{s\}$ for which the Laplace transforms of both terms converge is $\mathcal{R}e\{s\} > -1$, and thus, combining the two terms on the right-hand side of eq. (9.22), we obtain

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s-1}{s^2+3s+2}, \qquad \mathcal{R}e\{s\} > -1. \tag{9.23}$$

Example 9.4

In this example, we consider a signal that is the sum of a real and a complex exponential:

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t). \tag{9.24}$$

Using Euler's relation, we can write

$$x(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t}\right]u(t),$$

and the Laplace transform of $x(t)$ then can be expressed as

$$\begin{aligned} X(s) = & \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt \\ & + \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1-3j)t}u(t)e^{-st}\,dt \\ & + \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1+3j)t}u(t)e^{-st}\,dt. \end{aligned} \tag{9.25}$$

Each of the integrals in eq. (9.25) represents a Laplace transform of the type encountered in Example 9.1. It follows that

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathcal{R}e\{s\} > -2, \tag{9.26}$$

$$e^{-(1-3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1-3j)}, \quad \mathcal{R}e\{s\} > -1, \tag{9.27}$$

$$e^{-(1+3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1+3j)}, \quad \mathcal{R}e\{s\} > -1. \tag{9.28}$$

For all three Laplace transforms to converge simultaneously, we must have $\mathcal{R}e\{s\} > -1$. Consequently, the Laplace transform of $x(t)$ is

$$\frac{1}{s+2} + \frac{1}{2}\left(\frac{1}{s+(1-3j)}\right) + \frac{1}{2}\left(\frac{1}{s+(1+3j)}\right), \quad \mathcal{R}e\{s\} > -1, \tag{9.29}$$

or, with terms combined over a common denominator,

$$e^{-2t}u(t) + e^{-t}(\cos 3t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \quad \mathcal{R}e\{s\} > -1. \tag{9.30}$$

In each of the four preceding examples, the Laplace transform is rational, i.e., it is a ratio of polynomials in the complex variable s, so that

$$X(s) = \frac{N(s)}{D(s)}, \tag{9.31}$$

where $N(s)$ and $D(s)$ are the numerator polynomial and denominator polynomial, respectively. As suggested by Examples 9.3 and 9.4, $X(s)$ will be rational whenever $x(t)$ is a linear combination of real or complex exponentials. As we will see in Section 9.7, rational transforms also arise when we consider LTI systems specified in terms of linear constant-coefficient differential equations. Except for a scale factor, the numerator and denominator polynomials in a rational Laplace transform can be specified by their roots; thus, marking the locations of the roots of $N(s)$ and $D(s)$ in the s-plane and indicating the ROC provides a convenient pictorial way of describing the Laplace transform. For example, in Figure 9.2(a) we show the s-plane representation of the Laplace transform of Example 9.3, with the location of each root of the denominator polynomial in eq. (9.23) indicated with "×" and the location of the root of the numerator polynomial in eq. (9.23) indicated with "o." The corresponding plot of the roots of the numerator and denominator polynomials for the Laplace transform in Example 9.4 is given in Figure 9.2(b). The region of convergence for each of these examples is shaded in the corresponding plot.

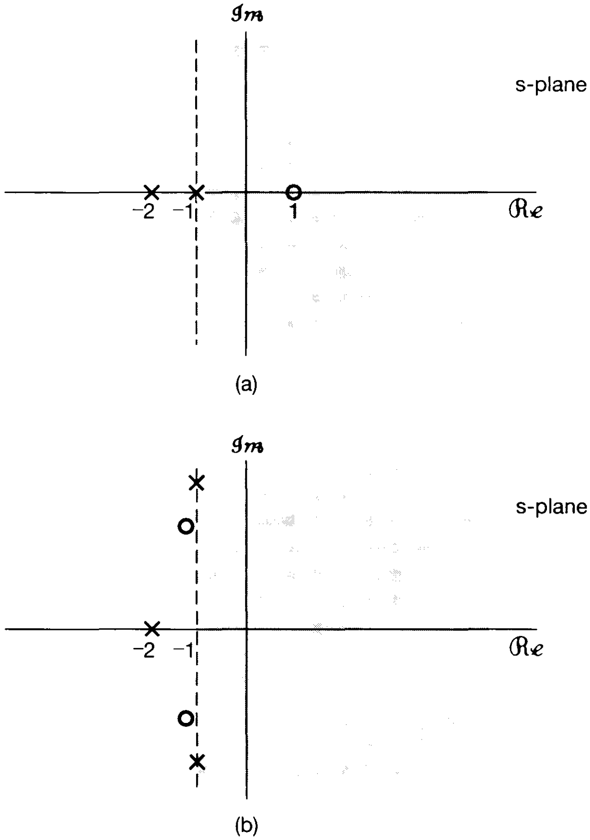


Figure 9.2 s-plane representation of the Laplace transforms for (a) Example 9.3 and (b) Example 9.4. Each × in these figures marks the location of a pole of the corresponding Laplace transform—i.e., a root of the denominator. Similarly, each o marks a zero—i.e., a root of the the numerator. The shaded regions indicate the ROCs.

For rational Laplace transforms, the roots of the numerator polynomial are commonly referred to as the *zeros* of $X(s)$, since, for those values of s, $X(s) = 0$. The roots of the denominator polynomial are referred to as the *poles* of $X(s)$, and for those values of s, $X(s)$ is infinite. The poles and zeros of $X(s)$ in the finite s-plane completely characterize the algebraic expression for $X(s)$ to within a scale factor. The representation of $X(s)$ through its poles and zeros in the s-plane is referred to as the *pole-zero plot* of $X(s)$.

However, as we saw in Examples 9.1 and 9.2, knowledge of the algebraic form of $X(s)$ does not by itself identify the ROC for the Laplace transform. That is, a complete specification, to within a scale factor, of a rational Laplace transform consists of the pole-zero plot of the transform, together with its ROC (which is commonly shown as a shaded region in the s-plane, as in Figures 9.1 and 9.2).

Also, while they are not needed to specify the algebraic form of a rational transform $X(s)$, it is sometimes convenient to refer to poles or zeros of $X(s)$ at infinity. Specifically, if the order of the denominator polynomial is greater than the order of the numerator polynomial, then $X(s)$ will become zero as s approaches infinity. Conversely, if the order of the numerator polynomial is greater than the order of the denominator, then $X(s)$ will become unbounded as s approaches infinity. This behavior can be interpreted as zeros or poles at infinity. For example, the Laplace transform in eq. (9.23) has a denominator of order 2 and a numerator of order only 1, so in this case $X(s)$ has one zero at infinity. The same is true for the transform in eq. (9.30), in which the numerator is of order 2 and the denominator is of order 3. In general, if the order of the denominator exceeds the order of the numerator by k, $X(s)$ will have k zeros at infinity. Similarly, if the order of the numerator exceeds the order of the denominator by k, $X(s)$ will have k poles at infinity.

Example 9.5

Let

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t). \tag{9.32}$$

The Laplace transform of the second and third terms on the right-hand side of eq. (9.32) can be evaluated from Example 9.1. The Laplace transform of the unit impulse can be evaluated directly as

$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t)e^{-st}\,dt = 1, \tag{9.33}$$

which is valid for any value of s. That is, the ROC of $\mathcal{L}\{\delta(t)\}$ is the entire s-plane. Using this result, together with the Laplace transforms of the other two terms in eq. (9.32), we obtain

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}, \quad \mathcal{R}e\{s\} > 2, \tag{9.34}$$

or

$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)}, \quad \mathcal{R}e\{s\} > 2, \tag{9.35}$$

where the ROC is the set of values of s for which the Laplace transforms of all three terms in $x(t)$ converge. The pole-zero plot for this example is shown in Figure 9.3, together with the ROC. Also, since the degrees of the numerator and denominator of $X(s)$ are equal, $X(s)$ has neither poles nor zeros at infinity.

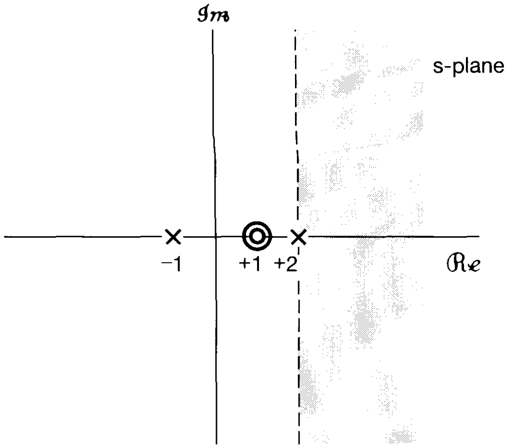


Figure 9.3 Pole-zero plot and ROC for Example 9.5.

Recall from eq. (9.6) that, for $s = j\omega$, the Laplace transform corresponds to the Fourier transform. However, if the ROC of the Laplace transform does not include the $j\omega$-axis, (i.e., if $\mathcal{Re}\{s\} = 0$), then the Fourier transform does not converge. As we see from Figure 9.3, this, in fact, is the case for Example 9.5, which is consistent with the fact that the term $(1/3)e^{2t}u(t)$ in $x(t)$ does not have a Fourier transform. Note also in this example that the two zeros in eq. (9.35) occur at the same value of s. In general, we will refer to the *order* of a pole or zero as the number of times it is repeated at a given location. In Example 9.5 there is a second-order zero at $s = 1$ and two first-order poles, one at $s = -1$, the other at $s = 2$. In this example the ROC lies to the right of the rightmost pole. In general, for rational Laplace transforms, there is a close relationship between the locations of the poles and the possible ROCs that can be associated with a given pole-zero plot. Specific constraints on the ROC are closely associated with time-domain properties of $x(t)$. In the next section, we explore some of these constraints and properties.

9.2 THE REGION OF CONVERGENCE FOR LAPLACE TRANSFORMS

In the preceding section, we saw that a complete specification of the Laplace transform requires not only the algebraic expression for $X(s)$, but also the associated region of convergence. As evidenced by Examples 9.1 and 9.2, two very different signals can have identical algebraic expressions for $X(s)$, so that their Laplace transforms are distinguishable *only* by the region of convergence. In this section, we explore some specific constraints on the ROC for various classes of signals. As we will see, an understanding of these constraints often permits us to specify implicitly or to reconstruct the ROC from knowledge of only the algebraic expression for $X(s)$ and certain general characteristics of $x(t)$ in the time domain.

Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

The validity of this property stems from the fact that the ROC of $X(s)$ consists of those values of $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-\sigma t}$ converges. That is, the ROC of the Laplace transform of $x(t)$ consists of those values of s for which $x(t)e^{-\sigma t}$ is absolutely integrable:[2]

$$\int_{-\infty}^{+\infty} |x(t)|e^{-\sigma t}\,dt < \infty. \tag{9.36}$$

Property 1 then follows, since this condition depends only on σ, the real part of s.

Property 2: For rational Laplace transforms, the ROC does not contain any poles.

Property 2 is easily observed in all the examples studied thus far. Since $X(s)$ is infinite at a pole, the integral in eq. (9.3) clearly does not converge at a pole, and thus the ROC cannot contain values of s that are poles.

Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

The intuition behind this result is suggested in Figures 9.4 and 9.5. Specifically, a finite-duration signal has the property that it is zero outside an interval of finite duration, as illustrated in Figure 9.4. In Figure 9.5(a), we have shown $x(t)$ of Figure 9.4 multiplied by a decaying exponential, and in Figure 9.5(b) the same signal multiplied by a growing

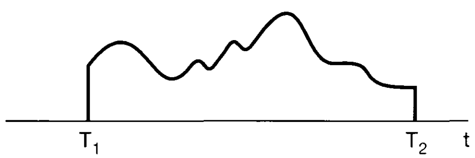


Figure 9.4 Finite-duration signal.

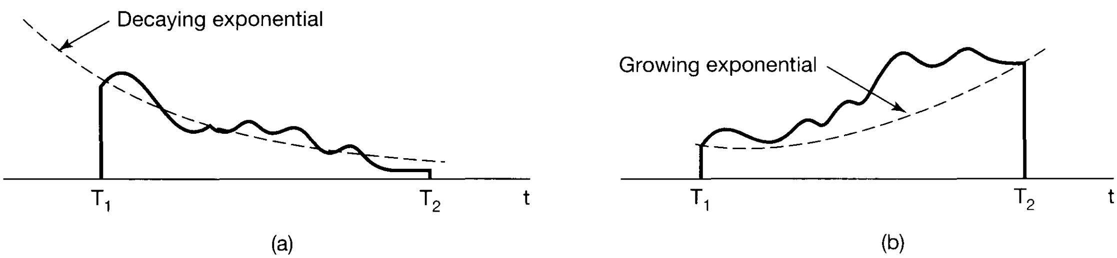


Figure 9.5 (a) Finite-duration signal of Figure 9.4 multiplied by a decaying exponential; (b) finite-duration signal of Figure 9.4 multiplied by a growing exponential.

[2]For a more thorough and formal treatment of Laplace transforms and their mathematical properties, including convergence, see E. D. Rainville, *The Laplace Transform: An Introduction* (New York: Macmillan, 1963), and R. V. Churchill and J. W. Brown, *Complex Variables and Applications* (5th ed.) (New York: McGraw-Hill, 1990). Note that the condition of absolute integrability is one of the Dirichlet conditions introduced in Section 4.1 in the context of our discussion of the convergence of Fourier transforms.

exponential. Since the interval over which $x(t)$ is nonzero is finite, the exponential weighting is never unbounded, and consequently, it is reasonable that the integrability of $x(t)$ not be destroyed by this exponential weighting.

A more formal verification of Property 3 is as follows: Suppose that $x(t)$ is absolutely integrable, so that

$$\int_{T_1}^{T_2} |x(t)|\,dt < \infty. \tag{9.37}$$

For $s = \sigma + j\omega$ to be in the ROC, we require that $x(t)e^{-\sigma t}$ be absolutely integrable, i.e.,

$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}\,dt < \infty. \tag{9.38}$$

Eq. (9.37) verifies that s is in the ROC when $\mathcal{R}e\{s\} = \sigma = 0$. For $\sigma > 0$, the maximum value of $e^{-\sigma t}$ over the interval on which $x(t)$ is nonzero is $e^{-\sigma T_1}$, and thus we can write

$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}\,dt < e^{-\sigma T_1}\int_{T_1}^{T_2} |x(t)|\,dt. \tag{9.39}$$

Since the right-hand side of eq.(9.39) is bounded, so is the left-hand side; therefore, the s-plane for $\mathcal{R}e\{s\} > 0$ must also be in the ROC. By a similar argument, if $\sigma < 0$, then

$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}\,dt < e^{-\sigma T_2}\int_{T_1}^{T_2} |x(t)|\,dt, \tag{9.40}$$

and again, $x(t)e^{-\sigma t}$ is absolutely integrable. Thus, the ROC includes the entire s-plane.

Example 9.6

Let

$$x(t) = \begin{cases} e^{-at}, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}. \tag{9.41}$$

Then

$$X(s) = \int_0^T e^{-at}e^{-st}\,dt = \frac{1}{s+a}[1 - e^{-(s+a)T}]. \tag{9.42}$$

Since in this example $x(t)$ is of finite length, it follows from Property 3 that the ROC is the entire s-plane. In the form of eq. (9.42), $X(s)$ would appear to have a pole at $s = -a$, which, from Property 2, would be inconsistent with an ROC that consists of the entire s-plane. In fact, however, in the algebraic expression in eq. (9.42), both numerator and denominator are zero at $s = -a$, and thus, to determine $X(s)$ at $s = -a$, we can use L'hôpital's rule to obtain

$$\lim_{s\to -a} X(s) = \lim_{s\to -a}\left[\frac{\frac{d}{ds}(1 - e^{-(s+a)T})}{\frac{d}{ds}(s+a)}\right] = \lim_{s\to -a} Te^{-aT}e^{-sT},$$

so that

$$X(-a) = T. \tag{9.43}$$

It is important to recognize that, to ensure that the exponential weighting is bounded over the interval in which $x(t)$ is nonzero, the preceding discussion relies heavily on the fact that $x(t)$ is of finite duration. In the next two properties, we consider modifications of the result in Property 3 when $x(t)$ is of finite extent in only the positive-time or negative-time direction.

Property 4: If $x(t)$ is right sided, and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e\{s\} > \sigma_0$ will also be in the ROC.

A *right-sided* signal is a signal for which $x(t) = 0$ prior to some finite time T_1, as illustrated in Figure 9.6. It is possible that, for such a signal, there is no value of s for which the Laplace transform will converge. One example is the signal $x(t) = e^{t^2}u(t)$. However, suppose that the Laplace transform converges for some value of σ, which we denote by σ_0. Then

$$\int_{-\infty}^{+\infty} |x(t)|e^{-\sigma_0 t}\,dt < \infty, \tag{9.44}$$

or equivalently, since $x(t)$ is right sided,

$$\int_{T_1}^{+\infty} |x(t)|e^{-\sigma_0 t}\,dt < \infty. \tag{9.45}$$

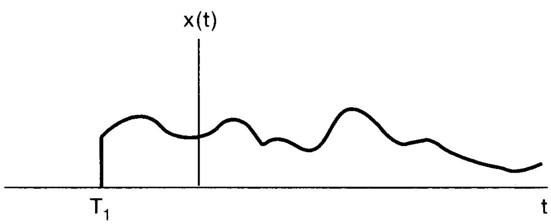


Figure 9.6 Right-sided signal.

Then if $\sigma_1 > \sigma_0$, it must also be true that $x(t)e^{-\sigma_1 t}$ is absolutely integrable, since $e^{-\sigma_1 t}$ decays faster than $e^{-\sigma_0 t}$ as $t \longrightarrow +\infty$, as illustrated in Figure 9.7. Formally, we can say that with $\sigma_1 > \sigma_0$,

$$\begin{aligned}\int_{T_1}^{\infty} |x(t)|e^{-\sigma_1 t}\,dt &= \int_{T_1}^{\infty} |x(t)|e^{-\sigma_0 t}e^{-(\sigma_1-\sigma_0)t}\,dt \\ &\leq e^{-(\sigma_1-\sigma_0)T_1}\int_{T_1}^{\infty} |x(t)|e^{-\sigma_0 t}\,dt.\end{aligned} \tag{9.46}$$

Since T_1 is finite, it follows from eq. (9.45) that the right side of the inequality in eq. (9.46) is finite, and hence, $x(t)e^{-\sigma_1 t}$ is absolutely integrable.

Note that in the preceding argument we explicitly rely on the fact that $x(t)$ is right sided, so that, although with $\sigma_1 > \sigma_0$, $e^{-\sigma_1 t}$ diverges faster than $e^{-\sigma_0 t}$ as $t \rightarrow -\infty$, $x(t)e^{-\sigma_1 t}$ cannot grow without bound in the negative-time direction, since $x(t) = 0$ for

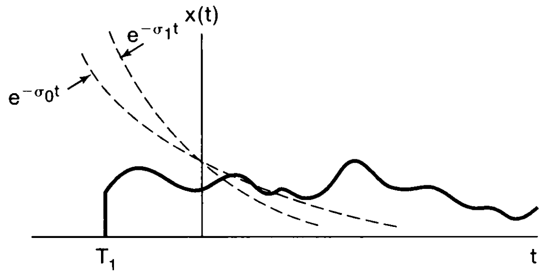


Figure 9.7 If $x(t)$ is right sided and $x(t)e^{-\sigma_0 t}$ is absolutely integrable, then $x(t)e^{-\sigma_1 t}, \sigma_1 > \sigma_0$, will also be absolutely integrable.

$t < T_1$. Also, in this case, if a point s is in the ROC, then all the points to the right of s, i.e., all points with larger real parts, are in the ROC. For this reason, the ROC in this case is commonly referred to as a *right-half plane.*

Property 5: If $x(t)$ is left sided, and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e\{s\} < \sigma_0$ will also be in the ROC.

A *left-sided* signal is a signal for which $x(t) = 0$ after some finite time T_2, as illustrated in Figure 9.8. The argument and intuition behind this property are exactly analogous to the argument and intuition behind Property 4. Also, for a left-sided signal, the ROC is commonly referred to as a *left-half plane,* as if a point s is in the ROC, then all points to the left of s are in the ROC.

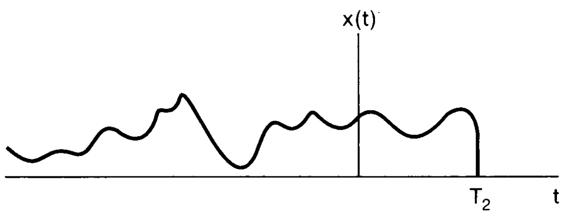


Figure 9.8 Left-sided signal.

Property 6: If $x(t)$ is two sided, and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\mathcal{R}e\{s\} = \sigma_0$.

A *two-sided* signal is a signal that is of infinite extent for both $t > 0$ and $t < 0$, as illustrated in Figure 9.9(a). For such a signal, the ROC can be examined by choosing an arbitrary time T_0 and dividing $x(t)$ into the sum of a right-sided signal $x_R(t)$ and a left-sided signal $x_L(t)$, as indicated in Figures 9.9(b) and 9.9(c). The Laplace transform of $x(t)$ converges for values of s for which the transforms of *both* $x_R(t)$ and $x_L(t)$ converge. From Property 4, the ROC of $\mathcal{L}\{x_R(t)\}$ consists of a half-plane $\mathcal{R}e\{s\} > \sigma_R$ for some value σ_R, and from Property 5, the ROC of $\mathcal{L}\{x_L(t)\}$ consists of a half-plane $\mathcal{R}e\{s\} < \sigma_L$ for some value σ_L. The ROC of $\mathcal{L}\{x(t)\}$ is then the overlap of these two half-planes, as indicated in Figure 9.10. This assumes, of course, that $\sigma_R < \sigma_L$, so that there is some overlap. If this is not the case, then even if the Laplace transforms of $x_R(t)$ and $x_L(t)$ individually exist, the Laplace transform of $x(t)$ does not.

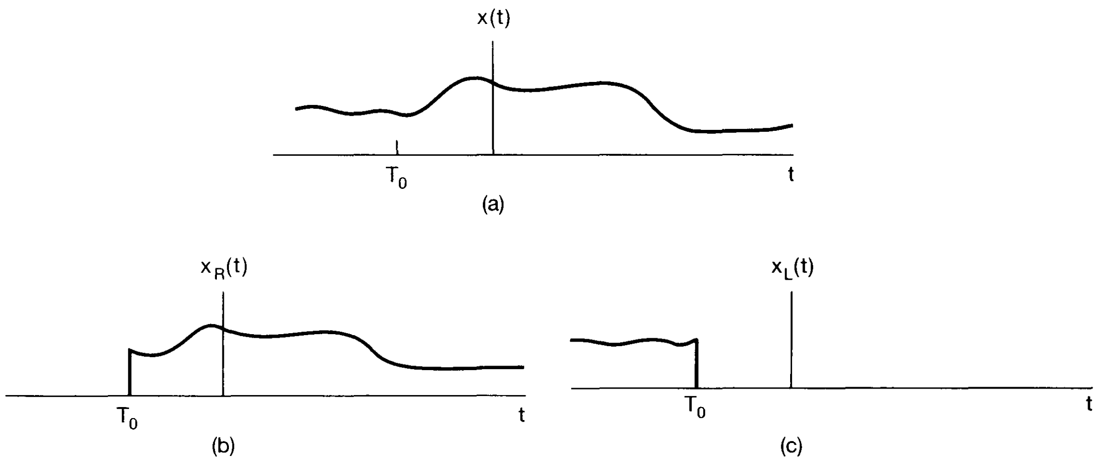


Figure 9.9 Two-sided signal divided into the sum of a right-sided and left-sided signal: (a) two-sided signal $x(t)$; (b) the right-sided signal equal to $x(t)$ for $t > T_0$ and equal to 0 for $t < T_0$; (c) the left-sided signal equal to $x(t)$ for $t < T_0$ and equal to 0 for $t > T_0$.

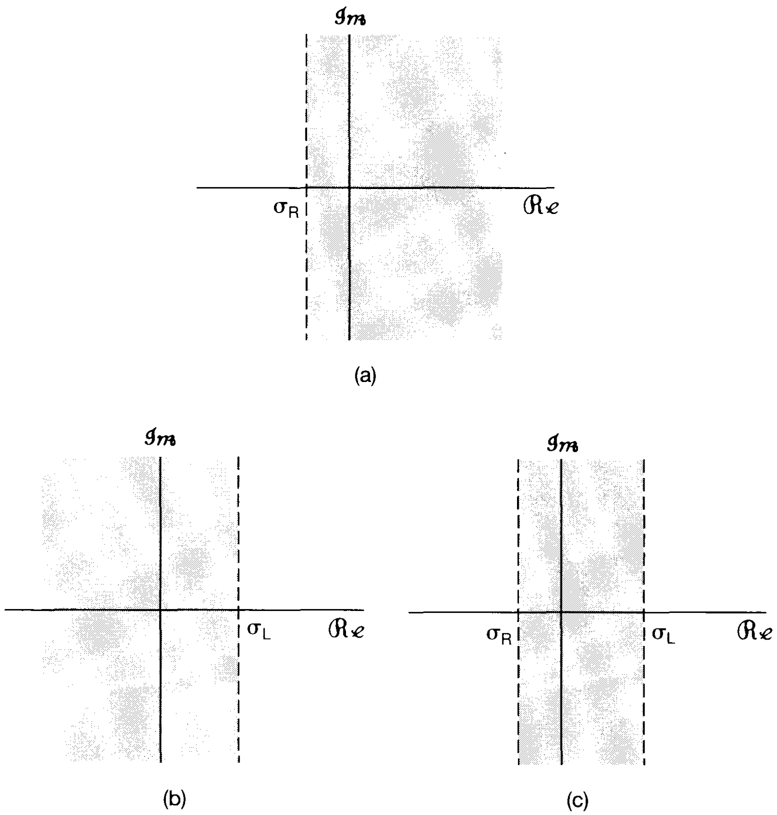


Figure 9.10 (a) ROC for $x_R(t)$ in Figure 9.9; (b) ROC for $x_L(t)$ in Figure 9.9; (c) the ROC for $x(t) = x_R(t) + x_L(t)$, assuming that the ROCs in (a) and (b) overlap.

Example 9.7

Let

$$x(t) = e^{-b|t|}, \tag{9.47}$$

as illustrated in Figure 9.11 for both $b > 0$ and $b < 0$. Since this is a two-sided signal, let us divide it into the sum of a right-sided and left-sided signal; that is,

$$x(t) = e^{-bt}u(t) + e^{+bt}u(-t). \tag{9.48}$$

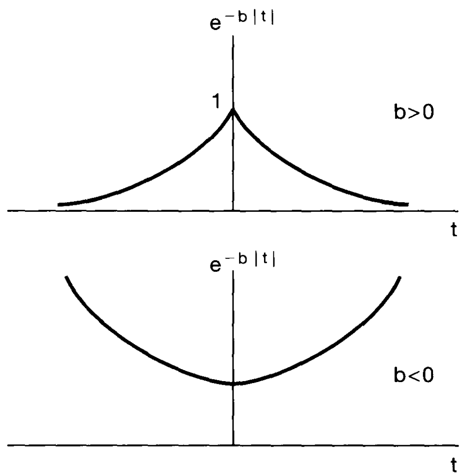


Figure 9.11 Signal $x(t) = e^{-b|t|}$ for both $b > 0$ and $b < 0$.

From Example 9.1,

$$e^{-bt}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b}, \quad \Re e\{s\} > -b, \tag{9.49}$$

and from Example 9.2,

$$e^{+bt}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b}, \quad \Re e\{s\} < +b. \tag{9.50}$$

Although the Laplace transforms of each of the individual terms in eq. (9.48) have a region of convergence, there is no *common* region of convergence if $b \leq 0$, and thus, for those values of b, $x(t)$ has no Laplace transform. If $b > 0$, the Laplace transform of $x(t)$ is

$$e^{-b|t|} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2-b^2}, \quad -b < \Re e\{s\} < +b. \tag{9.51}$$

The corresponding pole-zero plot is shown in Figure 9.12, with the shading indicating the ROC.

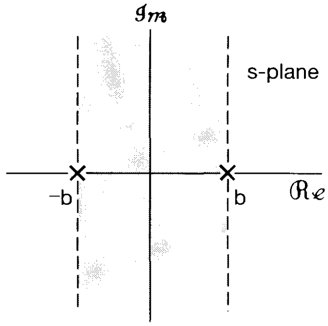


Figure 9.12 Pole-zero plot and ROC for Example 9.7.

A signal either does not have a Laplace transform or falls into one of the four categories covered by Properties 3 through 6. Thus, for any signal with a Laplace transform, the ROC *must* be the entire s-plane (for finite-length signals), a left-half plane (for left-sided signals), a right-half plane (for right-sided signals), or a single strip (for two-sided signals). In all the examples that we have considered, the ROC has the additional property that in each direction (i.e., $\mathcal{R}e\{s\}$ increasing and $\mathcal{R}e\{s\}$ decreasing) it is bounded by poles or extends to infinity. In fact, this is *always* true for rational Laplace transforms:

Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

A formal argument establishing this property is somewhat involved, but its validity is essentially a consequence of the facts that a signal with a rational Laplace transform consists of a linear combination of exponentials and, from Examples 9.1 and 9.2, that the ROC for the transform of individual terms in this linear combination must have the property. As a consequence of Property 7, together with Properties 4 and 5, we have

Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

To illustrate how different ROCs can be associated with the same pole-zero pattern, let us consider the following example:

Example 9.8

Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \tag{9.52}$$

with the associated pole-zero pattern in Figure 9.13(a). As indicated in Figures 9.13(b)–(d), there are three possible ROCs that can be associated with this algebraic expression, corresponding to three distinct signals. The signal associated with the pole-zero pattern in Figure 9.13(b) is right sided. Since the ROC includes the $j\omega$-axis, the Fourier

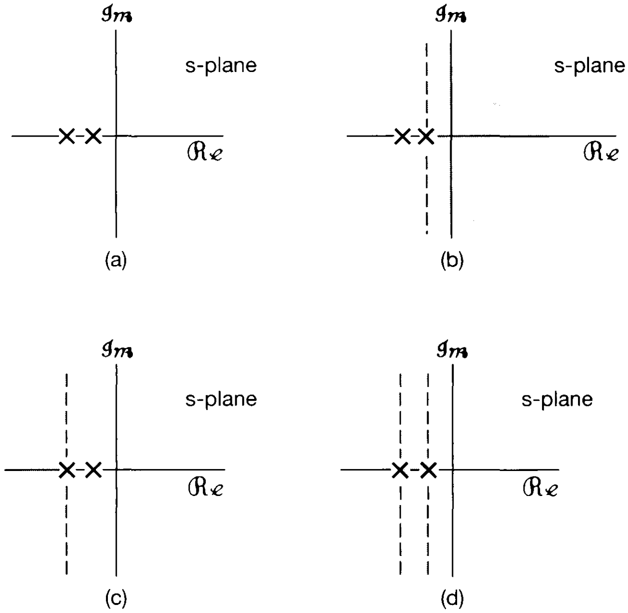


Figure 9.13 (a) Pole-zero pattern for Example 9.8; (b) ROC corresponding to a right-sided sequence; (c) ROC corresponding to a left-sided sequence; (d) ROC corresponding to a two-sided sequence.

transform of this signal converges. Figure 9.13(c) corresponds to a left-sided signal and Figure 9.13(d) to a two-sided signal. Neither of these two signals have Fourier transforms, since their ROCs do not include the $j\omega$-axis.

9.3 THE INVERSE LAPLACE TRANSFORM

In Section 9.1 we discussed the interpretation of the Laplace transform of a signal as the Fourier transform of an exponentially weighted version of the signal; that is, with s expressed as $s = \sigma + j\omega$, the Laplace transform of a signal $x(t)$ is

$$X(\sigma + j\omega) = \mathfrak{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt \tag{9.53}$$

for values of $s = \sigma + j\omega$ in the ROC. We can invert this relationship using the inverse Fourier transform as given in eq. (4.9). We have

$$x(t)e^{-\sigma t} = \mathfrak{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega, \tag{9.54}$$

or, multiplying both sides by $e^{\sigma t}$, we obtain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}\,d\omega. \tag{9.55}$$

That is, we can recover $x(t)$ from its Laplace transform evaluated for a set of values of $s = \sigma + j\omega$ in the ROC, with σ fixed and ω varying from $-\infty$ to $+\infty$. We can highlight this and gain additional insight into recovering $x(t)$ from $X(s)$ by changing the variable of integration in eq. (9.55) from ω to s and using the fact that σ is constant, so that $ds = j\,d\omega$. The result is the basic inverse Laplace transform equation:

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds.} \tag{9.56}$$

This equation states that $x(t)$ can be represented as a weighted integral of complex exponentials. The contour of integration in eq. (9.56) is the straight line in the s-plane corresponding to all points s satisfying $\mathcal{R}e\{s\} = \sigma$. This line is parallel to the $j\omega$-axis. Furthermore, we can choose any such line in the ROC—i.e., we can choose any value of σ such that $X(\sigma + j\omega)$ converges. The formal evaluation of the integral for a general $X(s)$ requires the use of contour integration in the complex plane, a topic that we will not consider here. However, for the class of rational transforms, the inverse Laplace transform can be determined without directly evaluating eq. (9.56) by using the technique of partial-fraction expansion in a manner similar to that used in Chapter 4 to determine the inverse Fourier transform. Basically, the procedure consists of expanding the rational algebraic expression into a linear combination of lower order terms.

For example, assuming no multiple-order poles, and assuming that the order of the denominator polynomial is greater than the order of the numerator polynomial, we can expand $X(s)$ in the form

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i}. \tag{9.57}$$

From the ROC of $X(s)$, the ROC of each of the individual terms in eq. (9.57) can be inferred, and then, from Examples 9.1 and 9.2, the inverse Laplace transform of each of these terms can be determined. There are two possible choices for the inverse transform of each term $A_i/(s + a_i)$ in the equation. If the ROC is to the right of the pole at $s = -a_i$, then the inverse transform of this term is $A_i e^{-a_i t}u(t)$, a right-sided signal. If the ROC is to the left of the pole at $s = -a_i$, then the inverse transform of the term is $-A_i e^{-a_i t}u(-t)$, a left-sided signal. Adding the inverse transforms of the individual terms in eq. (9.57) then yields the inverse transform of $X(s)$. The details of this procedure are best presented through a number of examples.

Example 9.9

Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathcal{R}e\{s\} > -1. \tag{9.58}$$

To obtain the inverse Laplace transform, we first perform a partial-fraction expansion to obtain

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}. \tag{9.59}$$

As discussed in the appendix, we can evaluate the coefficients A and B by multiplying both sides of eq. (9.59) by $(s + 1)(s + 2)$ and then equating coefficients of equal powers of s on both sides. Alternatively, we can use the relation

$$A = [(s + 1)X(s)]|_{s=-1} = 1, \tag{9.60}$$

$$B = [(s + 2)X(s)]|_{s=-2} = -1. \tag{9.61}$$

Thus, the partial-fraction expansion for $X(s)$ is

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}. \tag{9.62}$$

From Examples 9.1 and 9.2, we know that there are two possible inverse transforms for a transform of the form $1/(s + a)$, depending on whether the ROC is to the left or the right of the pole. Consequently, we need to determine which ROC to associate with each of the individual first-order terms in eq. (9.62). This is done by reference to the properties of the ROC developed in Section 9.2. Since the ROC for $X(s)$ is $\Re e\{s\} > -1$, the ROC for the individual terms in the partial-fraction expansion of eq. (9.62) includes $\Re e\{s\} > -1$. The ROC for each term can then be extended to the left or right (or both) to be bounded by a pole or infinity. This is illustrated in Figure 9.14. Figure 9.14(a) shows the pole-zero plot and ROC for $X(s)$, as specified in eq. (9.58). Figure 9.14(b) and 9.14(c) represent the individual terms in the partial-fraction expansion in eq. (9.62). The ROC for the sum is indicated with lighter shading. For the term represented by Figure 9.14(c), the ROC for the sum can be extended to the left as shown, so that it is bounded by a pole.

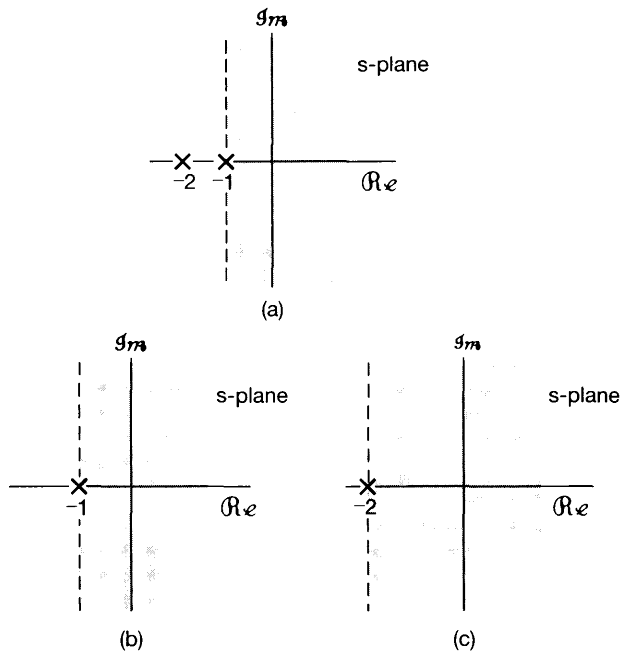


Figure 9.14 Construction of the ROCs for the individual terms in the partial-fraction expansion of $X(s)$ in Example 9.8: (a) pole-zero plot and ROC for $X(s)$; (b) pole at $s = -1$ and its ROC; (c) pole at $s = -2$ and its ROC.

Since the ROC is to the right of both poles, the same is true for each of the individual terms, as can be seen in Figures 9.14(b) and (c). Consequently, from Property 8 in the preceding section, we know that each of these terms corresponds to a right-sided signal. The inverse transform of the individual terms in eq. (9.62) can then be obtained by reference to Example 9.1:

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \qquad \mathcal{R}e\{s\} > -1, \tag{9.63}$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2. \tag{9.64}$$

We thus obtain

$$[e^{-t} - e^{-2t}]u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \qquad \mathcal{R}e\{s\} > -1. \tag{9.65}$$

Example 9.10

Let us now suppose that the algebraic expression for $X(s)$ is again given by eq. (9.58), but that the ROC is now the left-half plane $\mathcal{R}e\{s\} < -2$. The partial-fraction expansion for $X(s)$ relates only to the algebraic expression, so eq. (9.62) is still valid. With this new ROC, however, the ROC is to the *left* of both poles and thus, the same must be true for each of the two terms in the equation. That is, the ROC for the term corresponding to the pole at $s = -1$ is $\mathcal{R}e\{s\} < -1$, while the ROC for the term with pole at $s = -2$ is $\mathcal{R}e\{s\} < -2$. Then, from Example 9.2,

$$-e^{-t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \qquad \mathcal{R}e\{s\} < -1, \tag{9.66}$$

$$-e^{-2t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} < -2, \tag{9.67}$$

so that

$$x(t) = [-e^{-t} + e^{-2t}]u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \qquad \mathcal{R}e\{s\} < -2. \tag{9.68}$$

Example 9.11

Finally, suppose that the ROC of $X(s)$ in eq. (9.58) is $-2 < \mathcal{R}e\{s\} < -1$. In this case, the ROC is to the left of the pole at $s = -1$ so that this term corresponds to the left-sided signal in eq. (9.66), while the ROC is to the right of the pole at $s = -2$ so that this term corresponds to the right-sided signal in eq. (9.64). Combining these, we find that

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \qquad -2 < \mathcal{R}e\{s\} < -1. \tag{9.69}$$

As discussed in the appendix, when $X(s)$ has multiple-order poles, or when the denominator is not of higher degree than the numerator, the partial-fraction expansion of $X(s)$ will include other terms in addition to the first-order terms considered in Examples 9.9–9.11. In Section 9.5, after discussing properties of the Laplace transform, we develop some other Laplace transform pairs that, in conjunction with the properties, allow us to extend the inverse transform method outlined in Example 9.9 to arbitrary rational transforms.

9.4 GEOMETRIC EVALUATION OF THE FOURIER TRANSFORM FROM THE POLE-ZERO PLOT

As we saw in Section 9.1, the Fourier transform of a signal is the Laplace transform evaluated on the $j\omega$-axis. In this section we discuss a procedure for geometrically evaluating the Fourier transform and, more generally, the Laplace transform at any set of values from the pole-zero pattern associated with a rational Laplace transform. To develop the procedure, let us first consider a Laplace transform with a single zero [i.e., $X(s) = s - a$], which we evaluate at a specific value of s, say, $s = s_1$. The algebraic expression $s_1 - a$ is the sum of two complex numbers, s_1 and $-a$, each of which can be represented as a vector in the complex plane, as illustrated in Figure 9.15. The vector representing the complex number $s_1 - a$ is then the vector sum of $\mathbf{s}_1$ and $-\mathbf{a}$, which we see in the figure to be a vector from the zero at $s = a$ to the point s_1. The value of $X(s_1)$ then has a magnitude that is the length of this vector and an angle that is the angle of the vector relative to the real axis. If $X(s)$ instead has a single pole at $s = a$ [i.e., $X(s) = 1/(s - a)$], then the denominator would be represented by the same vector sum of $\mathbf{s}_1$ and $-\mathbf{a}$, and the value of $X(s_1)$ would have a magnitude that is the *reciprocal* of the length of the vector from the pole to $s = s_1$ and an angle that is the *negative* of the angle of the vector with the real axis.

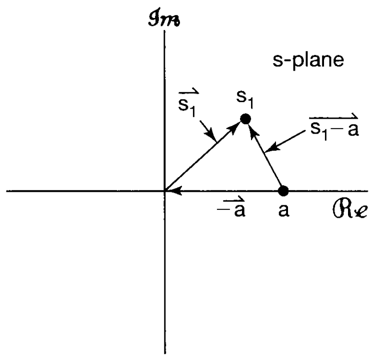


Figure 9.15 Complex plane representation of the vectors $\mathbf{s}_1$, $\mathbf{a}$, and $\mathbf{s}_1 - \mathbf{a}$ representing the complex numbers s_1, a, and $s_1 - a$, respectively.

A more general rational Laplace transform consists of a product of pole and zero terms of the form discussed in the preceding paragraph; that is, it can be factored into the form

$$X(s) = M\frac{\prod_{i=l}^{R}(s - \beta_i)}{\prod_{j=l}^{P}(s - \alpha_j)}. \tag{9.70}$$

To evaluate $X(s)$ at $s = s_1$, each term in the product is represented by a vector from the zero or pole to the point s_1. The magnitude of $X(s_1)$ is then the magnitude of the scale factor M, times the product of the lengths of the zero vectors (i.e., the vectors from the zeros to s_1) divided by the product of the lengths of the pole vectors (i.e., the vectors from the poles to s_1). The angle of the complex number $X(s_1)$ is the sum of the angles of the zero vectors minus the sum of the angles of the pole vectors. If the scale factor M in eq. (9.70) is negative, an additional angle of π would be included. If $X(s)$ has a multiple pole or zero (or both), corresponding to some of the α_j's being equal to each other or some of the β_i's being equal to each other (or both), the lengths and angles of the vectors from each of these poles or zeros must be included a number of times equal to the order of the pole or zero.

Example 9.12

Let

$$X(s) = \frac{1}{s + \frac{1}{2}}, \qquad \mathcal{R}e\{s\} > -\frac{1}{2}. \tag{9.71}$$

The Fourier transform is $X(s)|_{s=j\omega}$. For this example, then, the Fourier transform is

$$X(j\omega) = \frac{1}{j\omega + 1/2}. \tag{9.72}$$

The pole-zero plot for $X(s)$ is shown in Figure 9.16. To determine the Fourier transform graphically, we construct the pole vector as indicated. The magnitude of the Fourier transform at frequency ω is the reciprocal of the length of the vector from the pole to the point $j\omega$ on the imaginary axis. The phase of the Fourier transform is the negative of the angle of the vector. Geometrically, from Figure 9.16, we can write

$$|X(j\omega)|^2 = \frac{1}{\omega^2 + (1/2)^2} \tag{9.73}$$

and

$$\sphericalangle X(j\omega) = -\tan^{-1} 2\omega. \tag{9.74}$$

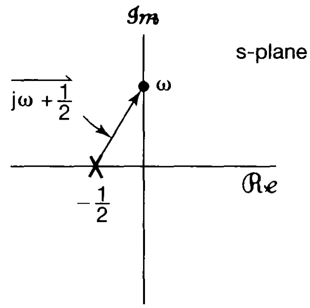


Figure 9.16 Pole-zero plot for Example 9.12. $|X(j\omega)|$ is the reciprocal of the length of the vector shown, and $\sphericalangle X(j\omega)$ is the negative of the angle of the vector.

Often, part of the value of the geometric determination of the Fourier transform lies in its usefulness in obtaining an approximate view of the overall characteristics of the transform. For example, in Figure 9.16, it is readily evident that the length of the pole vector monotonically increases with increasing ω, and thus, the magnitude of the Fourier transform will monotonically *decrease* with increasing ω. The ability to draw general conclusions about the behavior of the Fourier transform from the pole-zero plot is further illustrated by a consideration of general first- and second-order systems.

9.4.1 First-Order Systems

As a generalization of Example 9.12, let us consider the class of first-order systems that was discussed in some detail in Section 6.5.1. The impulse response for such a system is

$$h(t) = \frac{1}{\tau}e^{-t/\tau}u(t), \tag{9.75}$$

and its Laplace transform is

$$H(s) = \frac{1}{s\tau + 1}, \qquad \Re e\{s\} > -\frac{1}{\tau}. \tag{9.76}$$

The pole-zero plot is shown in Figure 9.17. Note from the figure that the length of the pole vector is minimal for $\omega = 0$ and increases monotonically as ω increases. Also, the angle of the pole increases monotonically from 0 to $\pi/2$ as ω increases from 0 to ∞.

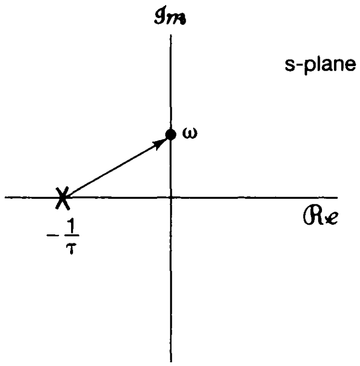


Figure 9.17 Pole-zero plot for first-order system of eq. (9.76).

From the behavior of the pole vector as ω varies, it is clear that the magnitude of the frequency response $H(j\omega)$ monotonically decreases as ω increases, while $\measuredangle H(j\omega)$ monotonically decreases from 0 to $-\pi/2$, as shown in the Bode plots for this system in Figure 9.18. Note also that when $\omega = 1/\tau$, the real and imaginary parts of the pole vector are equal, yielding a value of the magnitude of the frequency response that is reduced by a factor of $\sqrt{2}$, or approximately 3 dB, from its maximum at $\omega = 0$ and a value of $\pi/4$ for the angle of the frequency response. This is consistent with our examination of first-order systems in Section 6.5.1, where we noted that $\omega = 1/\tau$ is often referred to as the 3-dB point or the break frequency—i.e., the frequency at which the straight-line approximation of the Bode plot of $|H(j\omega)|$ has a break in its slope. As we also saw in Section 6.5.1, the time constant τ controls the speed of response of first-order systems, and we now see that

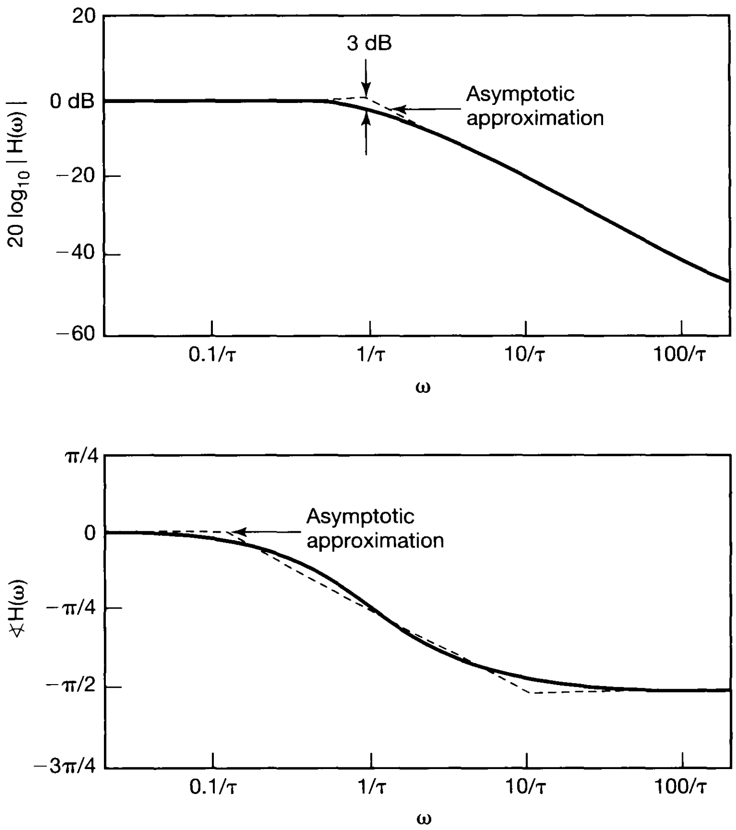


Figure 9.18 Frequency response for a first-order system.

the pole of such a system at $s = -1/\tau$ is on the negative real axis, at a distance to the origin that is the reciprocal of the time constant.

From our graphical interpretation, we can also see how changing the time constant or, equivalently, the position of the pole of $H(s)$ changes the characteristics of a first-order system. In particular, as the pole moves farther into the left-half plane, the break frequency and, hence, the effective cutoff frequency of the system increases. Also, from eq. (9.75) and from Figure 6.19, we see that this same movement of the pole to the left corresponds to a decrease in the time constant τ, resulting in a faster decay of the impulse response and a correspondingly faster rise time in the step response. This relationship between the real part of the pole locations and the speed of the system response holds more generally; that is, poles farther away from the $j\omega$-axis are associated with faster response terms in the impulse response.

9.4.2 Second-Order Systems

Let us next consider the class of second-order systems, which was discussed in some detail in Section 6.5.2. The impulse response and frequency response for the system, originally given in eqs. (6.37) and (6.33), respectively, are

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t), \tag{9.77}$$

where

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1},$$
$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}},$$

and

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}. \tag{9.78}$$

The Laplace transform of the impulse response is

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}. \tag{9.79}$$

For $\zeta > 1$, c_1 and c_2 are real and thus both poles lie on the real axis, as indicated in Figure 9.19(a). The case of $\zeta > 1$ is essentially a product of two first-order terms, as in Section 9.4.1. Consequently, in this case $|H(j\omega)|$ decreases monotonically as $|\omega|$ increases, while $\sphericalangle H(j\omega)$ varies from 0 at $\omega = 0$ to $-\pi$ as $\omega \to \infty$. This can be verified from Figure 9.19(a) by observing that the length of the vector from each of the two poles to the point $s = j\omega$ increases monotonically as ω increases from 0, and the angle of each of these vectors increases from 0 to $\pi/2$ as ω increases from 0 to ∞. Note also that as ζ increases, one pole moves closer to the $j\omega$-axis, indicative of a term in the impulse response that decays more slowly, and the other pole moves farther into the left-half plane, indicative of a term in the impulse response that decays more rapidly. Thus, for large values of ζ, it is the pole close to the $j\omega$-axis that dominates the system response for large time. Similarly, from a consideration of the pole vectors for $\zeta \gg 1$, as indicated in Figure 9.19(b), for low frequencies the length and angle of the vector for the pole close to the $j\omega$-axis are much more sensitive to changes in ω than the length and angle of the vector for the pole far from the $j\omega$-axis. Hence, we see that for low frequencies, the characteristics of the frequency response are influenced principally by the pole close to the $j\omega$-axis.

For $0 < \zeta < 1$, c_1 and c_2 are complex, so that the pole-zero plot is that shown in Figure 9.19(c). Correspondingly, the impulse response and step response have oscillatory parts. We note that the two poles occur in complex conjugate locations. In fact, as we discuss in Section 9.5.5, the complex poles (and zeros) for a real-valued signal always occur in complex conjugate pairs. From the figure—particularly when ζ is small, so that the poles are close to the $j\omega$-axis—as ω approaches $\omega_n\sqrt{1 - \zeta^2}$, the behavior of the frequency response is dominated by the pole vector in the second quadrant, and in

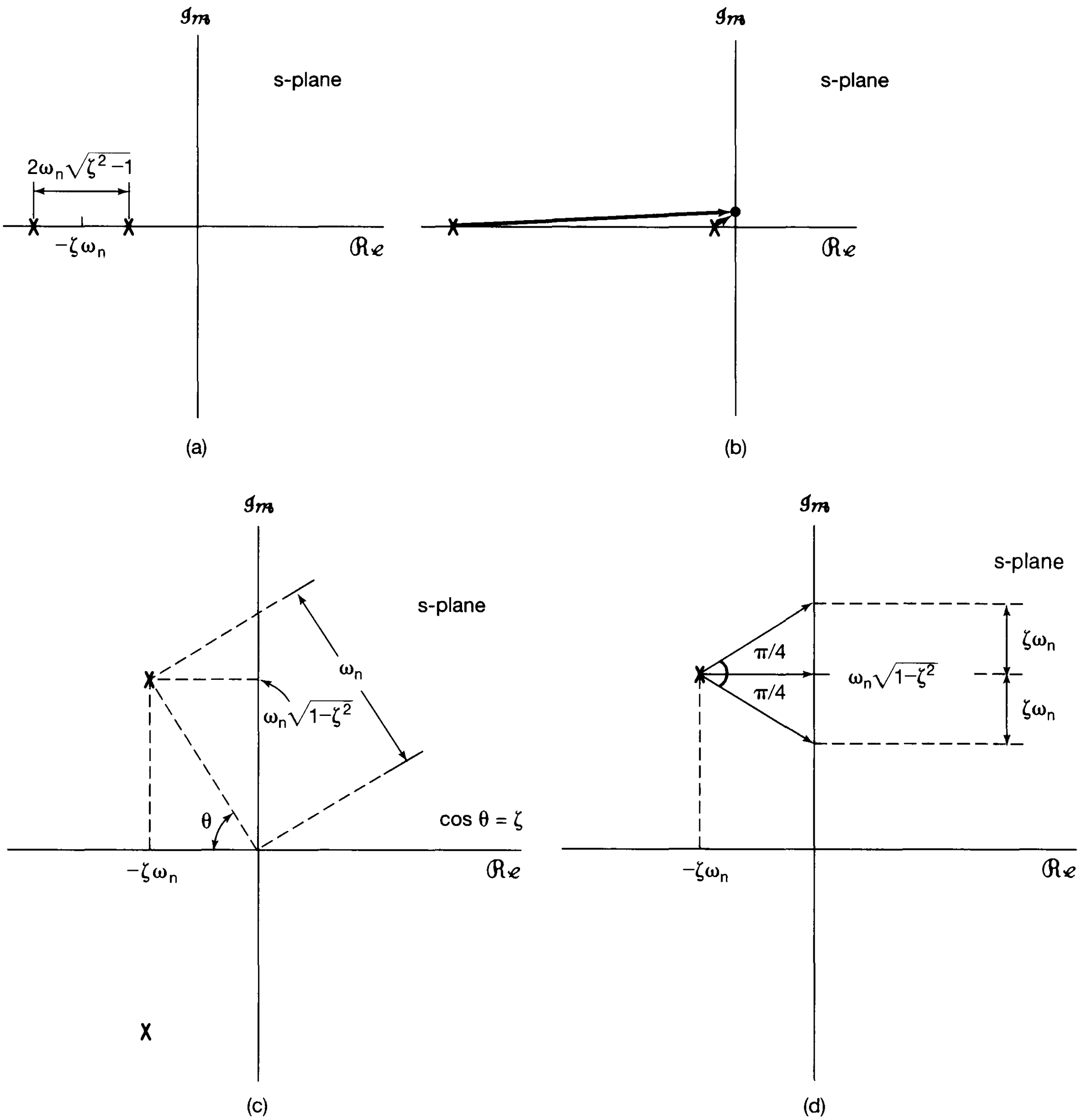


Figure 9.19 (a) Pole-zero plot for a second-order system with $\zeta > 1$; (b) pole vectors for $\zeta \gg 1$; (c) pole-zero plot for a second-order system with $0 < \zeta < 1$; (d) pole vectors for $0 < \zeta < 1$ and for $\omega = \omega_n\sqrt{1-\zeta^2}$ and $\omega = \omega_n\sqrt{1-\zeta^2} \pm \zeta\omega_n$.

particular, the length of that pole vector has a minimum at $\omega = \omega_n\sqrt{1-\zeta^2}$. Thus, qualitatively, we would expect the magnitude of the frequency response to exhibit a peak in the vicinity of that frequency. Because of the presence of the other pole, the peak will occur not exactly at $\omega = \omega_n\sqrt{1-\zeta^2}$, but at a frequency slightly less than this. A careful sketch of the magnitude of the frequency response is shown in Figure 9.20(a) for $\omega_n = 1$ and several values of ζ where the expected behavior in the vicinity of the poles is clearly evident. This is consistent with our analysis of second-order systems in Section 6.5.2.

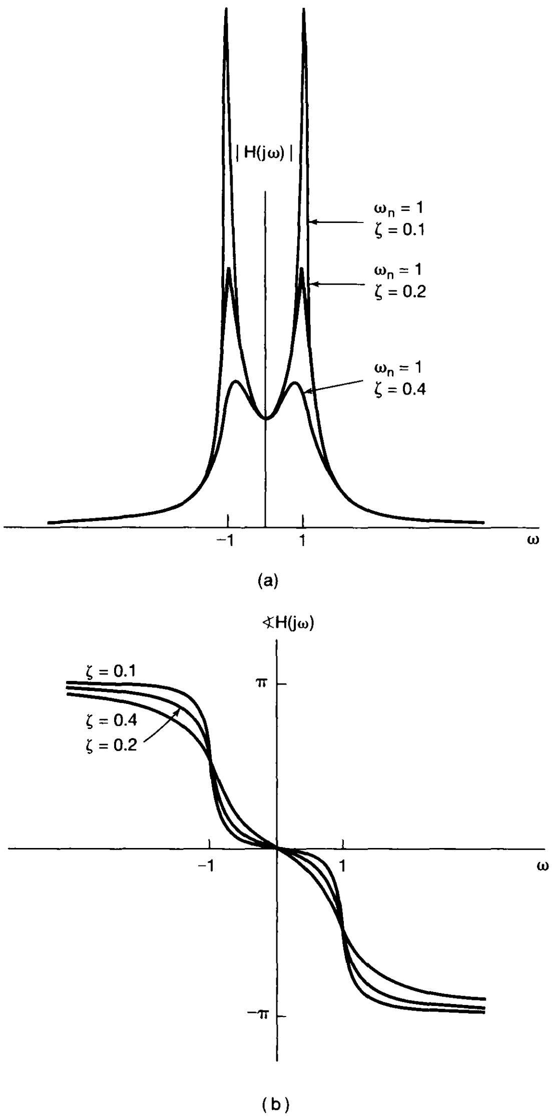


Figure 9.20 (a) Magnitude and (b) phase of the frequency response for a second-order system with $0 < \zeta < 1$.

Thus, for $0 < \zeta < 1$, the second-order system is a nonideal bandpass filter, with the parameter ζ controlling the sharpness and width of the peak in the frequency response. In particular, from the geometry in Figure 9.19(d), we see that the length of the pole vector from the second-quadrant pole increases by a factor of $\sqrt{2}$ from its minimum at $\omega = \omega_n\sqrt{1-\zeta^2}$ when ω increases or decreases from this value by $\zeta\omega_n$. Consequently, for small ζ, and neglecting the effect of the distant third-quadrant pole, $|H(j\omega)|$ is within a factor of $\sqrt{2}$ of its peak value over the frequency range

$$\omega_n\sqrt{1-\zeta^2} - \zeta\omega_n < \omega < \omega_n\sqrt{1-\zeta^2} + \zeta\omega_n.$$

If we define the relative bandwidth B as the length of this frequency interval divided by the undamped natural frequency ω_n, we see that

$$B = 2\zeta.$$

Thus, the closer ζ is to zero, the sharper and narrower the peak in the frequency response is. Note also that B is the reciprocal of the quality measure Q for second-order systems defined in Section 6.5.2. Thus, as the quality increases, the relative bandwidth decreases and the filter becomes increasingly frequency selective.

An analogous picture can be developed for $\sphericalangle H(\omega)$, which is plotted in Figure 9.20(b) for $\omega_n = 1$ and several values of ζ. As can be seen from Figure 9.19(d), the angle of the second-quadrant pole vector changes from $-\pi/4$ to 0 to $\pi/4$ as ω changes from $\omega_n\sqrt{1-\zeta^2} - \zeta\omega_n$ to $\omega_n\sqrt{1-\zeta^2}$ to $\omega_n\sqrt{1-\zeta^2} + \zeta\omega_n$. For small values of ζ, the angle for the third-quadrant pole changes very little over this frequency interval, resulting in a rapid change in $\sphericalangle H(j\omega)$ of approximately $\pi/2$ over the interval, as captured in the figure.

Varying ω_n with ζ fixed only changes the frequency scale in the preceding discussion—i.e., $|H(\omega)|$ and $\sphericalangle H(\omega)$ depend only on ω/ω_n. From Figure 9.19(c), we also can readily determine how the poles and system characteristics change as we vary ζ, keeping ω_n constant. Since $\cos\theta = \zeta$, the poles move along a semicircle with fixed radius ω_n. For $\zeta = 0$, the two poles are on the imaginary axis. Correspondingly, in the time domain, the impulse response is sinusoidal with no damping. As ζ increases from 0 to 1, the two poles remain complex and move into the left-half plane, and the vectors from the origin to the poles maintain a constant overall magnitude ω_n. As the real part of the poles becomes more negative, the associated time response will decay more quickly as $t \to \infty$. Also, as we have seen, as ζ increases from 0 toward 1, the relative bandwidth of the frequency response increases, and the frequency response becomes less sharp and less frequency selective.

9.4.3 All-Pass Systems

As a final illustration of the geometric evaluation of the frequency response, let us consider a system for which the Laplace transform of the impulse response has the pole-zero plot shown in Figure 9.21(a). From this figure, it is evident that for any point along the $j\omega$-axis, the pole and zero vectors have equal length, and consequently, the magnitude of the frequency response is constant and independent of frequency. Such a system is com-

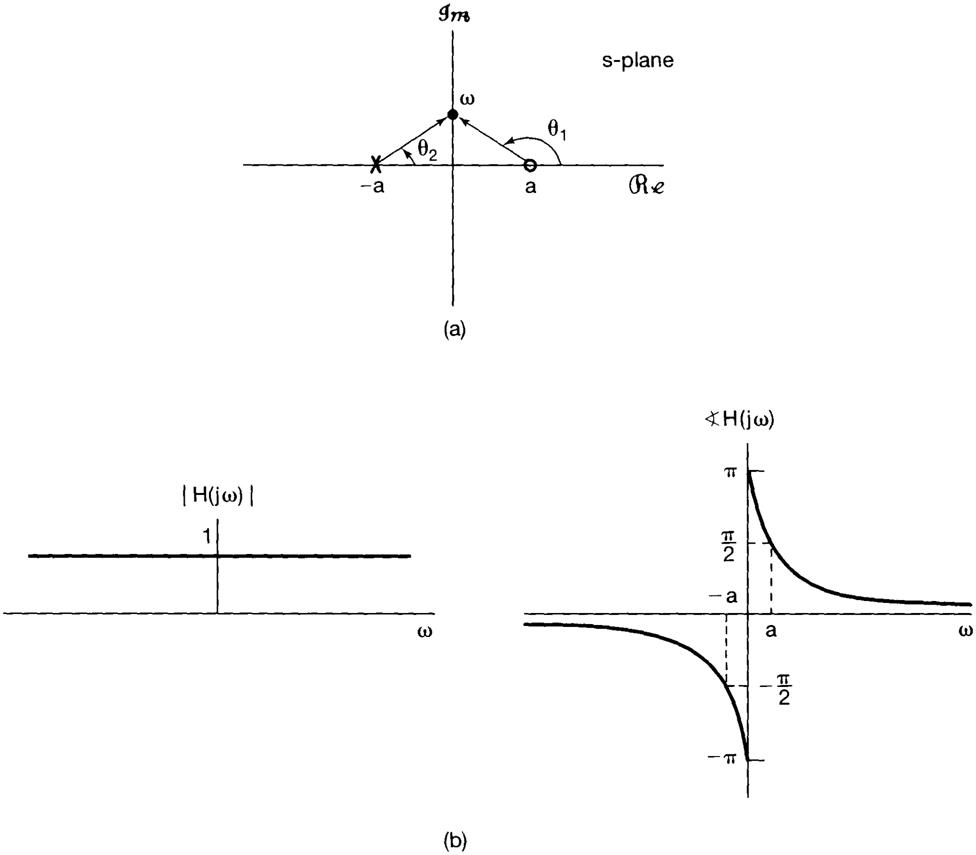


Figure 9.21 (a) Pole-zero plot for an all-pass system; (b) magnitude and phase of an all-pass frequency response.

monly referred to as an *all-pass system,* since it passes all frequencies with equal gain (or attenuation). The phase of the frequency response is $\theta_1 - \theta_2$, or, since $\theta_1 = \pi - \theta_2$,

$$\sphericalangle H(j\omega) = \pi - 2\theta_2. \tag{9.80}$$

From Figure 9.21(a), $\theta_2 = \tan^{-1}(\omega/a)$, and thus,

$$\sphericalangle H(j\omega) = \pi - 2\tan^{-1}\left(\frac{\omega}{a}\right). \tag{9.81}$$

The magnitude and phase of $H(j\omega)$ are illustrated in Figure 9.21(b).

9.5 PROPERTIES OF THE LAPLACE TRANSFORM

In exploiting the Fourier transform, we relied heavily on the set of properties developed in Section 4.3. In the current section, we consider the corresponding set of properties for the Laplace transform. The derivations of many of these results are analogous to those of the corresponding properties for the Fourier transform. Consequently, we will not present the derivations in detail, some of which are left as exercises at the end of the chapter. (See Problems 9.52–9.54.)

9.5.1 Linearity of the Laplace Transform

If

$$x_1(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s) \quad \text{with a region of convergence that will be denoted as } R_1$$

and

$$x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_2(s) \quad \text{with a region of convergence that will be denoted as } R_2,$$

then

$$\boxed{ax_1(t) + bx_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} aX_1(s) + bX_2(s), \text{ with ROC containing } R_1 \cap R_2.} \tag{9.82}$$

As indicated, the region of convergence of $X(s)$ is at least the intersection of R_1 and R_2, which could be empty, in which case $X(s)$ has no region of convergence—i.e., $x(t)$ has no Laplace transform. For example, for $x(t)$ as in eq. (9.47) of Example 9.7, with $b > 0$ the ROC for $X(s)$ is the intersection of the ROCs for the two terms in the sum. If $b < 0$, there are no common points in R_1 and R_2; that is, the intersection is empty, and thus, $x(t)$ has no Laplace transform. The ROC can also be larger than the intersection. As a simple example, for $x_1(t) = x_2(t)$ and $a = -b$ in eq. (9.82), $x(t) = 0$, and thus, $X(s) = 0$. The ROC of $X(s)$ is then the entire s-plane.

The ROC associated with a linear combination of terms can always be constructed by using the properties of the ROC developed in Section 9.2. Specifically, from the intersection of the ROCs for the individual terms (assuming that it is not empty), we can find a line or strip that is in the ROC of the linear combination. We then extend this to the right ($\mathcal{R}e\{s\}$ increasing) and to the left ($\mathcal{R}e\{s\}$ decreasing) to the nearest poles (which may be at infinity).

Example 9.13

In this example, we illustrate the fact that the ROC for the Laplace transform of a linear combination of signals can sometimes extend beyond the intersection of the ROCs for the individual terms. Consider

$$x(t) = x_1(t) - x_2(t), \tag{9.83}$$

where the Laplace transforms of $x_1(t)$ and $x_2(t)$ are, respectively,

$$X_1(s) = \frac{1}{s+1}, \qquad \mathcal{R}e\{s\} > -1, \tag{9.84}$$

and

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathcal{R}e\{s\} > -1. \tag{9.85}$$

The pole-zero plot, including the ROCs for $X_1(s)$ and $X_2(s)$, is shown in Figures 9.22(a) and (b). From eq. (9.82),

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}. \tag{9.86}$$

Thus, in the linear combination of $x_1(t)$ and $x_2(t)$, the pole at $s = -1$ is canceled by a zero at $s = -1$. The pole-zero plot for $X(s) = X_1(s) - X_2(s)$ is shown in Figure 9.22(c). The intersection of the ROCs for $X_1(s)$ and $X_2(s)$ is $\mathcal{R}e\{s\} > -1$. However, since the ROC is always bounded by a pole or infinity, for this example the ROC for $X(s)$ can be extended to the left to be bounded by the pole at $s = -2$, as a result of the pole-zero cancellation at $s = -1$.

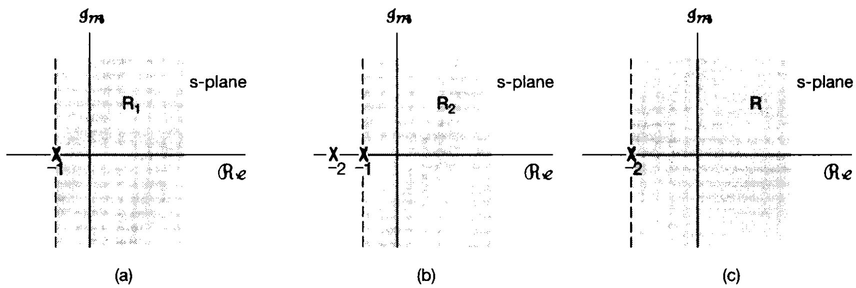


Figure 9.22 Pole-zero plots and ROCs for Example 9.13: (a) $X_1(s)$; (b) $X_2(s)$; (c) $X_1(s) - X_2(s)$. The ROC for $X_1(s) - X_2(s)$ includes the intersection of R_1 and R_2, which can then be extended to be bounded by the pole at $s = -2$.

9.5.2 Time Shifting

If

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \qquad \text{with ROC} = R,$$

then

$$\boxed{x(t-t_0) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-st_0}X(s), \qquad \text{with ROC} = R.} \tag{9.87}$$

9.5.3 Shifting in the s-Domain

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

then

$$e^{s_0 t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0), \quad \text{with ROC} = R + \mathcal{R}e\{s_0\}. \tag{9.88}$$

That is, the ROC associated with $X(s - s_0)$ is that of $X(s)$, shifted by $\mathcal{R}e\{s_0\}$. Thus, for any value s that is in R, the value $s + \mathcal{R}e\{s_0\}$ will be in R_1. This is illustrated in Figure 9.23. Note that if $X(s)$ has a pole or zero at $s = a$, then $X(s-s_0)$ has a pole or zero at $s-s_0 = a$—i.e., $s = a + s_0$.

An important special case of eq. (9.88) is when $s_0 = j\omega_0$—i.e., when a signal $x(t)$ is used to modulate a periodic complex exponential $e^{j\omega_0 t}$. In this case, eq. (9.88) becomes

$$e^{j\omega_0 t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - j\omega_0), \quad \text{with ROC} = R. \tag{9.89}$$

The right-hand side of eq. (9.89) can be interpreted as a shift in the s-plane parallel to the $j\omega$-axis. That is, if the Laplace transform of $x(t)$ has a pole or zero at $s = a$, then the Laplace transform of $e^{j\omega_0 t}x(t)$ has a pole or zero at $s = a + j\omega_0$.

9.5.4 Time Scaling

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R,$$

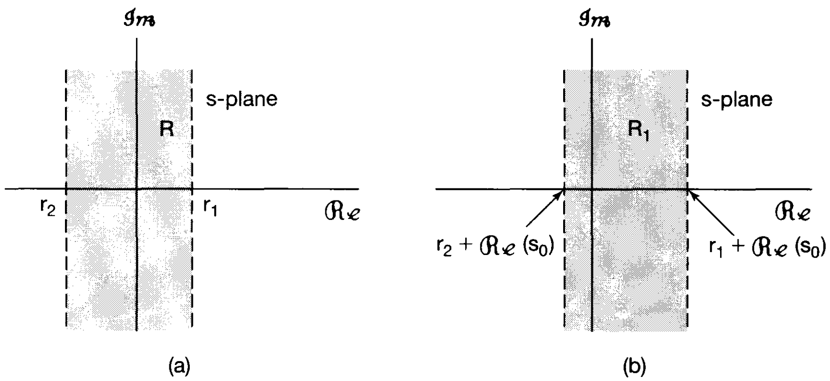


Figure 9.23 Effect on the ROC of shifting in the s-domain: (a) the ROC of $X(s)$; (b) the ROC of $X(s - s_0)$.

then

$$\boxed{x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right), \quad \text{with ROC } R_1 = aR.} \tag{9.90}$$

That is, for any value s in R [which is illustrated in Figure 9.24(a)], the value a/s will be in R_1, as illustrated in Figure 9.24(b) for a positive value of $a < 1$. Note that, for $0 < a < 1$, there is a compression in the size of the ROC of $X(s)$ by a factor of a, as depicted in Figure 9.24(b), while for a > 1, the ROC is expanded by a factor of a. Also, eq. (9.90) implies that if a is negative, the ROC undergoes a reversal plus a scaling. In particular, as depicted in Figure 9.24(c), the ROC of $1/|a|X(s/a)$ for $0 > a > -1$ involves

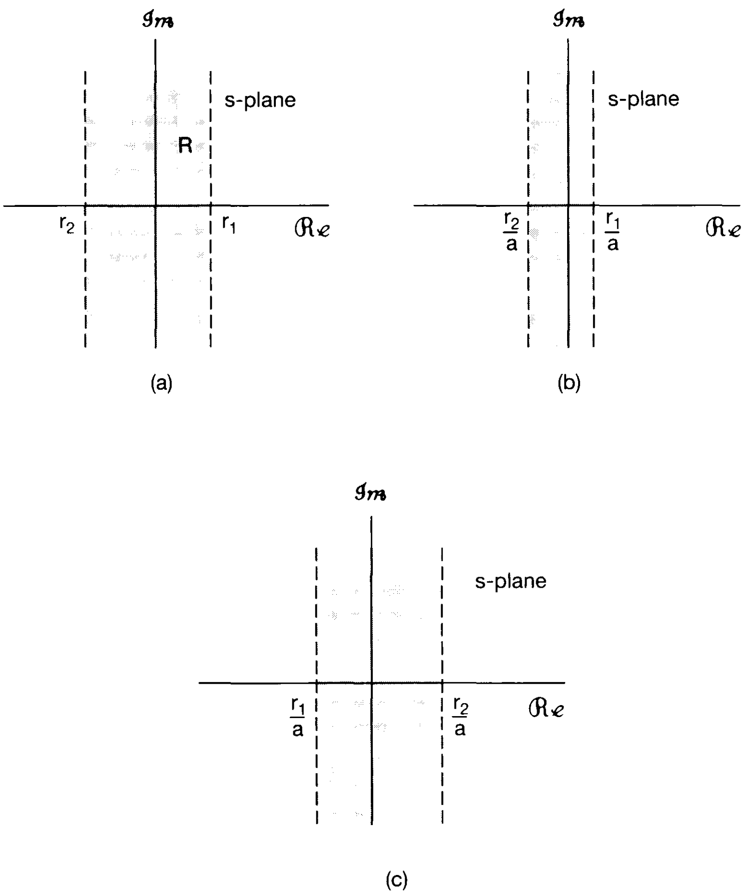


Figure 9.24 Effect on the ROC of time scaling: (a) ROC of $X(s)$; (b) ROC of $(1/|a|)X(s/a)$ for $0 < a < 1$; (c) ROC of $(1/|a|)X(s/a)$ for $0 > a > -1$.

a reversal about the $j\omega$-axis, together with a change in the size of the ROC by a factor of $|a|$. Thus, time reversal of $x(t)$ results in a reversal of the ROC. That is,

$$x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s), \quad \text{with ROC} = -R. \tag{9.91}$$

9.5.5 Conjugation

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{with ROC} = R, \tag{9.92}$$

then

$$\boxed{x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*), \quad \text{with ROC} = R.} \tag{9.93}$$

Therefore,

$$X(s) = X^*(s^*) \quad \text{when } x(t) \text{ is real.} \tag{9.94}$$

Consequently, if $x(t)$ is real and if $X(s)$ has a pole or zero at $s = s_0$ (i.e., if $X(s)$ is unbounded or zero at $s = s_0$), then $X(s)$ also has a pole or zero at the complex conjugate point $s = s_0^*$. For example, the transform $X(s)$ for the real signal $x(t)$ in Example 9.4 has poles at $s = 1 \pm 3j$ and zeros at $s = (-5 \pm j\sqrt{71})/2$.

9.5.6 Convolution Property

If

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s), \quad \text{with } ROC = R_1,$$

and

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s), \quad \text{with } ROC = R_2,$$

then

$$\boxed{x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s), \quad \text{with ROC containing } R_1 \cap R_2.} \tag{9.95}$$

In a manner similar to the linearity property set forth in Section 9.5.1, the ROC of $X_1(s)X_2(s)$ includes the intersection of the ROCs of $X_1(s)$ and $X_2(s)$ and may be larger if pole-zero cancellation occurs in the product. For example, if

$$X_1(s) = \frac{s+1}{s+2}, \qquad \mathcal{Re}\{s\} > -2, \tag{9.96}$$

and

$$X_2(s) = \frac{s+2}{s+1}, \qquad \Re e\{s\} > -1, \tag{9.97}$$

then $X_1(s)X_2(s) = 1$, and its ROC is the entire s-plane.

As we saw in Chapter 4, the convolution property in the context of the Fourier transform plays an important role in the analysis of linear time-invariant systems. In Sections 9.7 and 9.8 we will exploit in some detail the convolution property for Laplace transforms for the analysis of LTI systems in general and, more specifically, for the class of systems represented by linear constant-coefficient differential equations.

9.5.7 Differentiation in the Time Domain

If

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \qquad \text{with ROC} = R,$$

then

$$\boxed{\frac{dx(t)}{dt} \stackrel{\mathcal{L}}{\longleftrightarrow} sX(s), \qquad \text{with ROC containing } R.} \tag{9.98}$$

This property follows by differentiating both sides of the inverse Laplace transform as expressed in equation (9.56). Specifically, let

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds.$$

Then

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds. \tag{9.99}$$

Consequently, $dx(t)/dt$ is the inverse Laplace transform of $sX(s)$. The ROC of $sX(s)$ includes the ROC of $X(s)$ and may be larger if $X(s)$ has a first-order pole at $s = 0$ that is canceled by the multiplication by s. For example, if $x(t) = u(t)$, then $X(s) = 1/s$, with an ROC that is $\Re e\{s\} > 0$. The derivative of $x(t)$ is an impulse with an associated Laplace transform that is unity and an ROC that is the entire s-plane.

9.5.8 Differentiation in the s-Domain

Differentiating both sides of the Laplace transform equation (9.3), i.e.,

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt,$$

we obtain

$$\frac{dX(s)}{ds} = \int_{-\infty}^{+\infty} (-t)x(t)e^{-st}dt.$$

Consequently, if

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \qquad \text{with ROC} = R,$$

then

$$-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds}, \qquad \text{with ROC} = R. \tag{9.100}$$

The next two examples illustrate the use of this property.

Example 9.14

Let us find the Laplace transform of

$$x(t) = te^{-at}u(t). \tag{9.101}$$

Since

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \qquad \mathcal{R}e\{s\} > -a,$$

it follows from eq. (9.100) that

$$te^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \qquad \mathcal{R}e\{s\} > -a. \tag{9.102}$$

In fact, by repeated application of eq. (9.100), we obtain

$$\frac{t^2}{2}e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^3}, \qquad \mathcal{R}e\{s\} > -a, \tag{9.103}$$

and, more generally,

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n}, \qquad \mathcal{R}e\{s\} > -a. \tag{9.104}$$

As the next example illustrates, this specific Laplace transform pair is particularly useful when applying partial-fraction expansion to the determination of the inverse Laplace transform of a rational function with multiple-order poles.

Example 9.15

Consider the Laplace transform

$$X(s) = \frac{2s^2+5s+5}{(s+1)^2(s+2)}, \qquad \mathcal{R}e\{s\} > -1.$$

Applying the partial-fraction expansion method described in the appendix, we can write

$$X(s) = \frac{2}{(s+1)^2} - \frac{1}{(s+1)} + \frac{3}{s+2}, \qquad \mathcal{R}e\{s\} > -1. \tag{9.105}$$

Since the ROC is to the right of the poles at $s = -1$ and -2, the inverse transform of each of the terms is a right-sided signal, and, applying eqs. (9.14) and (9.104), we obtain the inverse transform

$$x(t) = [2te^{-t} - e^{-t} + 3e^{-2t}]u(t).$$

9.5.9 Integration in the Time Domain

If

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \qquad \text{with ROC} = R,$$

then

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s), \ \text{with ROC containing } R \cap \{\mathcal{R}e\{s\} > 0\}. \tag{9.106}$$

This property is the inverse of the differentiation property set forth in Section 9.5.7. It can be derived using the convolution property presented in Section 9.5.6. Specifically,

$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t). \tag{9.107}$$

From Example 9.1, with $a = 0$,

$$u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}, \qquad \mathcal{R}e\{s\} > 0, \tag{9.108}$$

and thus, from the convolution property,

$$u(t) * x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s), \tag{9.109}$$

with an ROC that contains the intersection of the ROC of $X(s)$ and the ROC of the Laplace transform of $u(t)$ in eq. (9.108), which results in the ROC given in eq. (9.106).

9.5.10 The Initial- and Final-Value Theorems

Under the specific constraints that $x(t) = 0$ for $t < 0$ and that $x(t)$ contains no impulses or higher order singularities at the origin, one can directly calculate, from the Laplace transform, the initial value $x(0^+)$—i.e., $x(t)$ as t approaches zero from positive values of t. Specifically the *initial-value theorem* states that

$$x(0^+) = \lim_{s\to\infty} sX(s), \tag{9.110}$$

Also, if $x(t) = 0$ for $t < 0$ and, in addition, $x(t)$ has a finite limit as $t \to \infty$, then the *final-value theorem* says that

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s). \tag{9.111}$$

The derivation of these results is considered in Problem 9.53.

Example 9.16

The initial- and final-value theorems can be useful in checking the correctness of the Laplace transform calculations for a signal. For example, consider the signal $x(t)$ in Example 9.4. From eq. (9.24), we see that $x(0^+) = 2$. Also, using eq. (9.29), we find that

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{2s^3 + 5s^2 + 12s}{s^3 + 4s^2 + 14s + 20} = 2,$$

which is consistent with the initial-value theorem in eq. (9.110).

9.5.11 Table of Properties

In Table 9.1, we summarize the properties developed in this section. In Section 9.7, many of these properties are used in applying the Laplace transform to the analysis and characterization of linear time-invariant systems. As we have illustrated in several examples, the various properties of Laplace transforms and their ROCs can provide us with

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to\infty} sX(s)$$

considerable information about a signal and its transform that can be useful either in characterizing the signal or in checking a calculation. In Sections 9.7 and 9.8 and in some of the problems at the end of this chapter, we give several other examples of the uses of these properties.

9.6 SOME LAPLACE TRANSFORM PAIRS

As we indicated in Section 9.3, the inverse Laplace transform can often be easily evaluated by decomposing $X(s)$ into a linear combination of simpler terms, the inverse transform of each of which can be recognized. Listed in Table 9.2 are a number of useful Laplace

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

transform pairs. Transform pair 1 follows directly from eq. (9.3). Transform pairs 2 and 6 follow directly from Example 9.1 with $a = 0$ and $a = \alpha$, respectively. Transform pair 4 was developed in Example 9.14 using the differentiation property. Transform pair 8 follows from transform pair 4 using the property set forth in Section 9.5.3. Transform pairs 3, 5, 7, and 9 are based on transform pairs 2, 4, 6 and 8, respectively, together with the time-scaling property of section 9.5.4 with $a = -1$. Similarly, transform pairs 10 through 16 can all be obtained from earlier ones in the table using appropriate properties in Table 9.1 (see Problem 9.55).

9.7 ANALYSIS AND CHARACTERIZATION OF LTI SYSTEMS USING THE LAPLACE TRANSFORM

One of the important applications of the Laplace transform is in the analysis and characterization of LTI systems. Its role for this class of systems stems directly from the convolution property (Section 9.5.6). Specifically, the Laplace transforms of the input and output of an LTI system are related through multiplication by the Laplace transform of the impulse response of the system. Thus,

$$Y(s) = H(s)X(s). \tag{9.112}$$

where $X(s)$, $Y(s)$, and $H(s)$ are the Laplace transforms of the input, output, and impulse response of the system, respectively. Equation (9.112) is the counterpart, in the context of Laplace transforms, of eq. (4.56) for Fourier transform. Also, from our discussion in Section 3.2 on the response of LTI systems to complex exponentials, if the input to an LTI system is $x(t) = e^{st}$, with s in the ROC of $H(s)$, then the output will be $H(s)e^{st}$; i.e., e^{st} is an eigenfunction of the system with eigenvalue equal to the Laplace transform of the impulse response.

If the ROC of $H(s)$ includes the imaginary axis, then for $s = j\omega$, $H(s)$ is the frequency response of the LTI system. In the broader context of the Laplace transform, $H(s)$ is commonly referred to as the system function or, alternatively, the *transfer function*. Many properties of LTI systems can be closely associated with the characteristics of the *system function* in the s-plane. We illustrate this next by examining several important properties and classes of systems.

9.7.1 Causality

For a causal LTI system, the impulse response is zero for $t < 0$ and thus is right sided. Consequently, from the discussion in Section 9.2, we see that

The ROC associated with the system function for a causal system is a right-half plane.

It should be stressed, however, that the converse of this statement is not necessarily true. That is, as illustrated in Example 9.19 to follow, an ROC to the right of the rightmost pole does not guarantee that a system is causal; rather, it guarantees only that the impulse response is right sided. However, if $H(s)$ is *rational*, then, as illustrated in Examples 9.17 and 9.18 to follow, we can determine whether the system is causal simply by checking to see if its ROC is a right-half plane. Specifically,

> For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

Example 9.17

Consider a system with impulse response

$$h(t) = e^{-t}u(t). \tag{9.113}$$

Since $h(t) = 0$ for $t < 0$, this system is causal. Also, the system function can be obtained from Example 9.1:

$$H(s) = \frac{1}{s+1}, \qquad \mathcal{Re}\{s\} > -1. \tag{9.114}$$

In this case, the system function is rational and the ROC in eq. (9.114) is to the right of the rightmost pole, consistent with our statement that causality for systems with rational system functions is equivalent to the ROC being to the right of the rightmost pole.

Example 9.18

Consider a system with impulse response

$$h(t) = e^{-|t|}.$$

Since $h(t) \neq 0$ for $t < 0$, this system is not causal. Also, from Example 9.7, the system function is

$$H(s) = \frac{-2}{s^2 - 1}, \qquad -1 < \mathcal{Re}\{s\} < +1.$$

Thus, $H(s)$ is rational and has an ROC that is *not* to the right of the the rightmost pole, consistent with the fact that the system is not causal.

Example 9.19

Consider the system function

$$H(s) = \frac{e^s}{s+1}, \qquad \mathcal{Re}\{s\} > -1. \tag{9.115}$$

For this system, the ROC is to the right of the rightmost pole. Therefore, the impulse response must be right sided. To determine the impulse response, we first use the result of Example 9.1:

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \qquad \mathcal{R}e\{s\} > -1. \tag{9.116}$$

Next, from the time-shifting property of Section 9.5.2 [eq. (9.87)], the factor e^s in eq. (9.115) can be accounted for by a shift in the time function in eq. (9.116). Then

$$e^{-(t+1)}u(t+1) \overset{\mathcal{L}}{\longleftrightarrow} \frac{e^s}{s+1}, \qquad \mathcal{R}e\{s\} > -1, \tag{9.117}$$

so that the impulse response associated with the system is

$$h(t) = e^{-(t+1)}u(t+1), \tag{9.118}$$

which is nonzero for $-1 < t < 0$. Hence, the system is not causal. This example serves as a reminder that causality implies that the ROC is to the right of the rightmost pole, but the converse is not in general true, unless the system function is rational.

In an exactly analogous manner, we can deal with the concept of anticausality. A system is *anticausal* if its impulse response $h(t) = 0$ for $t > 0$. Since in that case $h(t)$ would be left sided, we know from Section 9.2 that the ROC of the system function $H(s)$ would have to be a left-half plane. Again, in general, the converse is not true. That is, if the ROC of $H(s)$ is a left-half plane, all we know is that $h(t)$ is left sided. However, if $H(s)$ is rational, then having an ROC to the left of the leftmost pole is equivalent to the system being anticausal.

9.7.2 Stability

The ROC of $H(s)$ can also be related to the stability of a system. As mentioned in Section 2.3.7, the stability of an LTI system is equivalent to its impulse response being absolutely integrable, in which case (Section 4.4) the Fourier transform of the impulse response converges. Since the Fourier transform of a signal equals the Laplace transform evaluated along the $j\omega$-axis, we have the following:

> An LTI system is stable if and only if the ROC of its system function H(s) includes the entire $j\omega$-axis [i.e., $\mathcal{R}e\{s\} = 0$].

Example 9.20

Let us consider an LTI system with system function

$$H(s) = \frac{s-1}{(s+1)(s-2)}. \tag{9.119}$$

Since the ROC has not been specified, we know from our discussion in Section 9.2 that there are several different ROCs and, consequently, several different system impulse responses that can be associated with the algebraic expression for $H(s)$ given in eq. (9.119).

If, however, we have information about the causality or stability of the system, the appropriate ROC can be identified. For example, if the system is known to be *causal*, the ROC will be that indicated in Figure 9.25(a), with impulse response

$$h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t). \tag{9.120}$$

Note that this particular choice of ROC does not include the $j\omega$-axis, and consequently, the corresponding system is unstable (as can be checked by observing that $h(t)$ is not absolutely integrable). On the other hand, if the system is known to be *stable*, the ROC is that given in Figure 9.25(b), and the corresponding impulse response is

$$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t),$$

which is absolutely integrable. Finally, for the ROC in Figure 9.25(c), the system is anticausal and unstable, with

$$h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t).$$

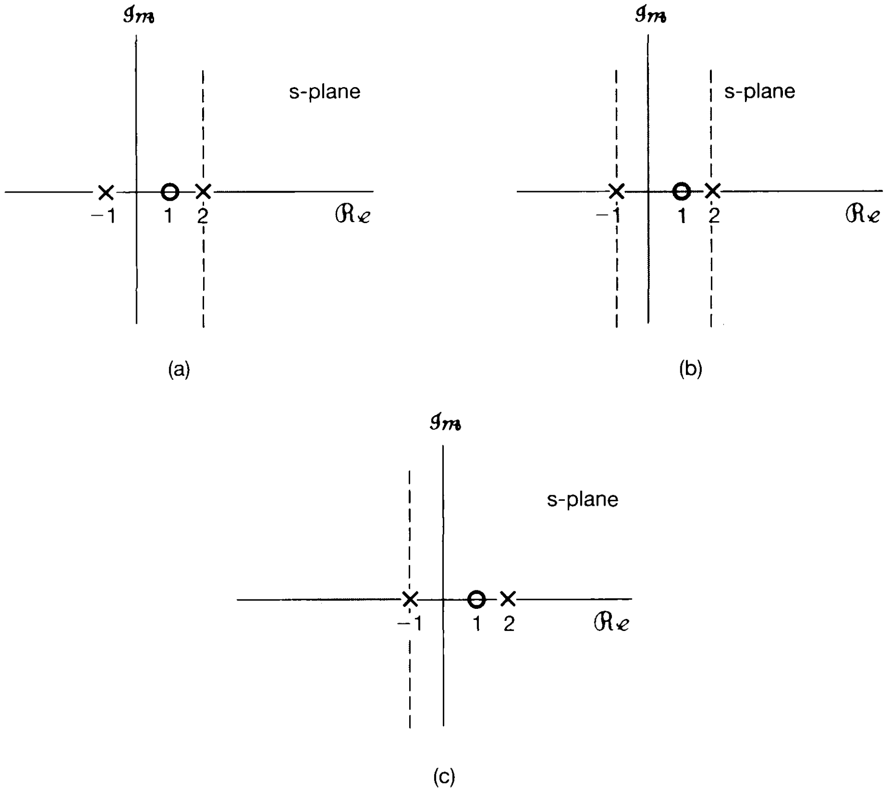


Figure 9.25 Possible ROCs for the system function of Example 9.20 with poles at $s = -1$ and $s = 2$ and a zero at $s = 1$: (a) causal, unstable system; (b) noncausal, stable system; (c) anticausal, unstable system.

It is perfectly possible, of course, for a system to be stable (or unstable) and have a system function that is not rational. For example, the system function in eq. (9.115) is not rational, and its impulse response in eq. (9.118) is absolutely integrable, indicating that the system is stable. However, for systems with rational system functions, stability is easily interpreted in terms of the poles of the system. For example, for the pole-zero plot in Figure 9.25, stability corresponds to the choice of an ROC that is between the two poles, so that the $j\omega$-axis is contained in the ROC.

For one particular and very important class of systems, stability can be characterized very simply in terms of the locations of the poles. Specifically, consider a causal LTI system with a rational system function $H(s)$. Since the system is causal, the ROC is to the right of the rightmost pole. Consequently, for this system to be stable (i.e., for the ROC to include the $j\omega$-axis), the rightmost pole of $H(s)$ must be to the *left* of the $j\omega$-axis. That is,

> A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane—i.e., all of the poles have negative real parts.

Example 9.21

Consider again the causal system in Example 9.17. The impulse response in eq. (9.113) is absolutely integrable, and thus the system is stable. Consistent with this, we see that the pole of $H(s)$ in eq. (9.114) is at $s = -1$, which is in the left-half of the s-plane. In contrast, the causal system with impulse response

$$h(t) = e^{2t}u(t)$$

is unstable, since $h(t)$ is not absolutely integrable. Also, in this case

$$H(s) = \frac{1}{s-2}, \qquad \Re e\{s\} > 2,$$

so the system has a pole at $s = 2$ in the right half of the s-plane.

Example 9.22

Let us consider the class of causal second-order systems previously discussed in Sections 9.4.2 and 6.5.2. The impulse response and system function are, respectively,

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t) \tag{9.121}$$

and

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s-c_1)(s-c_2)}, \tag{9.122}$$

where

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \tag{9.123}$$

$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}, \tag{9.124}$$

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}. \tag{9.125}$$

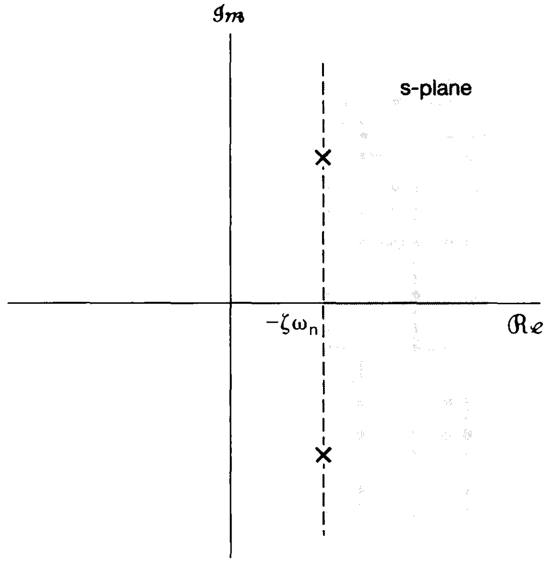


Figure 9.26 Pole locations and ROC for a causal second-order system with $\zeta < 0$.

In Figure 9.19, we illustrated the pole locations for $\zeta > 0$. In Figure 9.26, we illustrate the pole locations for $\zeta < 0$. As is evident from the latter figure and from eqs. (9.124) and (9.125), for $\zeta < 0$ both poles have positive real parts. Consequently, for $\zeta < 0$, the causal second-order system cannot be stable. This is also evident in eq. (9.121), since, with $\mathcal{Re}\{c_1\} > 0$ and $\mathcal{Re}\{c_2\} > 0$, each term grows exponentially as t increases, and thus $h(t)$ cannot be absolutely integrable.

9.7.3 LTI Systems Characterized by Linear Constant-Coefficient Differential Equations

In Section 4.7, we discussed the use of the Fourier transform to obtain the frequency response of an LTI system characterized by a linear constant-coefficient differential equation without first obtaining the impulse response or time-domain solution. In an exactly analogous manner, the properties of the Laplace transform can be exploited to directly obtain the system function for an LTI system characterized by a linear constant-coefficient differential equation. We illustrate this procedure in the next example.

Example 9.23

Consider an LTI system for which the input $x(t)$ and output $y(t)$ satisfy the linear constant-coefficient differential equation

$$\frac{dy(t)}{dt} + 3y(t) = x(t). \tag{9.126}$$

Applying the Laplace transform to both sides of eq. (9.126), and using the linearity and differentiation properties set forth in Sections 9.5.1 and 9.5.7, respectively [(eqs. (9.82) and (9.98)], we obtain the algebraic equation

$$sY(s) + 3Y(s) = X(s). \tag{9.127}$$

Since, from eq. (9.112), the system function is

$$H(s) = \frac{Y(s)}{X(s)},$$

we obtain, for this system,

$$H(s) = \frac{1}{s+3}. \tag{9.128}$$

This, then, provides the algebraic expression for the system function, but not the region of convergence. In fact, as we discussed in Section 2.4, the differential equation itself is not a complete specification of the LTI system, and there are, in general, different impulse responses, all consistent with the differential equation. If, in addition to the differential equation, we know that the system is causal, then the ROC can be inferred to be to the right of the rightmost pole, which in this case corresponds to $\mathcal{Re}\{s\} > -3$. If the system were known to be anticausal, then the ROC associated with $H(s)$ would be $\mathcal{Re}\{s\} < -3$. The corresponding impulse response in the causal case is

$$h(t) = e^{-3t}u(t), \tag{9.129}$$

whereas in the anticausal case it is

$$h(t) = -e^{-3t}u(-t). \tag{9.130}$$

The same procedure used to obtain $H(s)$ from the differential equation in Example 9.23 can be applied more generally. Consider a general linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{9.131}$$

Applying the Laplace transform to both sides and using the linearity and differentiation properties repeatedly, we obtain

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s), \tag{9.132}$$

or

$$H(s) = \frac{\left\{\sum_{k=0}^{M} b_k s^k\right\}}{\left\{\sum_{k=0}^{N} a_k s^k\right\}}. \tag{9.133}$$

Thus, the system function for a system specified by a differential equation is always rational, with zeros at the solutions of

$$\sum_{k=0}^{M} b_k s^k = 0 \tag{9.134}$$

and poles at the solutions of

$$\sum_{k=0}^{N} a_k s^k = 0. \tag{9.135}$$

Consistently with our previous discussion, eq. (9.133) does not include a specification of the region of convergence of $H(s)$, since the linear constant-coefficient differential equation by itself does not constrain the region of convergence. However, with additional information, such as knowledge about the stability or causality of the system, the region of convergence can be inferred. For example, if we impose the condition of initial rest on the system, so that it is causal, the ROC will be to the right of the rightmost pole.

Example 9.24

An *RLC* circuit whose capacitor voltage and inductor current are initially zero constitutes an LTI system describable by a linear constant-coefficient differential equation. Consider the series *RLC* circuit in Figure 9.27. Let the voltage across the voltage source be the input signal $x(t)$, and let the voltage measured across the capacitor be the output signal $y(t)$. Equating the sum of the voltages across the resistor, inductor, and capacitor with the source voltage, we obtain

$$RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) = x(t). \tag{9.136}$$

Applying eq. (9.133), we obtain

$$H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}. \tag{9.137}$$

As shown in Problem 9.64, if the values of R, L, and C are all positive, the poles of this system function will have negative real parts, and consequently, the system will be stable.

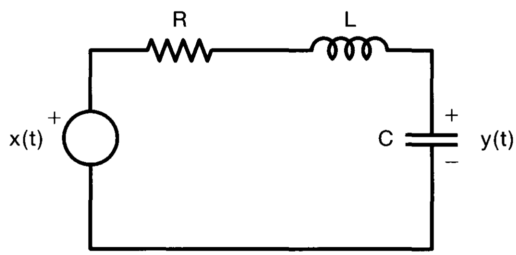


Figure 9.27 A series *RLC* circuit.

9.7.4 Examples Relating System Behavior to the System Function

As we have seen, system properties such as causality and stability can be directly related to the system function and its characteristics. In fact, each of the properties of Laplace transforms that we have described can be used in this way to relate the behavior of the system to the system function. In this section, we give several examples illustrating this.

Example 9.25

Suppose we know that if the input to an LTI system is

$$x(t) = e^{-3t}u(t),$$

then the output is

$$y(t) = [e^{-t} - e^{-2t}]u(t).$$

As we now show, from this knowledge we can determine the system function for this system and from this can immediately deduce a number of other properties of the system.

Taking Laplace transforms of $x(t)$ and $y(t)$, we get

$$X(s) = \frac{1}{s+3}, \qquad \Re e\{s\} > -3,$$

and

$$Y(s) = \frac{1}{(s+1)(s+2)}, \qquad \Re e\{s\} > -1.$$

From eq. (9.112), we can then conclude that

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}.$$

Furthermore, we can also determine the ROC for this system. In particular, we know from the convolution property set forth in Section 9.5.6 that the ROC of $Y(s)$ must include at least the intersections of the ROCs of $X(s)$ and $H(s)$. Examining the three possible choices for the ROC of $H(s)$ (i.e., to the left of the pole at $s = -2$, between the poles at -2 and -1, and to the right of the pole at $s = -1$), we see that the only choice that is consistent with the ROCs of $X(s)$ and $Y(s)$ is $\Re e\{s\} > -1$. Since this is to the right of the rightmost pole of $H(s)$, we conclude that $H(s)$ is causal, and since both poles of $H(s)$ have negative real parts, it follows that the system is stable. Moreover, from the relationship between eqs. (9.131) and (9.133), we can specify the differential equation that, together with the condition of initial rest, characterizes the system:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t).$$

Example 9.26

Suppose that we are given the following information about an LTI system:

1. The system is causal.
2. The system function is rational and has only two poles, at $s = -2$ and $s = 4$.

3. If $x(t) = 1$, then $y(t) = 0$.

4. The value of the impulse response at $t = 0^+$ is 4.

From this information we would like to determine the system function of the system.

From the first two facts, we know that the system is unstable (since it is causal and has a pole at $s = 4$ with positive real part) and that the system function is of the form

$$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2 - 2s - 8},$$

where $p(s)$ is a polynomial in s. Because the response $y(t)$ to the input $x(t) = 1 = e^{0 \cdot t}$ must equal $H(0) \cdot e^{0 \cdot t} = H(0)$, we conclude, from fact 3, that $p(0) = 0$—i.e., that $p(s)$ must have a root at $s = 0$ and thus is of the form

$$p(s) = sq(s),$$

where $q(s)$ is another polynomial in s.

Finally, from fact 4 and the initial-value theorem in Section 9.5.10, we see that

$$\lim_{s \to \infty} sH(s) = \lim_{s \to \infty} \frac{s^2 q(s)}{s^2 - 2s - 8} = 4. \tag{9.138}$$

As $s \to \infty$, the terms of highest power in s in both the numerator and the denominator of $sH(s)$ dominate and thus are the only ones of importance in evaluating eq. (9.138). Furthermore, if the numerator has higher degree than the denominator, the limit will diverge. Consequently, we can obtain a finite nonzero value for the limit only if the degree of the numerator of $sH(s)$ is the same as the degree of the denominator. Since the degree of the denominator is 2, we conclude that, for eq. (9.138) to hold, $q(s)$ must be a constant—i.e., $q(s) = K$. We can evaluate this constant by evaluating

$$\lim_{s \to \infty} \frac{Ks^2}{s^2 - 2s - 8} = \lim_{s \to \infty} \frac{Ks^2}{s^2} = K. \tag{9.139}$$

Equating eqs. (9.138) and (9.139), we see that $K = 4$, and thus,

$$H(s) = \frac{4s}{(s+2)(s-4)}.$$

Example 9.27

Consider a stable and causal system with impulse response $h(t)$ and system function $H(s)$. Suppose $H(s)$ is rational, contains a pole at $s = -2$, and does not have a zero at the origin. The location of all other poles and zeros is unknown. For each of the following statements let us determine whether we can definitely say that it is true, whether we can definitely say that it is false, or whether there is insufficient information to ascertain the statement's truth:

(a) $\mathcal{F}\{h(t)e^{3t}\}$ converges.

(b) $\int_{-\infty}^{+\infty} h(t)\,dt = 0$.

(c) $th(t)$ is the impulse response of a causal and stable system.

(d) $dh(t)/dt$ contains at least one pole in its Laplace transform.

(e) $h(t)$ has finite duration.

(f) $H(s) = H(-s)$.

(g) $\lim_{s\to\infty} H(s) = 2$.

Statement (a) is false, since $\mathfrak{F}\{h(t)e^{3t}\}$ corresponds to the value of the Laplace transform of $h(t)$ at $s = -3$. If this converges, it implies that $s = -3$ is in the ROC. A causal and stable system must always have its ROC to the *right* of all of its poles. However, $s = -3$ is not to the right of the pole at $s = -2$.

Statement (b) is false, because it is equivalent to stating that $H(0) = 0$. This contradicts the fact that $H(s)$ does not have a zero at the origin.

Statement (c) is true. According to Table 9.1, the property set forth in Section 9.5.8, the Laplace transform of $th(t)$ has the same ROC as that of $H(s)$. This ROC includes the $j\omega$-axis, and therefore, the corresponding system is stable. Also, $h(t) = 0$ for $t < 0$ implies that $th(t) = 0$ for $t < 0$. Thus, $th(t)$ represents the impulse response of a causal system.

Statement (d) is true. According to Table 9.1, $dh(t)/dt$ has the Laplace transform $sH(s)$. The multiplication by s does not eliminate the pole at $s = -2$.

Statement (e) is false. If $h(t)$ is of finite duration, then if its Laplace transform has any points in its ROC, the ROC must be the entire s-plane. However, this is not consistent with $H(s)$ having a pole at $s = -2$.

Statement (f) is false. If it were true, then, since $H(s)$ has a pole at $s = -2$, it must also have a pole at $s = 2$. This is inconsistent with the fact that all the poles of a causal and stable system must be in the left half of the s-plane.

The truth of statement (g) cannot be ascertained with the information given. The statement requires that the degree of the numerator and denominator of $H(s)$ be equal, and we have insufficient information about $H(s)$ to determine whether this is the case.

9.7.5 Butterworth Filters

In Example 6.3 we briefly introduced the widely-used class of LTI systems known as Butterworth filters. The filters in this class have a number of properties, including the characteristics of the magnitude of the frequency response of each of these filters in the passband, that make them attractive for practical implementation. As a further illustration of the usefulness of Laplace transforms, in this section we use Laplace transform techniques to determine the system function of a Butterworth filter from the specification of its frequency response magnitude.

An Nth-order lowpass Butterworth filter has a frequency response the square of whose magnitude is given by

$$|B(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}, \tag{9.140}$$

where N is the order of the filter. From eq. (9.140), we would like to determine the system function $B(s)$ that gives rise to $|B(j\omega)|^2$. We first note that, by definition,

$$|B(j\omega)|^2 = B(j\omega)B^*(j\omega). \tag{9.141}$$

If we restrict the impulse response of the Butterworth filter to be real, then from the property of conjugate symmetry for Fourier transforms,

$$B^*(j\omega) = B(-j\omega), \tag{9.142}$$

so that

$$B(j\omega)B(-j\omega) = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}. \tag{9.143}$$

Next, we note that $B(s)|_{s=j\omega} = B(j\omega)$, and consequently, from eq. (9.143),

$$B(s)B(-s) = \frac{1}{1 + (s/j\omega_c)^{2N}}. \tag{9.144}$$

The roots of the denominator polynomial corresponding to the combined poles of $B(s)B(-s)$ are at

$$s = (-1)^{1/2N}(j\omega_c). \tag{9.145}$$

Equation (9.145) is satisfied for any value $s = s_p$ for which

$$|s_p| = \omega_c \tag{9.146}$$

and

$$\sphericalangle s_p = \frac{\pi(2k+1)}{2N} + \frac{\pi}{2}, \qquad k \text{ an integer}; \tag{9.147}$$

that is,

$$s_p = \omega_c \exp\left(j\left[\frac{\pi(2k+1)}{2N} + \pi/2\right]\right). \tag{9.148}$$

In Figure 9.28 we illustrate the positions of the poles of $B(s)B(-s)$ for $N = 1, 2, 3$, and 6. In general, the following observations can be made about these poles:

1. There are $2N$ poles equally spaced in angle on a circle of radius ω_c in the s-plane.
2. A pole never lies on the $j\omega$-axis and occurs on the σ-axis for N odd, but not for N even.
3. The angular spacing between the poles of $B(s)B(-s)$ is π/N radians.

To determine the poles of $B(s)$ given the poles of $B(s)B(-s)$, we observe that the poles of $B(s)B(-s)$ occur in pairs, so that if there is a pole at $s = s_p$, then there is also a pole at $s = -s_p$. Consequently, to construct $B(s)$, we choose one pole from each pair. If we restrict the system to be stable and causal, then the poles that we associate with $B(s)$ are the poles along the semicircle in the left-half plane. The pole locations specify $B(s)$ only to within a scale factor. However, from eq. (9.144), we see that $B^2(s)|_{s=0} = 1$, or equivalently, from eq. (9.140), the scale factor is chosen so that the square of the magnitude of the frequency response has unity gain at $\omega = 0$.

To illustrate the determination of $B(s)$, let us consider the cases $N = 1, N = 2$, and $N = 3$. In Figure 9.28 we showed the poles of $B(s)B(-s)$, as obtained from eq. (9.148).

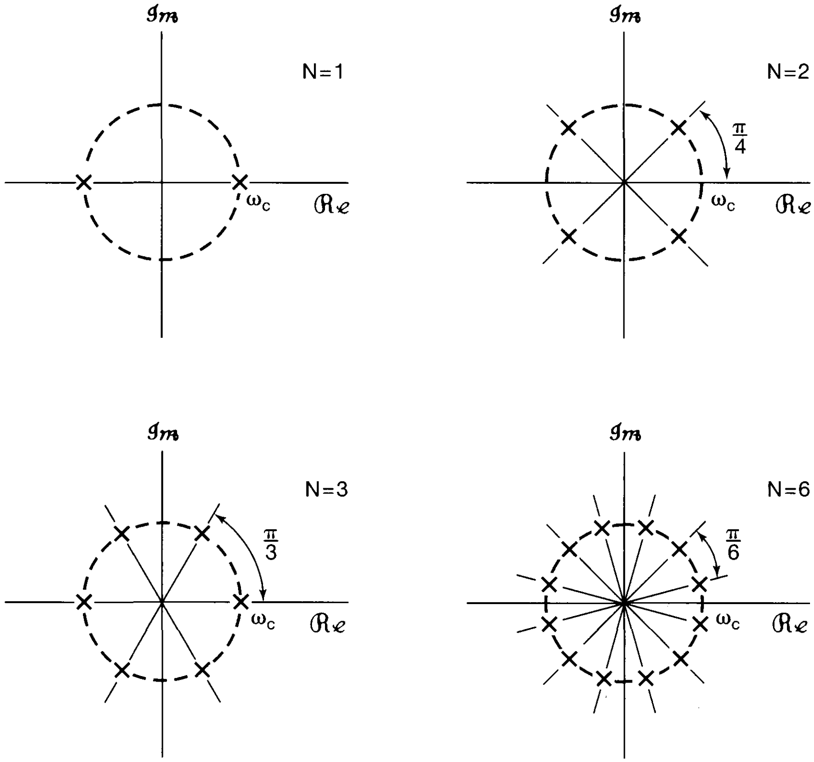


Figure 9.28 Position of the poles of $B(s)B(-s)$ for $N = 1, 2, 3$, and 6.

In Figure 9.29 we show the poles associated with $B(s)$ for each of these values of N. The corresponding transfer functions are:

$$N = 1: \qquad B(s) = \frac{\omega_c}{s + \omega_c}; \tag{9.149}$$

$$N = 2: \qquad B(s) = \frac{\omega_c^2}{(s + \omega_c e^{j(\pi/4)})(s + \omega_c e^{-j(\pi/4)})}$$

$$= \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}; \tag{9.150}$$

$$N = 3: \qquad B(s) = \frac{\omega_c^3}{(s + \omega_c)(s + \omega_c e^{j(\pi/3)})(s + \omega_c e^{-j(\pi/3)})}$$

$$= \frac{\omega_c^3}{(s + \omega_c)(s^2 + \omega_c s + \omega_c^2)} \tag{9.151}$$

$$= \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}.$$

Based on the discussion in Section 9.7.3, from $B(s)$ we can determine the associated linear constant-coefficient differential equation. Specifically, for the foregoing three values

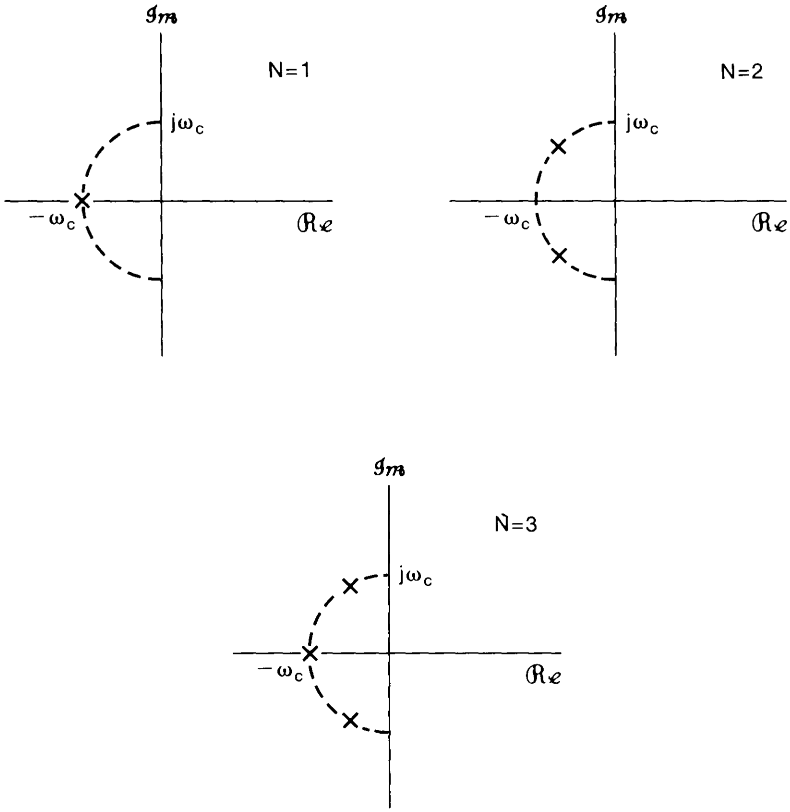


Figure 9.29 Position of the poles of $B(s)$ for $N = 1$, 2, and 3.

of N, the corresponding differential equations are:

$$N = 1: \quad \frac{dy(t)}{dt} + \omega_c y(t) = \omega_c x(t); \tag{9.152}$$

$$N = 2: \quad \frac{d^2y(t)}{dt^2} + \sqrt{2}\omega_c \frac{dy(t)}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t); \tag{9.153}$$

$$N = 3: \quad \frac{d^3y(t)}{dt^3} + 2\omega_c \frac{d^2y(t)}{dt^2} + 2\omega_c^2 \frac{dy(t)}{dt} + \omega_c^3 y(t) = \omega_c^3 x(t). \tag{9.154}$$

9.8 SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATIONS

The use of the Laplace transform allows us to replace time-domain operations such as differentiation, convolution, time shifting, and so on, with algebraic operations. We have already seen many of the benefits of this in terms of analyzing LTI systems, and in this section we take a look at another important use of system function algebra, namely, in analyzing interconnections of LTI systems and synthesizing systems as interconnections of elementary system building blocks.

9.8.1 System Functions for Interconnections of LTI Systems

Consider the parallel interconnection of two systems, as shown in Figure 9.30(a). The impulse response of the overall system is

$$h(t) = h_1(t) + h_2(t), \tag{9.155}$$

and from the linearity of the Laplace transform,

$$H(s) = H_1(s) + H_2(s). \tag{9.156}$$

Similarly, the impulse response of the series interconnection in Figure 9.30(b) is

$$h(t) = h_1(t) * h_2(t), \tag{9.157}$$

and the associated system function is

$$H(s) = H_1(s)H_2(s). \tag{9.158}$$

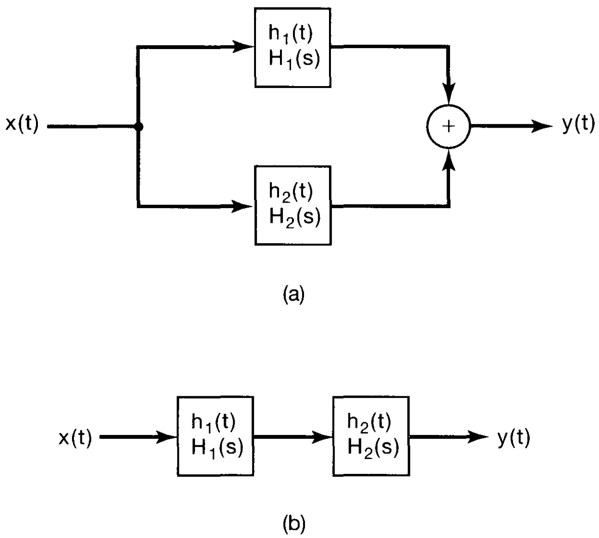


Figure 9.30 (a) Parallel interconnection of two LTI systems; (b) series combination of two LTI systems.

The utility of the Laplace transform in representing combinations of linear systems through algebraic operations extends to far more complex interconnections than the simple parallel and series combinations in Figure 9.30. To illustrate this, consider the feedback interconnection of two systems, as indicated in Figure 9.31. The design, applications, and analysis of such interconnections are treated in detail in Chapter 11. While analysis of the system in the time domain is not particularly simple, determining the overall system function from input $x(t)$ to output $y(t)$ is a straightforward algebraic manipulation. Specifically, from Figure 9.31,

$$Y(s) = H_1(s)E(s), \tag{9.159}$$

$$E(s) = X(s) - Z(s), \tag{9.160}$$

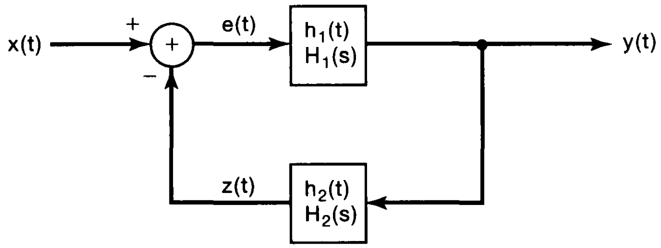


Figure 9.31 Feedback interconnection of two LTI systems.

and

$$Z(s) = H_2(s)Y(s), \tag{9.161}$$

from which we obtain the relation

$$Y(s) = H_1(s)[X(s) - H_2(s)Y(s)], \tag{9.162}$$

or

$$\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}. \tag{9.163}$$

9.8.2 Block Diagram Representations for Causal LTI Systems Described by Differential Equations and Rational System Functions

In Section 2.4.3, we illustrated the block diagram representation of an LTI system described by a first-order differential equation using the basic operations of addition, multiplication by a coefficient, and integration. These same three operations can also be used to build block diagrams for higher order systems, and in this section we illustrate this in several examples.

Example 9.28

Consider the causal LTI system with system function

$$H(s) = \frac{1}{s+3}.$$

From Section 9.7.3, we know that this system can also be described by the differential equation

$$\frac{dy(t)}{dt} + 3y(t) = x(t),$$

together with the condition of initial rest. In Section 2.4.3 we constructed a block diagram representation, shown in Figure 2.32, for a first-order system such as this. An equivalent block diagram (corresponding to Figure 2.32 with $a = 3$ and $b = 1$) is shown in

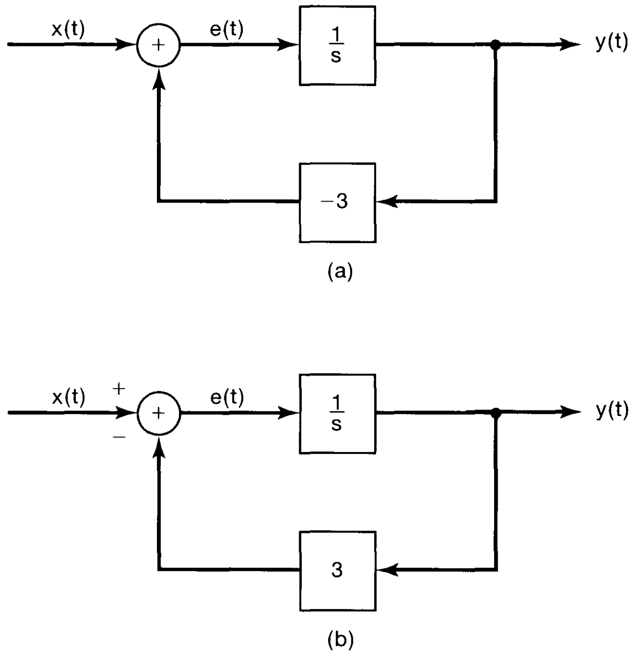


Figure 9.32 (a) Block diagram representation of the causal LTI system in Example 9.28; (b) equivalent block diagram representation.

Figure 9.32(a). Here, $1/s$ is the system function of a system with impulse response $u(t)$, i.e., it is the system function of an integrator. Also, the system function -3 in the feedback path in Figure 9.32(a) corresponds to multiplication by the coefficient -3. The block diagram in the figure involves a feedback loop much as we considered in the previous subsection and as pictured in Figure 9.31, the sole difference being that the two signals that are the inputs to the adder in Figure 9.32(a) are added, rather than subtracted as in Figure 9.31. However, as illustrated in Figure 9.32(b), by changing the sign of the coefficient in the multiplication in the feedback path, we obtain a block diagram representation of exactly the same form as Figure 9.31. Consequently, we can apply eq. (9.163) to verify that

$$H(s) = \frac{1/s}{1 + 3/s} = \frac{1}{s+3}.$$

Example 9.29

Consider now the causal LTI system with system function

$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+3}\right)(s+2). \tag{9.164}$$

As suggested by eq. (9.164), this system can be thought of as a cascade of a system with system function $1/(s + 3)$ followed by a system with system function $s + 2$, and we have illustrated this in Figure 9.33(a), in which we have used the block diagram in Figure 9.32(a) to represent $1/(s + 3)$.

It is also possible to obtain an alternative block diagram representation for the system in eq. (9.164). Using the linearity and differentiation properties of the Laplace transform, we know that $y(t)$ and $z(t)$ in Figure 9.33 (a) are related by

$$y(t) = \frac{dz(t)}{dt} + 2z(t).$$

However, the input $e(t)$ to the integrator is exactly the derivative of the output $z(t)$, so that

$$y(t) = e(t) + 2z(t),$$

which leads directly to the alternative block diagram representation shown in Figure 9.33(b). Note that the block diagram in Figure 9.33(a) requires the differentiation of $z(t)$, since

$$y(t) = \frac{dz(t)}{dt} + 2z(t)$$

In contrast, the block diagram in Figure 9.33(b) does not involve the explicit differentiation of any signal.

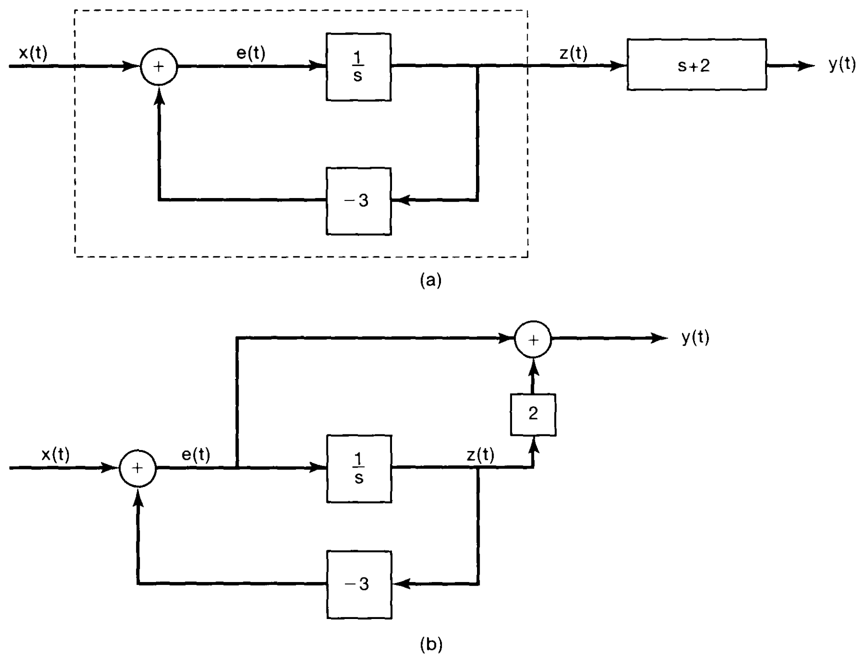


Figure 9.33 (a) Block diagram representations for the system in Example 9.29; (b) equivalent block diagram representation.

Example 9.30

Consider next a causal second-order system with system function

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}. \tag{9.165}$$

The input $x(t)$ and output $y(t)$ for this system satisfy the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t). \tag{9.166}$$

By employing similar ideas to those used in the preceding examples, we obtain the block diagram representation for this system shown in Figure 9.34(a). Specifically, since the

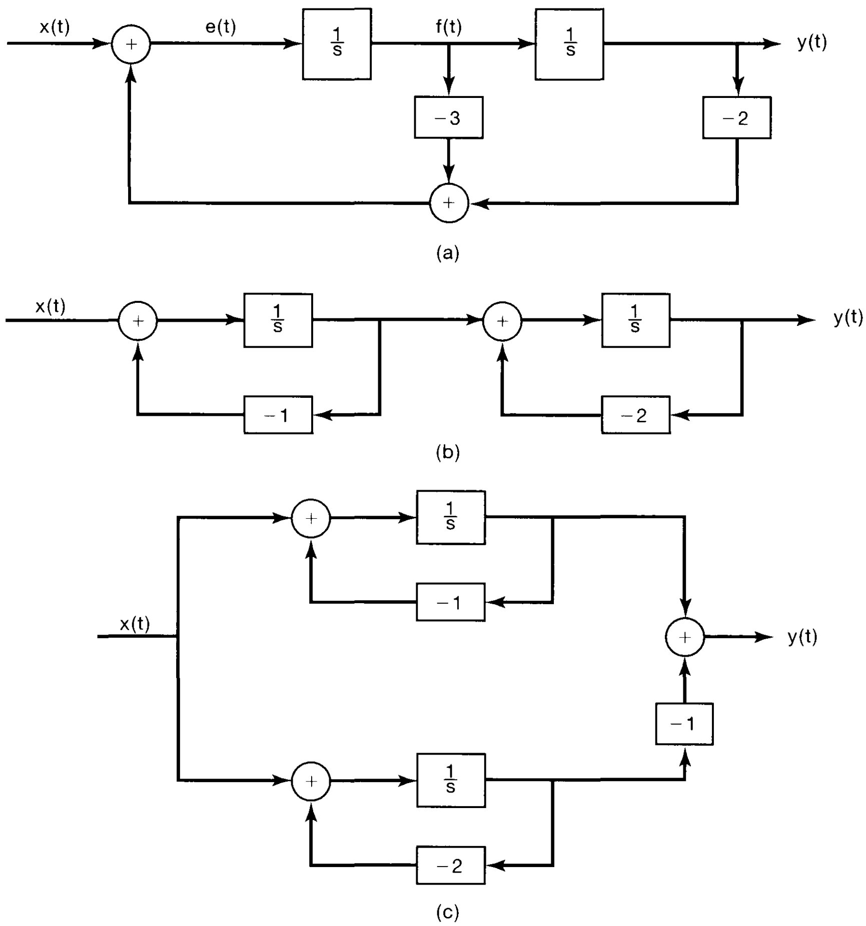


Figure 9.34 Block diagram representations for the system in Example 9.30: (a) direct form; (b) cascade form; (c) parallel form.

input to an integrator is the derivative of the output of the integrator, the signals in the block diagram are related by

$$f(t) = \frac{dy(t)}{dt},$$
$$e(t) = \frac{df(t)}{dt} = \frac{d^2y(t)}{dt}.$$

Also, eq. (9.166) can be rewritten as

$$\frac{d^2y(t)}{dt^2} = -3\frac{dy(t)}{dt} - 2y(t) + x(t),$$

or

$$e(t) = -3f(t) - 2y(t) + x(t),$$

which is exactly what is represented in Figure 9.34(a).

The block diagram in this figure is sometimes referred to as a *direct-form* representation, since the coefficients appearing in the diagram can be directly identified with the coefficients appearing in the system function or, equivalently, the differential equation. Other block diagram representations of practical importance also can be obtained after a modest amount of system function algebra. Specifically, $H(s)$ in eq. (9.165) can be rewritten as

$$H(s) = \left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right),$$

which suggests that this system can be represented as the cascade of two first-order systems. The *cascade-form* representation corresponding to $H(s)$ is shown in Figure 9.34(b).

Alternatively, by performing a partial-fraction expansion of $H(s)$, we obtain

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2},$$

which leads to the *parallel-form* representation depicted in Figure 9.34(c).

Example 9.31

As a final example, consider the system function

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2}. \tag{9.167}$$

Once again, using system function algebra, we can write $H(s)$ in several different forms, each of which suggests a block diagram representation. In particular, we can write

$$H(s) = \left(\frac{1}{s^2 + 3s + 2}\right)(2s^2 + 4s - 6),$$

which suggests the representation of $H(s)$ as the cascade of the system depicted in Figure 9.34(a) and the system with system function $2s^2 + 4s - 6$. However, exactly as we did in Example 9.29, we can extract the derivatives required for this second system by "tapping" the signals appearing as the inputs to the integrators in the first system. The details of this construction are examined in Problem 9.36, and the result is the direct-form block diagram shown in Figure 9.35. Once again, in the direct-form representation the coefficients appearing in the block diagram can be determined by inspection from the coefficients in the system function in eq. (9.167).

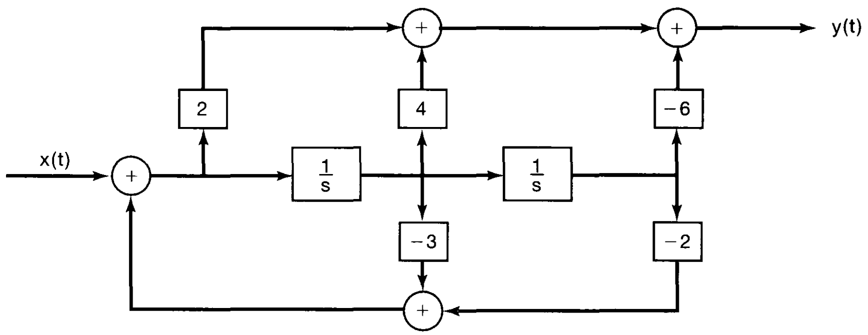


Figure 9.35 Direct-form representation for the system in Example 9.31.

Alternatively, we can write $H(s)$ in the form

$$H(s) = \left(\frac{2(s-1)}{s+2}\right)\left(\frac{s+3}{s+1}\right) \tag{9.168}$$

or

$$H(s) = 2 + \frac{6}{s+2} - \frac{8}{s+1}. \tag{9.169}$$

The first of these suggests a cascade-form representation, while the second leads to a parallel-form block diagram. These are also considered in Problem 9.36.

The methods for constructing block diagram representations for causal LTI systems described by differential equations and rational system functions can be applied equally well to higher order systems. In addition, there is often considerable flexibility in how this is done. For example, by reversing the numerators in eq. (9.168), we can write

$$H(s) = \left(\frac{s+3}{s+2}\right)\left(\frac{2(s-1)}{s+2}\right),$$

which suggests a different cascade form. Also, as illustrated in Problem 9.38, a fourth-order system function can be written as the product of two second-order system functions, each of which can be represented in a number of ways (e.g., direct form, cascade, or parallel), and it can also be written as the sum of lower order terms, each of which has several different representations. In this way, simple low-order systems can serve as building blocks for the implementation of more complex, higher order systems.

9.9 THE UNILATERAL LAPLACE TRANSFORM

In the preceding sections of this chapter, we have dealt with what is commonly called the bilateral Laplace transform. In this section, we introduce and examine a somewhat different transform, the *unilateral Laplace transform*, which is of considerable value in analyzing causal systems and, particularly, systems specified by linear constant-coefficient differential equations with nonzero initial conditions (i.e., systems that are not initially at rest).

The unilateral Laplace transform of a continuous-time signal $x(t)$ is defined as

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}\,dt, \tag{9.170}$$

where the lower limit of integration, 0^-, signifies that we include in the interval of integration any impulses or higher order singularity functions concentrated at $t = 0$. Once again we adopt a convenient shorthand notation for a signal and its unilateral Laplace transform:

$$x(t) \stackrel{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s) = \mathcal{UL}\{x(t)\}. \tag{9.171}$$

Comparing eqs. (9.170) and (9.3), we see that the difference in the definitions of the unilateral and bilateral Laplace transform lies in the lower limit on the integral. The bilateral transform depends on the entire signal from $t = -\infty$ to $t = +\infty$, whereas the unilateral transform depends only on the signal from $t = 0^-$ to ∞. Consequently, two signals that differ for $t < 0$, but that are identical for $t \geq 0$, will have different bilateral Laplace transforms, but identical unilateral transforms. Similarly, any signal that is identically zero for $t < 0$ has identical bilateral and unilateral transforms.

Since the unilateral transform of $x(t)$ is identical to the bilateral transform of the signal obtained from $x(t)$ by setting its value to 0 for all $t < 0$, many of the insights, concepts, and results pertaining to bilateral transforms can be directly adapted to the unilateral case. For example, using Property 4 in Section 9.2 for right-sided signals, we see that the ROC for eq. (9.170) is always a right-half plane. The evaluation of the inverse unilateral Laplace transforms is also the same as for bilateral transforms, with the constraint that the ROC for a unilateral transform must always be a right-half plane.

9.9.1 Examples of Unilateral Laplace Transforms

To illustrate the unilateral Laplace transform, let us consider the following examples:

Example 9.32

Consider the signal

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t). \tag{9.172}$$

Since $x(t) = 0$ for $t < 0$, the unilateral and bilateral transforms are identical. Thus, from Table 9.2,

$$\mathcal{X}(s) = \frac{1}{(s+a)^n}, \qquad \mathcal{R}e\{s\} > -a. \tag{9.173}$$

Example 9.33

Consider next

$$x(t) = e^{-a(t+1)}u(t+1). \tag{9.174}$$

The *bilateral* transform $X(s)$ for this example can be obtained from Example 9.1 and the time-shifting property (Section 9.5.2):

$$X(s) = \frac{e^s}{s+a}, \qquad \mathcal{R}e\{s\} > -a. \tag{9.175}$$

By contrast, the unilateral transform is

$$\begin{aligned} \mathcal{X}(s) &= \int_{0^-}^{\infty} e^{-a(t+1)}u(t+1)e^{-st}\,dt \\ &= \int_{0^-}^{\infty} e^{-a}e^{-t(s+a)}\,dt \\ &= e^{-a}\frac{1}{s+a}, \qquad \mathcal{R}e\{s\} > -a. \end{aligned} \tag{9.176}$$

Thus, in this example, the unilateral and bilateral Laplace transforms are clearly different. In fact, we should recognize $\mathcal{X}(s)$ as the bilateral transform not of $x(t)$, but of $x(t)u(t)$, consistent with our earlier comment that the unilateral transform is the bilateral transform of a signal whose values for $t < 0^-$ have been set to zero.

Example 9.34

Consider the signal

$$x(t) = \delta(t) + 2u_1(t) + e^t u(t). \tag{9.177}$$

Since $x(t) = 0$ for $t < 0$, and since singularities at the origin are included in the interval of integration, the unilateral transform for $x(t)$ is the same as the bilateral transform. Specifically, using the fact (transform pair 15 in Table 9.2) that the bilateral transform of $u_n(t)$ is s^n, we have

$$\mathcal{X}(s) = X(s) = 1 + 2s + \frac{1}{s-1} = \frac{s(2s-1)}{s-1}, \qquad \mathcal{R}e\{s\} > 1. \tag{9.178}$$

Example 9.35

Consider the unilateral Laplace transform

$$\mathcal{X}(s) = \frac{1}{(s+1)(s+2)}. \tag{9.179}$$

In Example 9.9, we considered the inverse transform for a bilateral Laplace transform of the exact form as that in eq. (9.179) and for several ROCs. For the unilateral transform, the ROC must be the right-half plane to the right of the rightmost pole of $\mathcal{X}(s)$; i.e., in this case, the ROC consists of all points s with $\Re e\{s\} > -1$. We can then invert this unilateral transform exactly as in Example 9.9 to obtain

$$x(t) = [e^{-t} - e^{-2t}]u(t) \quad \text{for} \quad t > 0^-, \tag{9.180}$$

where we have emphasized the fact that unilateral Laplace transforms provide us with information about signals only for $t > 0^-$.

Example 9.36

Consider the unilateral transform

$$\mathcal{X}(s) = \frac{s^2 - 3}{s + 2}. \tag{9.181}$$

Since the degree of the numerator of $\mathcal{X}(s)$ is not strictly less than the degree of the denominator, we expand $\mathcal{X}(s)$ as

$$\mathcal{X}(s) = A + Bs + \frac{C}{s + 2}. \tag{9.182}$$

Equating eqs. (9.181) and (9.182), and clearing denominators, we obtain

$$s^2 - 3 = (A + Bs)(s + 2) + C, \tag{9.183}$$

and equating coefficients for each power of s yields

$$\mathcal{X}(s) = -2 + s + \frac{1}{s + 2}, \tag{9.184}$$

with an ROC of $\Re e\{s\} > -2$. Taking inverse transforms of each term results in

$$x(t) = -2\delta(t) + u_1(t) + e^{-2t}u(t) \quad \text{for} \quad t > 0^-. \tag{9.185}$$

9.9.2 Properties of the Unilateral Laplace Transform

As with the bilateral Laplace transform, the unilateral Laplace transform has a number of important properties, many of which are the same as their bilateral counterparts and several of which differ in significant ways. Table 9.3 summarizes these properties. Note that we have not included a column explicitly identifying the ROC for the unilateral Laplace transform for each signal, since the ROC of any unilateral Laplace transform is always a right-half plane. For example the ROC for a rational unilateral Laplace transform is always to the right of the rightmost pole.

Contrasting Table 9.3 with Table 9.1 for the bilateral transform, we see that, with the caveat that ROCs for unilateral Laplace transforms are always right-half planes, the linearity, s-domain shifting, time-scaling, conjugation and differentiation in the s-domain

TABLE 9.3 PROPERTIES OF THE UNILATERAL LAPLACE TRANSFORM

Property	Signal	Unilateral Laplace Transform
	$x(t)$ $x_1(t)$ $x_2(t)$	$\mathcal{X}(s)$ $\mathcal{X}_1(s)$ $\mathcal{X}_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\mathcal{X}_1(s) + b\mathcal{X}_2(s)$
Shifting in the s-domain	$e^{s_0 t}x(t)$	$\mathcal{X}(s - s_0)$
Time scaling	$x(at), \quad a > 0$	$\frac{1}{a}\mathcal{X}\left(\frac{s}{a}\right)$
Conjugation	$x*(t)$	$x*(s)$
Convolution (assuming that $x_1(t)$ and $x_2(t)$ are identically zero for $t < 0$)	$x_1(t) * x_2(t)$	$\mathcal{X}_1(s)\mathcal{X}_2(s)$
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$s\mathcal{X}(s) - x(0^-)$
Differentiation in the s-domain	$-tx(t)$	$\frac{d}{ds}\mathcal{X}(s)$
Integration in the time domain	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}\mathcal{X}(s)$

Initial- and Final-Value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} s\mathcal{X}(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\mathcal{X}(s)$$

properties are identical to their bilateral counterparts. Similarly, the initial- and final-value theorems stated in Section 9.5.10 also hold for unilateral Laplace transforms.[3] The derivation of each of these properties is identical to that of its bilateral counterpart.

The convolution property for unilateral transforms also is quite similar to the corresponding property for bilateral transforms. This property states that if

$$x_1(t) = x_2(t) = 0 \quad \text{for all} \quad t < 0, \tag{9.186}$$

[3]In fact, the initial- and final-value theorems are basically unilateral transform properties, as they apply only to signals $x(t)$ that are identically 0 for $t < 0$.

then

$$x_1(t) * x_2(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}_1(s)\mathcal{X}_2(s). \tag{9.187}$$

Equation (9.187) follows immediately from the bilateral convolution property, since, under the conditions of eq. (9.186), the unilateral and bilateral transforms are identical for each of the signals $x_1(t)$ and $x_2(t)$. Thus, the system analysis tools and system function algebra developed and used in this chapter apply without change to unilateral transforms, as long as we deal with causal LTI systems (for which the system function is *both* the bilateral *and* the unilateral transform of the impulse response) with inputs that are identically zero for $t < 0$. An example of this is the integration property in Table 9.3: If $x(t) = 0$ for $t < 0$, then

$$\int_{0^-}^{t} x(\tau)\,d\tau = x(t) * u(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s)\mathcal{U}(s) = \frac{1}{s}\mathcal{X}(s) \tag{9.188}$$

As a second case in point, consider the following example:

Example 9.37

Suppose a causal LTI system is described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t), \tag{9.189}$$

together with the condition of initial rest. Using eq. (9.133), we find that the system function for this system is

$$\mathcal{H}(s) = \frac{1}{s^2 + 3s + 2}. \tag{9.190}$$

Let the input to this system be $x(t) = \alpha u(t)$. In this case, the unilateral (and bilateral) Laplace transform of the output $y(t)$ is

$$\begin{aligned}\mathcal{Y}(s) = \mathcal{H}(s)\mathcal{X}(s) &= \frac{\alpha}{s(s+1)(s+2)} \\ &= \frac{\alpha/2}{s} - \frac{\alpha}{s+1} + \frac{\alpha/2}{s+2}.\end{aligned} \tag{9.191}$$

Applying Example 9.32 to each term of eq. (9.191) yields

$$y(t) = \alpha\left[\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right]u(t). \tag{9.192}$$

It is important to note that the convolution property for unilateral Laplace transforms applies only if the signals $x_1(t)$ and $x_2(t)$ in eq. (9.187) are both zero for $t < 0$. That is, while we have seen that the bilateral Laplace transform of $x_1(t) * x_2(t)$ always equals the product of the bilateral transforms of $x_1(t)$ and $x_2(t)$, the unilateral transform of $x_1(t)*x_2(t)$ in general does *not* equal the product of the unilateral transforms if either $x_1(t)$ or $x_2(t)$ is nonzero for $t < 0$. (See, for example, Problem 9.39).

A particularly important difference between the properties of the unilateral and bilateral transforms is the differentiation property. Consider a signal $x(t)$ with unilateral Laplace transform $\mathcal{X}(s)$. Then, integrating by parts, we find that the unilateral transform of $dx(t)/dt$ is given by

$$\begin{aligned}\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt &= x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}\, dt \\ &= s\mathcal{X}(s) - x(0^-).\end{aligned} \tag{9.193}$$

Similarly, a second application of this would yield the unilateral Laplace transform of $d^2x(t)/dt^2$, i.e.,

$$s^2\mathcal{X}(s) - sx(0^-) - x'(0^-), \tag{9.194}$$

where $x'(0^-)$ denotes the derivative of $x(t)$ evaluated at $t = 0^-$. Clearly, we can continue the procedure to obtain the unilateral transform of higher derivatives.

9.9.3 Solving Differential Equations Using the Unilateral Laplace Transform

A primary use of the unilateral Laplace transform is in obtaining the solution of linear constant-coefficient differential equations with nonzero initial conditions. We illustrate this in the following example:

Example 9.38

Consider the system characterized by the differential equation (9.189) with initial conditions

$$y(0^-) = \beta, \qquad y'(0^-) = \gamma. \tag{9.195}$$

Let $x(t) = \alpha u(t)$. Then, applying the unilateral transform to both sides of eq. (9.189), we obtain

$$s^2\mathcal{Y}(s) - \beta s - \gamma + 3s\mathcal{Y}(s) - 3\beta + 2\mathcal{Y}(s) = \frac{\alpha}{s}, \tag{9.196}$$

or

$$\mathcal{Y}(s) = \frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)} + \frac{\alpha}{s(s+1)(s+2)}, \tag{9.197}$$

where $\mathcal{Y}(s)$ is the unilateral Laplace transform of $y(t)$.

Referring to Example 9.37 and, in particular, to eq. (9.191), we see that the last term on the right-hand side of eq. (9.197) is precisely the unilateral Laplace transform of the response of the system when the initial conditions in eq. (9.195) are both zero ($\beta = \gamma = 0$). That is, the last term represents the response of the causal LTI system described by eq. (9.189) and the condition of initial rest. This response is often referred to as the *zero-state response*—i.e., the response when the initial state (the set of initial conditions in eq. (9.195)) is zero.

An analogous interpretation applies to the first two terms on the right-hand side of eq. (9.197). These terms represent the unilateral transform of the response of the system when the input is zero ($\alpha = 0$). This response is commonly referred to as the *zero-input response*. Note that the zero-input response is a linear function of the values of the initial conditions (e.g., doubling the values of both β and γ doubles the zero-input response). Furthermore, eq. (9.197) illustrates an important fact about the solution of linear constant-coefficient differential equations with nonzero initial conditions, namely, that the overall response is simply the superposition of the zero-state and the zero-input responses. The zero-state response is the response obtained by setting the initial conditions to zero—i.e., it is the response of an LTI system defined by the differential equation and the condition of initial rest. The zero-input response is the response to the initial conditions with the input set to zero. Other examples illustrating this can be found in Problems 9.20, 9.40, and 9.66.

Finally, for any values of α, β, and γ, we can, of course, expand $\mathcal{Y}(s)$ in eq. (9.197) in a partial-fraction expansion and invert to obtain $y(t)$. For example, if $\alpha = 2$, $\beta = 3$, and $\gamma = -5$, then performing a partial-fraction expansion for eq. (9.197) we find that

$$\mathcal{Y}(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}. \tag{9.198}$$

Applying Example 9.32 to each term then yields

$$y(t) = [1 - e^{-t} + 3e^{-2t}]u(t) \quad \text{for} \quad t > 0. \tag{9.199}$$

9.10 SUMMARY

In this chapter, we have developed and studied the Laplace transform, which can be viewed as a generalization of the Fourier transform. It is particularly useful as an analytical tool in the analysis and study of LTI systems. Because of the properties of Laplace transforms, LTI systems, including those represented by linear constant-coefficient differential equations, can be characterized and analyzed in the transform domain by algebraic manipulations. In addition, system function algebra provides a convenient tool both for analyzing interconnections of LTI systems and for constructing block diagram representations of LTI systems described by differential equations.

For signals and systems with rational Laplace transforms, the transform is often conveniently represented in the complex plane (s-plane) by marking the locations of the poles and zeros and indicating the region of convergence. From the pole-zero plot, the Fourier transform can be geometrically obtained, within a scaling factor. Causality, stability, and other characteristics are also easily identified from knowledge of the pole locations and the region of convergence.

In this chapter, we have been concerned primarily with the bilateral Laplace transform. However, we also introduced a somewhat different form of the Laplace transform known as the unilateral Laplace transform. The unilateral transform can be interpreted as the bilateral transform of a signal whose values prior to $t = 0^-$ have been set to zero. This form of the Laplace transform is especially useful for analyzing systems described by linear constant-coefficient differential equations with nonzero initial conditions.

Chapter 9 Problems

The first section of problems belongs to the basic category, and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

9.1. For each of the following integrals, specify the values of the real parameter σ which ensure that the integral converges:

(a) $\int_0^{\infty} e^{-5t} e^{-(\sigma+j\omega)t}\, dt$ **(b)** $\int_{-\infty}^{0} e^{-5t} e^{-(\sigma+j\omega)t}\, dt$

(c) $\int_{-5}^{5} e^{-5t} e^{-(\sigma+j\omega)t}\, dt$ **(d)** $\int_{-\infty}^{\infty} e^{-5t} e^{-(\sigma+j\omega)t}\, dt$

(e) $\int_{-\infty}^{\infty} e^{-5|t|} e^{-(\sigma+j\omega)t}\, dt$ **(f)** $\int_{-\infty}^{0} e^{-5|t|} e^{-(\sigma+j\omega)t}\, dt$

9.2. Consider the signal

$$x(t) = e^{-5t}u(t-1),$$

and denote its Laplace transform by $X(s)$.

(a) Using eq. (9.3), evaluate $X(s)$ and specify its region of convergence.

(b) Determine the values of the finite numbers A and t_0 such that the Laplace transform $G(s)$ of

$$g(t) = Ae^{-5t}u(-t-t_0)$$

has the same algebraic form as $X(s)$. What is the region of convergence corresponding to $G(s)$?

9.3. Consider the signal

$$x(t) = e^{-5t}u(t) + e^{-\beta t}u(t),$$

and denote its Laplace transform by $X(s)$. What are the constraints placed on the real and imaginary parts of β if the region of convergence of $X(s)$ is $\Re e\{s\} > -3$?

9.4. For the Laplace transform of

$$x(t) = \begin{cases} e^t \sin 2t, & t \le 0 \\ 0, & t > 0 \end{cases},$$

indicate the location of its poles and its region of convergence.

9.5. For each of the following algebraic expressions for the Laplace transform of a signal, determine the number of zeros located in the finite s-plane and the number of zeros located at infinity:

(a) $\dfrac{1}{s+1} + \dfrac{1}{s+3}$

(b) $\dfrac{s+1}{s^2-1}$

(c) $\dfrac{s^3-1}{s^2+s+1}$

9.6. An absolutely integrable signal $x(t)$ is known to have a pole at $s = 2$. Answer the following questions:

(a) Could $x(t)$ be of finite duration?
(b) Could $x(t)$ be left sided?
(c) Could $x(t)$ be right sided?
(d) Could $x(t)$ be two sided?

9.7. How many signals have a Laplace transform that may be expressed as

$$\frac{(s-1)}{(s+2)(s+3)(s^2+s+1)}$$

in its region of convergence?

9.8. Let $x(t)$ be a signal that has a rational Laplace transform with exactly two poles, located at $s = -1$ and $s = -3$. If $g(t) = e^{2t}x(t)$ and $G(j\omega)$ [the Fourier transform of $g(t)$] converges, determine whether $x(t)$ is left sided, right sided, or two sided.

9.9. Given that

$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \qquad \mathcal{Re}\{s\} > \mathcal{Re}\{-a\},$$

determine the inverse Laplace transform of

$$X(s) = \frac{2(s+2)}{s^2+7s+12}, \qquad \mathcal{Re}\{s\} > -3.$$

9.10. Using geometric evaluation of the magnitude of the Fourier transform from the corresponding pole-zero plot, determine, for each of the following Laplace transforms, whether the magnitude of the corresponding Fourier transform is approximately lowpass, highpass, or bandpass:

(a) $H_1(s) = \dfrac{1}{(s+1)(s+3)}, \quad \mathcal{Re}\{s\} > -1$

(b) $H_2(s) = \dfrac{s}{s^2+s+1}, \quad \mathcal{Re}\{s\} > -\frac{1}{2}$

(c) $H_3(s) = \dfrac{s^2}{s^2+2s+1}, \quad \mathcal{Re}\{s\} > -1$

9.11. Use geometric evaluation from the pole-zero plot to determine the magnitude of the Fourier transform of the signal whose Laplace transform is specified as

$$X(s) = \frac{s^2-s+1}{s^2+s+1}, \qquad \mathcal{Re}\{s\} > -\frac{1}{2}.$$

9.12. Suppose we are given the following three facts about the signal $x(t)$:

1. $x(t) = 0$ for $t < 0$.
2. $x(k/80) = 0$ for $k = 1, 2, 3, \ldots$.
3. $x(1/160) = e^{-120}$.

Let $X(s)$ denote the Laplace transform of $x(t)$, and determine which of the following statements is consistent with the given information about $x(t)$:

(a) $X(s)$ has only one pole in the finite s-plane.
(b) $X(s)$ has only two poles in the finite s-plane.
(c) $X(s)$ has more than two poles in the finite s-plane.

9.13. Let

$$g(t) = x(t) + \alpha x(-t),$$

where

$$x(t) = \beta e^{-t}u(t)$$

and the Laplace transform of $g(t)$ is

$$G(s) = \frac{s}{s^2 - 1}, \qquad -1 < \mathcal{Re}\{s\} < 1.$$

Determine the values of the constants α and β.

9.14. Suppose the following facts are given about the signal $x(t)$ with Laplace transform $X(s)$:

1. $x(t)$ is real and even.
2. $X(s)$ has four poles and no zeros in the finite s-plane.
3. $X(s)$ has a pole at $s = (1/2)e^{j\pi/4}$.
4. $\int_{-\infty}^{\infty} x(t)\,dt = 4$.

Determine $X(s)$ and its ROC.

9.15. Consider two right-sided signals $x(t)$ and $y(t)$ related through the differential equations

$$\frac{dx(t)}{dt} = -2y(t) + \delta(t)$$

and

$$\frac{dy(t)}{dt} = 2x(t).$$

Determine $Y(s)$ and $X(s)$, along with their regions of convergence.

9.16. A causal LTI system S with impulse response $h(t)$ has its input $x(t)$ and output $y(t)$ related through a linear constant-coefficient differential equation of the form

$$\frac{d^3y(t)}{dt^3} + (1+\alpha)\frac{d^2y(t)}{dt^2} + \alpha(\alpha+1)\frac{dy(t)}{dt} + \alpha^2 y(t) = x(t).$$

(a) If

$$g(t) = \frac{dh(t)}{dt} + h(t),$$

how many poles does $G(s)$ have?

(b) For what real values of the parameter α is S guaranteed to be stable?

9.17. A causal LTI system S has the block diagram representation shown in Figure P9.17. Determine a differential equation relating the input $x(t)$ to the output $y(t)$ of this system.

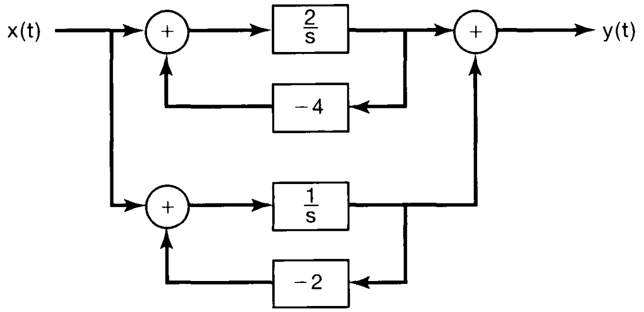


Figure P9.17

9.18. Consider the causal LTI system represented by the *RLC* circuit examined in Problem 3.20.

(a) Determine $H(s)$ and specify its region of convergence. Your answer should be consistent with the fact that the system is causal and stable.

(b) Using the pole-zero plot of $H(s)$ and geometric evaluation of the magnitude of the Fourier transform, determine whether the magnitude of the corresponding Fourier transform has an approximately lowpass, highpass, or bandpass characteristic.

(c) If the value of R is now changed to $10^{-3}\ \Omega$, determine $H(s)$ and specify its region of convergence.

(d) Using the pole-zero plot of $H(s)$ obtained in part (c) and geometric evaluation of the magnitude of the Fourier transform, determine whether the magnitude of the corresponding Fourier transform has an approximately lowpass, highpass, or bandpass characteristic.

9.19. Determine the unilateral Laplace transform of each of the following signals, and specify the corresponding regions of convergence:

(a) $x(t) = e^{-2t}u(t+1)$

(b) $x(t) = \delta(t+1) + \delta(t) + e^{-2(t+3)}u(t+1)$

(c) $x(t) = e^{-2t}u(t) + e^{-4t}u(t)$

9.20. Consider the *RL* circuit of Problem 3.19.

(a) Determine the zero-state response of this circuit when the input current is $x(t) = e^{-2t}u(t)$.

(b) Determine the zero-input response of the circuit for $t > 0^-$, given that

$$y(0^-) = 1.$$

(c) Determine the output of the circuit when the input current is $x(t) = e^{-2t}u(t)$ and the initial condition is the same as the one specified in part (b).

BASIC PROBLEMS

9.21. Determine the Laplace transform and the associated region of convergence and pole-zero plot for each of the following functions of time:

(a) $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$ **(b)** $x(t) = e^{-4t}u(t) + e^{-5t}(\sin 5t)u(t)$

(c) $x(t) = e^{2t}u(-t) + e^{3t}u(-t)$ **(d)** $x(t) = te^{-2|t|}$

(e) $x(t) = |t|e^{-2|t|}$ **(f)** $x(t) = |t|e^{2t}u(-t)$

(g) $x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$ **(h)** $x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \end{cases}$

(i) $x(t) = \delta(t) + u(t)$ **(j)** $x(t) = \delta(3t) + u(3t)$

9.22. Determine the function of time, $x(t)$, for each of the following Laplace transforms and their associated regions of convergence:

(a) $\frac{1}{s^2+9}$, $\mathcal{R}e\{s\} > 0$

(b) $\frac{s}{s^2+9}$, $\mathcal{R}e\{s\} < 0$

(c) $\frac{s+1}{(s+1)^2+9}$, $\mathcal{R}e\{s\} < -1$

(d) $\frac{s+2}{s^2+7s+12}$, $-4 < \mathcal{R}e\{s\} < -3$

(e) $\frac{s+1}{s^2+5s+6}$, $-3 < \mathcal{R}e\{s\} < -2$

(f) $\frac{(s+1)^2}{s^2-s+1}$, $\mathcal{R}e\{s\} > \frac{1}{2}$

(g) $\frac{s^2-s+1}{(s+1)^2}$, $\mathcal{R}e\{s\} > -1$

9.23. For each of the following statements about $x(t)$, and for each of the four pole-zero plots in Figure P9.23, determine the corresponding constraint on the ROC:

1. $x(t)e^{-3t}$ is absolutely integrable.
2. $x(t) * (e^{-t}u(t))$ is absolutely integrable.
3. $x(t) = 0,\ t > 1$.
4. $x(t) = 0,\ t < -1$.

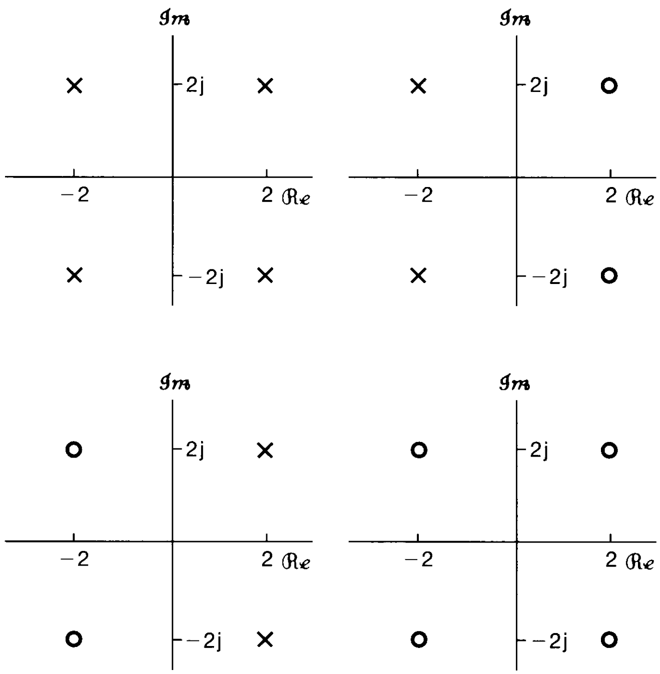


Figure P9.23

9.24. Throughout this problem, we will consider the region of convergence of the Laplace transforms always to include the $j\omega$-axis.

(a) Consider a signal $x(t)$ with Fourier transform $X(j\omega)$ and Laplace transform $X(s) = s + 1/2$. Draw the pole-zero plot for $X(s)$. Also, draw the vector whose length represents $|X(j\omega)|$ and whose angle with respect to the real axis represents $\sphericalangle X(j\omega)$ for a given ω.

(b) By examining the pole-zero plot and vector diagram in part (a), determine a different Laplace transform $X_1(s)$ corresponding to the function of time, $x_1(t)$, so that

$$|X_1(j\omega)| = |X(j\omega)|,$$

but

$$x_1(t) \neq x(t).$$

Show the pole-zero plot and associated vectors that represent $X_1(j\omega)$.

(c) For your answer in part (b), determine, again by examining the related vector diagrams, the relationship between $\sphericalangle X(j\omega)$ and $\sphericalangle X_1(j\omega)$.

(d) Determine a Laplace transform $X_2(s)$ such that

$$\sphericalangle X_2(j\omega) = \sphericalangle X(j\omega),$$

but $x_2(t)$ is not proportional to $x(t)$. Show the pole-zero plot for $X_2(s)$ and the associated vectors that represent $X_2(j\omega)$.

(e) For your answer in part (d), determine the relationship between $|X_2(j\omega)|$ and $|X(j\omega)|$.

(f) Consider a signal $x(t)$ with Laplace transform $X(s)$ for which the pole-zero plot is shown in Figure P9.24. Determine $X_1(s)$ such that $|X(j\omega)| = |X_1(j\omega)|$ and all poles and zeros of $X_1(s)$ are in the left-half of the s-plane [i.e., $\mathcal{R}e\{s\} < 0$]. Also, determine $X_2(s)$ such that $\sphericalangle X(j\omega) = \sphericalangle X_2(j\omega)$ and all poles and zeros of $X_2(s)$ are in the left-half of the s-plane.

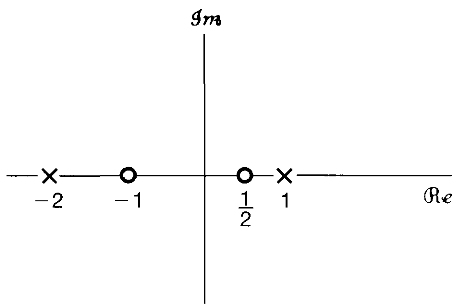


Figure P9.24

9.25. By considering the geometric determination of the Fourier transform, as developed in Section 9.4, sketch, for each of the pole-zero plots in Figure P9.25, the magnitude of the associated Fourier transform.

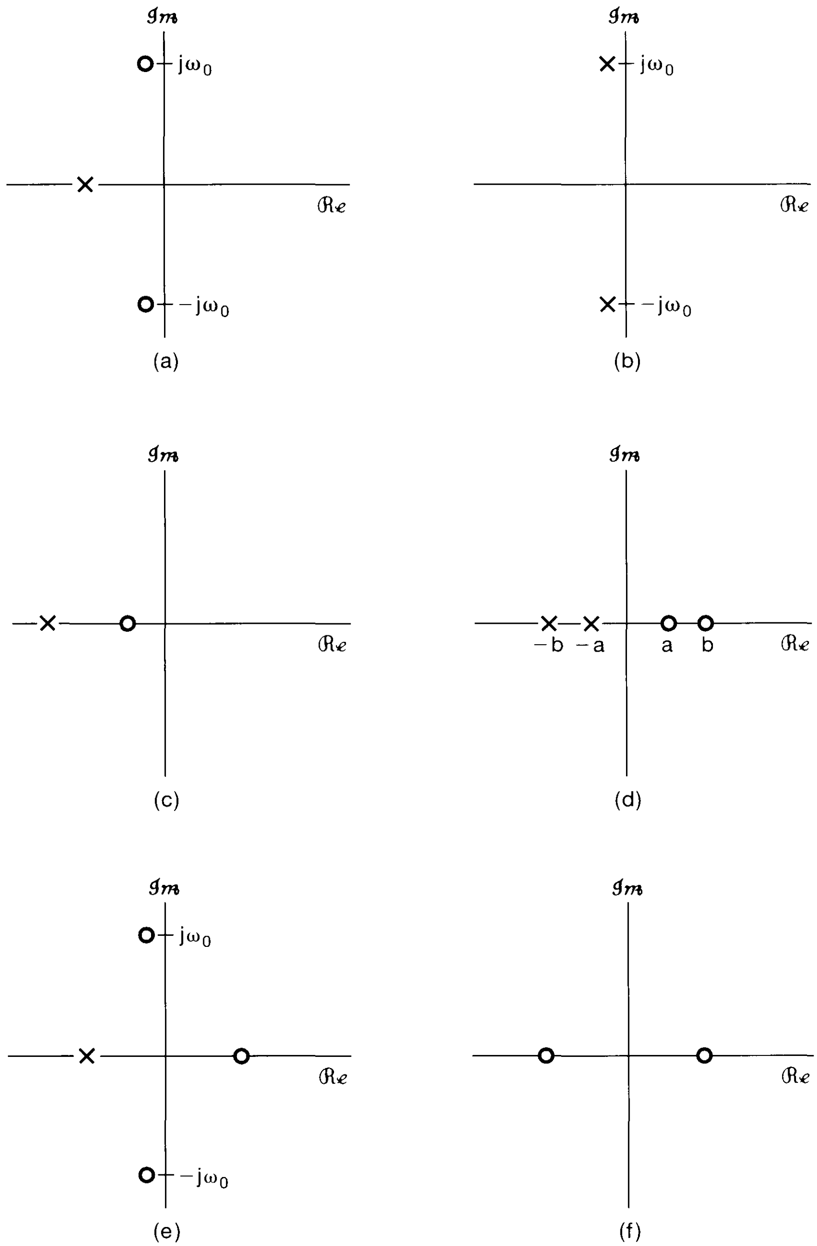


Figure P9.25

9.26. Consider a signal $y(t)$ which is related to two signals $x_1(t)$ and $x_2(t)$ by

$$y(t) = x_1(t-2) * x_2(-t+3)$$

where

$$x_1(t) = e^{-2t}u(t) \qquad \text{and} \qquad x_2(t) = e^{-3t}u(t).$$

Given that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > a,$$

use properties of the Laplace transform to determine the Laplace transform $Y(s)$ of $y(t)$.

9.27. We are given the following five facts about a real signal $x(t)$ with Laplace transform $X(s)$:

1. $X(s)$ has exactly two poles.
2. $X(s)$ has no zeros in the finite s-plane.
3. $X(s)$ has a pole at $s = -1 + j$.
4. $e^{2t}x(t)$ is not absolutely integrable.
5. $X(0) = 8$.

Determine $X(s)$ and specify its region of convergence.

9.28. Consider an LTI system for which the system function $H(s)$ has the pole-zero pattern shown in Figure P9.28.

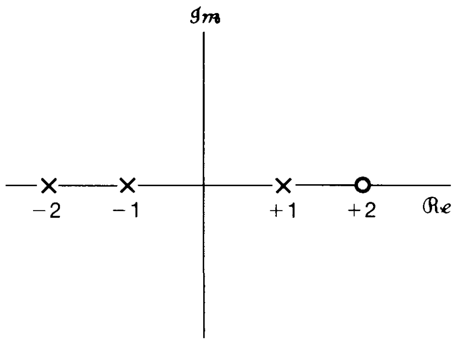


Figure P9.28

(a) Indicate all possible ROCs that can be associated with this pole-zero pattern.
(b) For each ROC identified in part (a), specify whether the associated system is stable and/or causal.

9.29. Consider an LTI system with input $x(t) = e^{-t}u(t)$ and impulse response $h(t) = e^{-2t}u(t)$.

(a) Determine the Laplace transforms of $x(t)$ and $h(t)$.
(b) Using the convolution property, determine the Laplace transform $Y(s)$ of the output $y(t)$.
(c) From the Laplace transform of $y(t)$ as obtained in part (b), determine $y(t)$.
(d) Verify your result in part (c) by explicitly convolving $x(t)$ and $h(t)$.

9.30. A pressure gauge that can be modeled as an LTI system has a time response to a unit step input given by $(1 - e^{-t} - te^{-t})u(t)$. For a certain input $x(t)$, the output is observed to be $(2 - 3e^{-t} + e^{-3t})u(t)$.

For this observed measurement, determine the true pressure input to the gauge as a function of time.

9.31. Consider a continuous-time LTI system for which the input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t).$$

Let $X(s)$ and $Y(s)$ denote Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system impulse response.

(a) Determine $H(s)$ as a ratio of two polynomials in s. Sketch the pole-zero pattern of $H(s)$.

(b) Determine $h(t)$ for each of the following cases:

1. The system is stable.
2. The system is causal.
3. The system is *neither* stable *nor* causal.

9.32. A causal LTI system with impulse response $h(t)$ has the following properties:

1. When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = (1/6)e^{2t}$ for all t.
2. The impulse response $h(t)$ satisfies the differential equation

$$\frac{dh(t)}{dt} + 2h(t) = (e^{-4t})u(t) + bu(t),$$

where b is an unknown constant.

Determine the system function $H(s)$ of the system, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should *not* appear in the answer.

9.33. The system function of a causal LTI system is

$$H(s) = \frac{s+1}{s^2+2s+2}.$$

Determine and sketch the response $y(t)$ when the input is

$$x(t) = e^{-|t|}, \qquad -\infty < t < \infty.$$

9.34. Suppose we are given the following information about a causal and stable LTI system S with impulse response $h(t)$ and a rational system function $H(s)$:

1. $H(1) = 0.2$.
2. When the input is $u(t)$, the output is absolutely integrable.
3. When the input is $tu(t)$, the output is not absolutely integrable.
4. The signal $d^2h(t)/dt^2 + 2\,dh(t)/dt + 2h(t)$ is of finite duration.
5. $H(s)$ has exactly one zero at infinity.

Determine $H(s)$ and its region of convergence.

9.35. The input $x(t)$ and output $y(t)$ of a causal LTI system are related through the block-diagram representation shown in Figure P9.35.

(a) Determine a differential equation relating $y(t)$ and $x(t)$.

(b) Is this system stable?

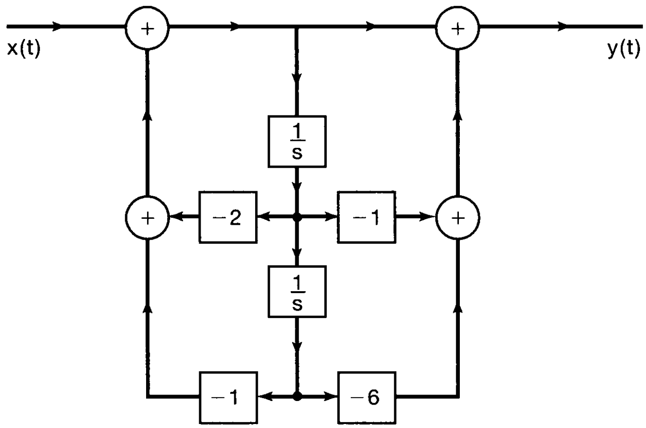


Figure P9.35

9.36. In this problem, we consider the construction of various types of block diagram representations for a causal LTI system S with input $x(t)$, output $y(t)$, and system function

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2}.$$

To derive the direct-form block diagram representation of S, we first consider a causal LTI system S_1 that has the same input $x(t)$ as S, but whose system function is

$$H_1(s) = \frac{1}{s^2 + 3s + 2}.$$

With the output of S_1 denoted by $y_1(t)$, the direct-form block diagram representation of S_1 is shown in Figure P9.36. The signals $e(t)$ and $f(t)$ indicated in the figure represent respective inputs into the two integrators.

(a) Express $y(t)$ (the output of S) as a linear combination of $y_1(t)$, $dy_1(t)/dt$, and $d^2y_1(t)/dt^2$.

(b) How is $dy_1(t)/dt$ related to $f(t)$?

(c) How is $d^2y_1(t)/dt^2$ related to $e(t)$?

(d) Express $y(t)$ as a linear combination of $e(t)$, $f(t)$, and $y_1(t)$.

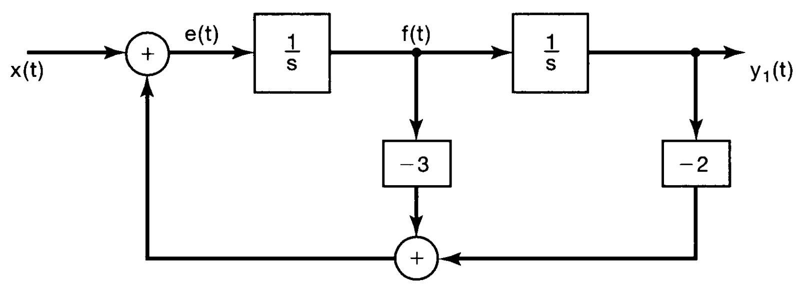


Figure P9.36

(e) Use the result from the previous part to extend the direct-form block diagram representation of S_1 and create a block diagram representation of S.

(f) Observing that

$$H(s) = \left(\frac{2(s-1)}{s+2}\right)\left(\frac{s+3}{s+1}\right),$$

draw a block diagram representation for S as a cascade combination of two subsystems.

(g) Observing that

$$H(s) = 2 + \frac{6}{s+2} - \frac{8}{s+1},$$

draw a block-diagram representation for S as a parallel combination of three subsystems.

9.37. Draw a direct-form representation for the causal LTI systems with the following system functions:

(a) $H_1(s) = \frac{s+1}{s^2+5s+6}$ **(b)** $H_2(s) = \frac{s^2-5s+6}{s^2+7s+10}$ **(c)** $H_3(s) = \frac{s}{(s+2)^2}$

9.38. Consider a fourth-order causal LTI system S whose system function is specified as

$$H(s) = \frac{1}{(s^2 - s + 1)(s^2 + 2s + 1)}.$$

(a) Show that a block diagram representation for S consisting of a cascade of four first-order sections will contain multiplications by coefficients that are not purely real.

(b) Draw a block diagram representation for S as a *cascade* interconnection of two second-order systems, each of which is represented in direct form. There should be no multiplications by nonreal coefficients in the resulting block diagram.

(c) Draw a block diagram representation for S as a *parallel* interconnection of two second-order systems, each of which is represented in direct form. There should be no multiplications by nonreal coefficients in the resulting block diagram.

9.39. Let

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = e^{-3(t+1)}u(t+1).$$

(a) Determine the unilateral Laplace transform $\mathcal{X}_1(s)$ and the bilateral Laplace transform $X_1(s)$ for the signal $x_1(t)$.

(b) Determine the unilateral Laplace transform $\mathcal{X}_2(s)$ and the bilateral Laplace transform $X_2(s)$ for the signal $x_2(t)$.

(c) Take the inverse bilateral Laplace transform of the product $X_1(s)X_2(s)$ to determine the signal $g(t) = x_1(t) * x_2(t)$.

(d) Show that the inverse unilateral Laplace transform of the product $\mathcal{X}_1(s)\mathcal{X}_2(s)$ is not the same as $g(t)$ for $t > 0^-$.

9.40. Consider the system S characterized by the differential equation

$$\frac{d^3y(t)}{dt^3} + 6\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 6y(t) = x(t).$$

(a) Determine the zero-state response of this system for the input $x(t) = e^{-4t}u(t)$.
(b) Determine the zero-input response of the system for $t > 0^-$, given that

$$y(0^-) = 1, \qquad \left.\frac{dy(t)}{dt}\right|_{t=0^-} = -1, \qquad \left.\frac{d^2y(t)}{dt^2}\right|_{t=0^-} = 1.$$

(c) Determine the output of S when the input is $x(t) = e^{-4t}u(t)$ and the initial conditions are the same as those specified in part (b).

ADVANCED PROBLEMS

9.41. **(a)** Show that, if $x(t)$ is an even function, so that $x(t) = x(-t)$, then $X(s) = X(-s)$.
(b) Show that, if $x(t)$ is an odd function, so that $x(t) = -x(-t)$, then $X(s) = -X(-s)$.
(c) Determine which, if any, of the pole-zero plots in Figure P9.41 could correspond to an even function of time. For those that could, indicate the required ROC.

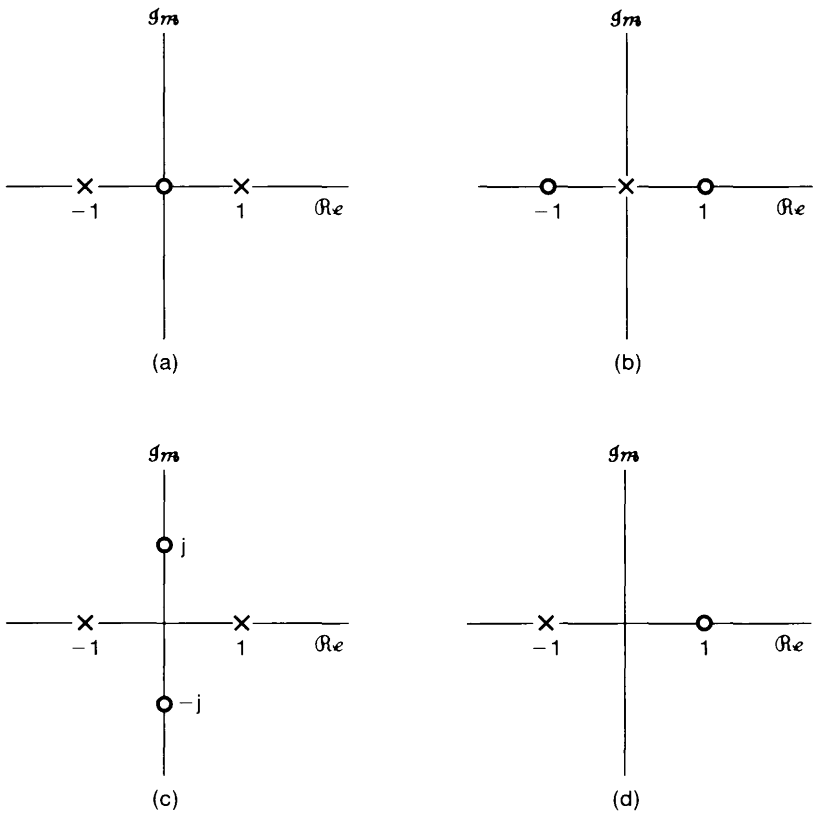


Figure P9.41

9.42. Determine whether each of the following statements is true or false. If a statement is true, construct a convincing argument for it. If it is false, give a counterexample.

(a) The Laplace transform of $t^2u(t)$ does not converge anywhere on the s-plane.

(b) The Laplace transform of $e^{t^2}u(t)$ does not converge anywhere on the s-plane.

(c) The Laplace transform of $e^{j\omega_0 t}$ does not converge anywhere on the s-plane.

(d) The Laplace transform of $e^{j\omega_0 t}u(t)$ does not converge anywhere on the s-plane.

(e) The Laplace transform of $|t|$ does not converge anywhere on the s-plane.

9.43. Let $h(t)$ be the impulse response of a causal and stable LTI system with a rational system function.

(a) Is the system with impulse response $dh(t)/dt$ guaranteed to be causal and stable?

(b) Is the system with impulse response $\int_{-\infty}^{t} h(\tau)\,d\tau$ guaranteed to be causal and unstable?

9.44. Let $x(t)$ be the sampled signal specified as

$$x(t) = \sum_{n=0}^{\infty} e^{-nT}\,\delta(t - nT),$$

where $T > 0$.

(a) Determine $X(s)$, including its region of convergence.

(b) Sketch the pole-zero plot for $X(s)$.

(c) Use geometric interpretation of the pole-zero plot to argue that $X(j\omega)$ is periodic.

9.45. Consider the LTI system shown in Figure P9.45(a) for which we are given the following information:

$$X(s) = \frac{s+2}{s-2},$$

$$x(t) = 0, \qquad t > 0,$$

and

$$y(t) = -\frac{2}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(t). \qquad \text{[See Figure P9.45(b).]}$$

(a) Determine $H(s)$ and its region of convergence.

(b) Determine $h(t)$.

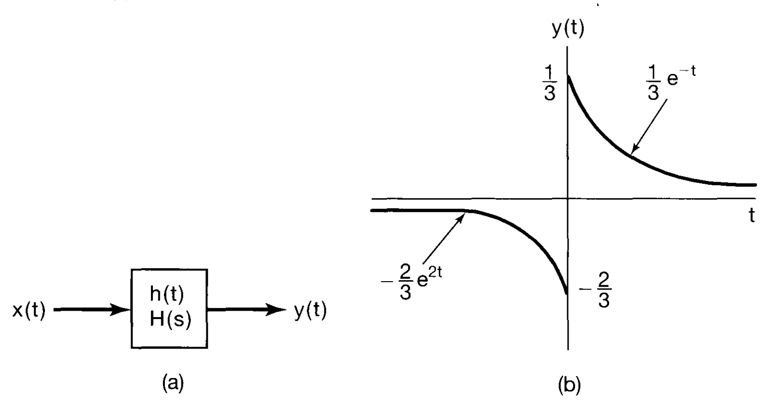


Figure P9.45

(c) Using the system function $H(s)$ found in part (a), determine the output $y(t)$ if the input is

$$x(t) = e^{3t}, \qquad -\infty < t < +\infty.$$

9.46. Let $H(s)$ represent the system function for a causal, stable system. The input to the system consists of the sum of three terms, one of which is an impulse $\delta(t)$ and another a complex exponential of the form $e^{s_0 t}$, where s_0 is a complex constant. The output is

$$y(t) = -6e^{-t}u(t) + \frac{4}{34}e^{4t}\cos 3t + \frac{18}{34}e^{4t}\sin 3t + \delta(t).$$

Determine $H(s)$, consistently with this information.

9.47. The signal

$$y(t) = e^{-2t}u(t)$$

is the output of a causal all-pass system for which the system function is

$$H(s) = \frac{s-1}{s+1}.$$

(a) Find and sketch at least two possible inputs $x(t)$ that could produce $y(t)$.

(b) What is the input $x(t)$ if it is known that

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty?$$

(c) What is the input $x(t)$ if it is known that a stable (but not necessarily causal) system exists that will have $x(t)$ as an output if $y(t)$ is the input? Find the impulse response $h(t)$ of this filter, and show by direct convolution that it has the property claimed [i.e., that $y(t) * h(t) = x(t)$].

9.48. The *inverse* of an LTI system $H(s)$ is defined as a system that, when cascaded with $H(s)$, results in an overall transfer function of unity or, equivalently, an overall impulse response that is an impulse.

(a) If $H_1(s)$ denotes the transfer function of an inverse system for $H(s)$, determine the general algebraic relationship between $H(s)$ and $H_1(s)$.

(b) Shown in Figure P9.48 is the pole-zero plot for a stable, causal system $H(s)$. Determine the pole-zero plot for the associated inverse system.

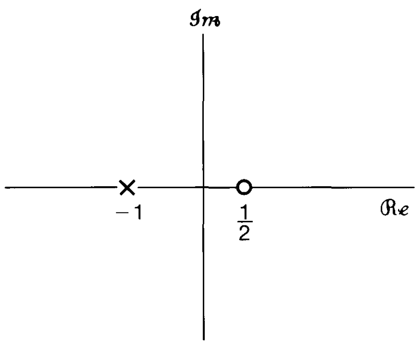


Figure P9.48

9.49. A class of systems, referred to as minimum-delay or minimum-phase systems, is sometimes defined through the statement that these systems are causal and stable and that the inverse systems are also causal and stable.

Based on the preceding definition, develop an argument to demonstrate that all poles *and* zeros of the transfer function of a minimum-delay system must be in the left half of the s-plane [i.e., $\mathcal{R}e\{s\} < 0$].

9.50. Determine whether or not each of the following statements about LTI systems is true. If a statement is true, construct a convincing argument for it. If it is false, give a counterexample.

(a) A stable continuous-time system must have all its poles in the left half of the s-plane [i.e., $\mathcal{R}e\{s\} < 0$].

(b) If a system function has more poles than zeros, and the system is causal, the step response will be continuous at $t = 0$.

(c) If a system function has more poles than zeros, and the system is not restricted to be causal, the step response can be discontinuous at $t = 0$.

(d) A stable, causal system must have all its poles and zeros in the left half of the s-plane.

9.51. Consider a stable and causal system with a real impulse response $h(t)$ and system function $H(s)$. It is known that $H(s)$ is rational, one of its poles is at $-1 + j$, one of its zeros is at $3 + j$, and it has exactly two zeros at infinity. For each of the following statements, determine whether it is true, whether it is false, or whether there is insufficient information to determine the statement's truth.

(a) $h(t)e^{-3t}$ is absolutely integrable.

(b) The ROC for $H(s)$ is $\mathcal{R}e\{s\} > -1$.

(c) The differential equation relating inputs $x(t)$ and outputs $y(t)$ for S may be written in a form having only real coefficients.

(d) $\lim_{s \to \infty} H(s) = 1$.

(e) $H(s)$ does not have fewer than four poles.

(f) $H(j\omega) = 0$ for at least one finite value of ω.

(g) If the input to S is $e^{3t} \sin t$, the output is $e^{3t} \cos t$.

9.52. As indicated in Section 9.5, many of the properties of the Laplace transform and their derivation are analogous to corresponding properties of the Fourier transform and their derivation, as developed in Chapter 4. In this problem, you are asked to outline the derivation of a number of the Laplace transform properties.

Observing the derivation for the corresponding property in Chapter 4 for the Fourier transform, derive each of the following Laplace transform properties. Your derivation must include a consideration of the region of convergence.

(a) Time shifting (Section 9.5.2)

(b) Shifting in the s-domain (Section 9.5.3)

(c) Time scaling (Section 9.5.4)

(d) Convolution property (Section 9.5.6)

9.53. As presented in Section 9.5.10, the initial-value theorem states that, for a signal $x(t)$ with Laplace transform $X(s)$ and for which $x(t) = 0$ for $t < 0$, the initial value of $x(t)$ [i.e., $x(0+)$] can be obtained from $X(s)$ through the relation

$$x(0+) = \lim_{s \to \infty} sX(s). \qquad \text{[eq. (9.110)]}$$

First, we note that, since $x(t) = 0$ for $t < 0$, $x(t) = x(t)u(t)$. Next, expanding $x(t)$ as a Taylor series at $t = 0+$, we obtain

$$x(t) = \left[x(0+) + x^{(1)}(0+)t + \cdots + x^{(n)}(0+)\frac{t^n}{n!} + \cdots\right]u(t), \qquad \text{(P9.53–1)}$$

where $x^{(n)}(0+)$ denotes the nth derivative of $x(t)$ evaluated at $t = 0+$.

(a) Determine the Laplace transform of an arbitrary term $x^{(n)}(0+)(t^n/n!)u(t)$ on the right-hand side of eq. (P9.53–1). (You may find it helpful to review Example 9.14.)

(b) From your result in part (a) and the expansion in eq. (P9.53–1), show that $X(s)$ can be expressed as

$$X(s) = \sum_{n=0}^{\infty} x^{(n)}(0+)\frac{1}{s^{n+1}}.$$

(c) Demonstrate that eq. (9.110) follows from the result of part (b).

(d) By first determining $x(t)$, verify the initial-value theorem for each of the following examples:

(1) $X(s) = \frac{1}{s+2}$

(2) $X(s) = \frac{s+1}{(s+2)(s+3)}$

(e) A more general form of the initial-value theorem states that if $x^{(n)}(0+) = 0$ for $n < N$, then $x^{(N)}(0+) = \lim_{s\to\infty} s^{N+1}X(s)$. Demonstrate that this more general statement also follows from the result in part (b).

9.54. Consider a real-valued signal $x(t)$ with Laplace transform $X(s)$.

(a) By applying complex conjugation to both sides of eq. (9.56), show that $X(s) = X^*(s^*)$.

(b) From your result in (a), show that if $X(s)$ has a pole (zero) at $s = s_0$, it must also have a pole (zero) at $s = s_0^*$; i.e., for $x(t)$ real, the poles and zeros of $X(s)$ that are not on the real axis must occur in complex conjugate pairs.

9.55. In Section 9.6, Table 9.2, we listed a number of Laplace transform pairs, and we indicated specifically how transform pairs 1 through 9 follow from Examples 9.1 and 9.14, together with various properties from Table 9.1.

By exploiting properties from Table 9.1, show how transform pairs 10 through 16 follow from transform pairs 1 through 9 in Table 9.2.

9.56. The Laplace transform is said to exist for a specific complex s if the magnitude of the transform is finite—that is, if $|X(s)| < \infty$.

Show that a *sufficient* condition for the existence of the transform $X(s)$ at $s = s_0 = \sigma_0 + j\omega_0$ is that

$$\int_{-\infty}^{+\infty} |x(t)|e^{-\sigma_0 t}\,dt < \infty.$$

In other words, show that $x(t)$ exponentially weighted by $e^{-\sigma_0 t}$ is absolutely integrable. You will need to use the result that, for a complex function $f(t)$,

$$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt. \qquad \text{(P9.56–1)}$$

Without rigorously proving eq. (P9.56–1), argue its plausibility.

9.57. The Laplace transform $X(s)$ of a signal $x(t)$ has four poles and an unknown number of zeros. The signal $x(t)$ is known to have an impulse at $t = 0$. Determine what information, if any, this provides about the number of zeros and their locations.

9.58. Let $h(t)$ be the impulse response of a causal and stable LTI system with rational system function $H(s)$. Show that $g(t) = \mathcal{R}e\{h(t)\}$ is also the impulse response of a causal and stable system.

9.59. If $\mathcal{X}(s)$ denotes the unilateral Laplace transform of $x(t)$, determine, in terms of $\mathcal{X}(s)$, the unilateral Laplace transform of:

(a) $x(t-1)$ **(b)** $x(t+1)$

(c) $\int_{-\infty}^{\infty} x(\tau)\,d\tau$ **(d)** $\frac{d^3x(t)}{dt^3}$

EXTENSION PROBLEMS

9.60. In long-distance telephone communication, an echo is sometimes encountered due to the transmitted signal being reflected at the receiver, sent back down the line, reflected again at the transmitter, and returned to the receiver. The impulse response for a system that models this effect is shown in Figure P9.60, where we have assumed that only one echo is received. The parameter T corresponds to the one-way travel time along the communication channel, and the parameter α represents the attenuation in amplitude between transmitter and receiver.

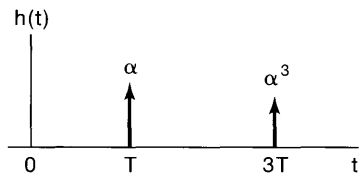


Figure P9.60

(a) Determine the system function $H(s)$ and associated region of convergence for the system.

(b) From your result in part (a), you should observe that $H(s)$ does not consist of a ratio of polynomials. Nevertheless, it is useful to represent it in terms of poles and zeros, where, as usual, the zeros are the values of s for which $H(s) = 0$

and the poles are the values of s for which $1/H(s) = 0$. For the system function found in part (a), determine the zeros and demonstrate that there are no poles.

(c) From your result in part (b), sketch the pole-zero plot for $H(s)$.

(d) By considering the appropriate vectors in the s-plane, sketch the magnitude of the frequency response of the system.

9.61. The autocorrelation function of a signal $x(t)$ is defined as

$$\phi_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)\,dt.$$

(a) Determine, in terms of $x(t)$, the impulse response $h(t)$ of an LTI system for which, when the input is $x(t)$, the output is $\phi_{xx}(t)$ [Figure P9.61(a)].

(b) From your answer in part (a), determine $\Phi_{xx}(s)$, the Laplace transform of $\phi_{xx}(\tau)$ in terms of $X(s)$. Also, express $\Phi_{xx}(j\omega)$, the Fourier transform of $\phi_{xx}(\tau)$, in terms of $X(j\omega)$.

(c) If $X(s)$ has the pole-zero pattern and ROC shown in Figure P9.61(b), sketch the pole-zero pattern and indicate the ROC for $\Phi_{xx}(s)$.

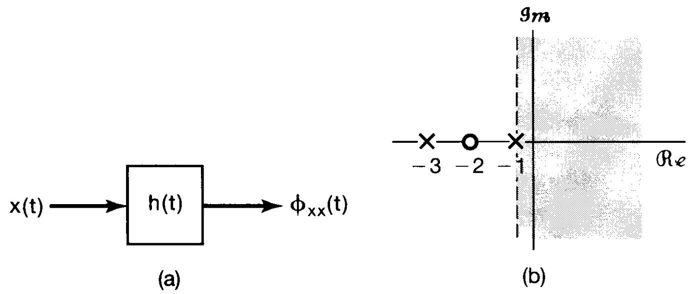


Figure P9.61

9.62. In a number of applications in signal design and analysis, the class of signals

$$\phi_n(t) = e^{-t/2}L_n(t)u(t), \quad n = 0, 1, 2, \ldots, \tag{P9.62–1}$$

where

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \tag{P9.62–2}$$

is encountered.

(a) The functions $L_n(t)$ are referred to as *Laguerre polynomials*. To verify that they in fact have the form of polynomials, determine $L_0(t)$, $L_1(t)$, and $L_2(t)$ explicitly.

(b) Using the properties of the Laplace transform in Table 9.1 and Laplace transform pairs in Table 9.2, determine the Laplace transform $\Phi_n(s)$ of $\phi_n(t)$.

(c) The set of signals $\phi_n(t)$ can be generated by exciting a network of the form in Figure P9.62 with an impulse. From your result in part (b), determine $H_1(s)$ and $H_2(s)$ so that the impulse responses along the cascade chain are the signals $\phi_n(t)$ as indicated.

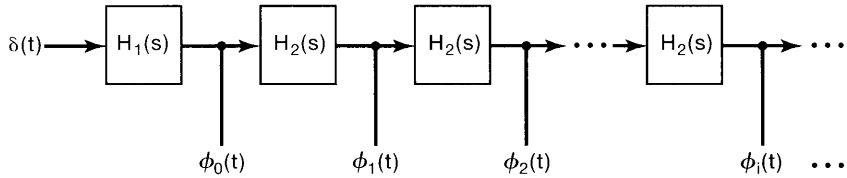


Figure P9.62

9.63. In filter design, it is often possible and convenient to transform a lowpass filter to a highpass filter and vice versa. With $H(s)$ denoting the transfer function of the original filter and $G(s)$ that of the transformed filter, one such commonly used transformation consists of replacing s by $1/s$; that is,

$$G(s) = H\left(\frac{1}{s}\right).$$

(a) For $H(s) = 1/(s + 1/2)$, sketch $|H(j\omega)|$ and $|G(j\omega)|$.

(b) Determine the linear constant-coefficient differential equation associated with $H(s)$ and with $G(s)$.

(c) Now consider a more general case in which $H(s)$ is the transfer function associated with the linear constant-coefficient differential equation in the general form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}. \qquad \text{(P9.63–1)}$$

Without any loss of generality, we have assumed that the number of derivatives N is the same on both sides of the equation, although in any particular case, some of the coefficients may be zero. Determine $H(s)$ and $G(s)$.

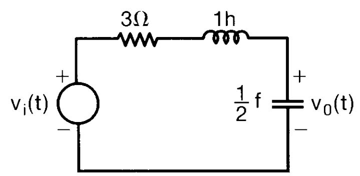
(d) From your result in part (c), determine, in terms of the coefficients in eq. (P9.63–1), the linear constant-coefficient differential equation associated with $G(s)$.

9.64. Consider the *RLC* circuit shown in Figure 9.27 with input $x(t)$ and output $y(t)$.

(a) Show that if R, L, and C are all positive, then this LTI system is stable.

(b) How should R, L, and C be related to each other so that the system represents a second-order Butterworth filter?

9.65. **(a)** Determine the differential equation relating $v_i(t)$ and $v_0(t)$ for the *RLC* circuit of Figure P9.65.



$v_0(0+) = 1$

$\left.\frac{dv_0(t)}{dt}\right|_{t=0+} = 2$

Figure P9.65

(b) Suppose that $v_i(t) = e^{-3t}u(t)$. Using the unilateral Laplace transform, determine $v_o(t)$ for $t > 0$.

9.66. Consider the RL circuit shown in Figure P9.66. Assume that the current $i(t)$ has reached a steady state with the switch at position A. At time $t = 0$, the switch is moved from position A to position B.

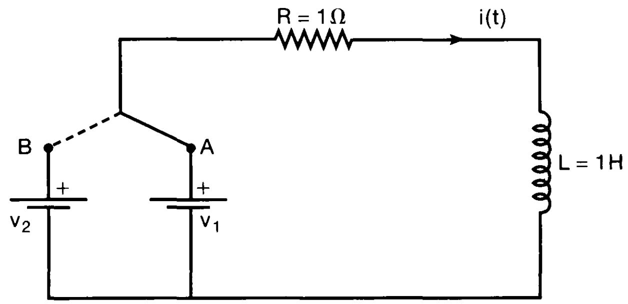


Figure P9.66

(a) Find the differential equation relating $i(t)$ and v_2 for $t > 0^-$. Specify the initial condition (i.e., the value of $i(0^-)$) for this differential equation in terms of v_1.

(b) Using the properties of the unilateral Laplace transform in Table 9.3, determine and plot the current $i(t)$ for each of the following values of v_1 and v_2:

(i) $v_1 = 0\ V, v_2 = 2\ V$
(ii) $v_1 = 4\ V, v_2 = 0\ V$
(iii) $v_1 = 4\ V, v_2 = 2\ V$

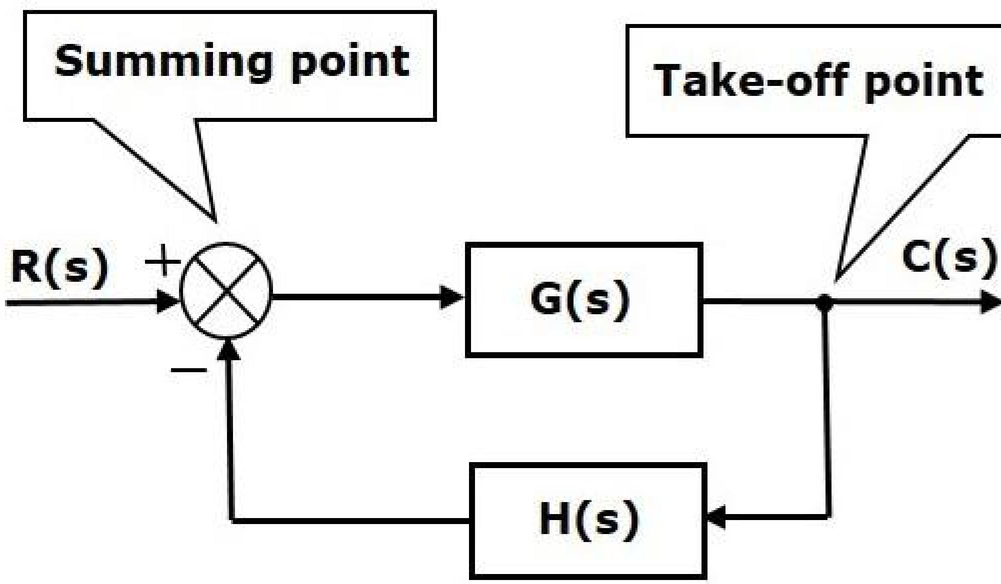
Using your answers for (i), (ii), and (iii), argue that the current $i(t)$ may be expressed as a sum of the circuit's zero-state response and zero-input response.

Control Systems - Block Diagrams

Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.

Basic Elements of Block Diagram

The basic elements of a block diagram are a block, the summing point and the take-off point. Let us consider the block diagram of a closed loop control system as shown in the following figure to identify these elements.



The above block diagram consists of two blocks having transfer functions G(s) and H(s). It is also having one summing point and one take-off point. Arrows indicate the direction of the flow of signals. Let us now discuss these elements one by one.

Block

The transfer function of a component is represented by a block. Block has single input and single output.

The following figure shows a block having input X(s), output Y(s) and the transfer function G(s).

Transfer Function, $G(s) = \frac{Y(s)}{X(s)}$

$$\Rightarrow Y(s) = G(s)X(s)$$

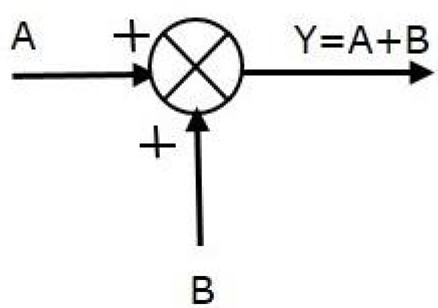
Output of the block is obtained by multiplying transfer function of the block with input.

Summing Point

The summing point is represented with a circle having cross (X) inside it. It has two or more inputs and single output. It produces the algebraic sum of the inputs. It also performs the summation or subtraction or combination of summation and subtraction of the inputs based on the polarity of the inputs. Let us see these three operations one by one.

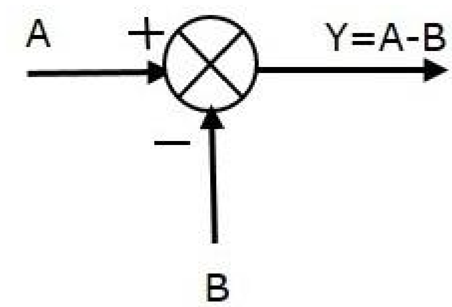
The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B have a positive sign. So, the summing point produces the output, Y as **sum of A and B.**

i.e.,Y = A + B.



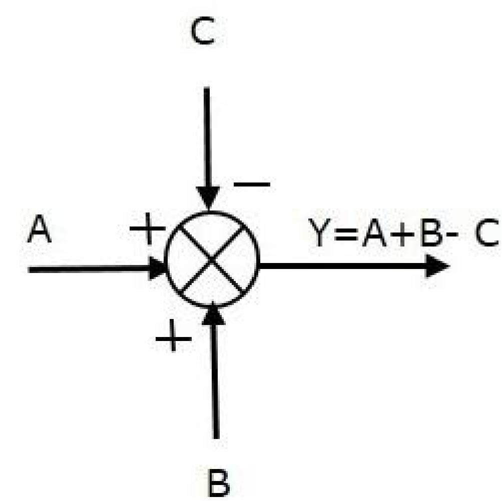
The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B is having negative sign. So, the summing point produces the output **Y** as the **difference of A and B**.

Y = A + (-B) = A - B.



The following figure shows the summing point with three inputs (A, B, C) and one output (Y). Here, the inputs A and B are having positive signs and C is having a negative sign. So, the summing point produces the output **Y** as

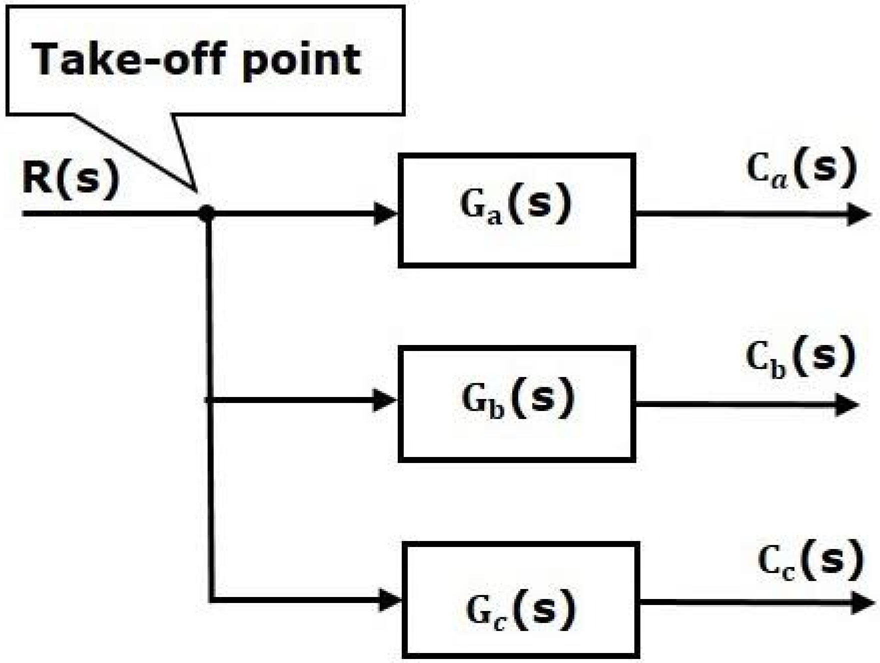
Y = A + B + (−C) = A + B − C.



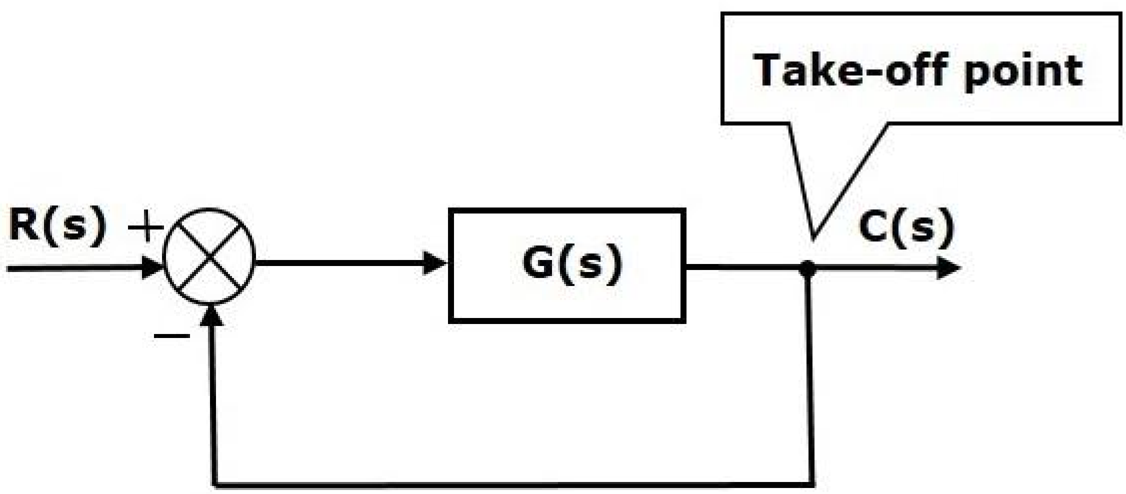
Take-off Point

The take-off point is a point from which the same input signal can be passed through more than one branch. That means with the help of take-off point, we can apply the same input to one or more blocks, summing points.

In the following figure, the take-off point is used to connect the same input, R(s) to two more blocks.



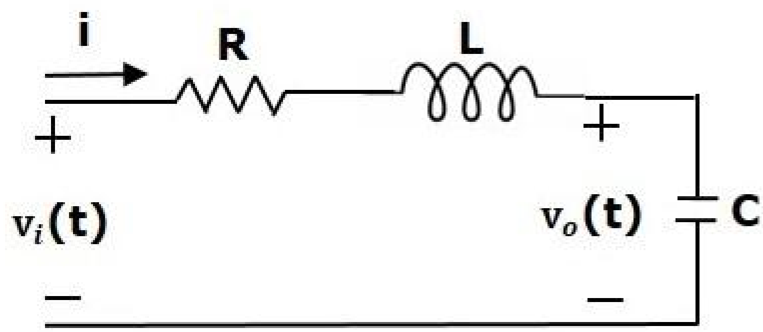
In the following figure, the take-off point is used to connect the output C(s), as one of the inputs to the summing point.



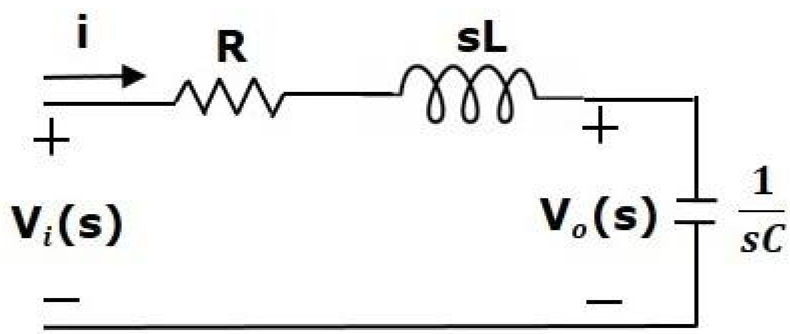
Block Diagram Representation of Electrical Systems

In this section, let us represent an electrical system with a block diagram. Electrical systems contain mainly three basic elements — **resistor, inductor and capacitor**.

Consider a series of RLC circuit as shown in the following figure. Where, $V_i(t)$ and $V_o(t)$ are the input and output voltages. Let i(t) be the current passing through the circuit. This circuit is in time domain.



By applying the Laplace transform to this circuit, will get the circuit in s-domain. The circuit is as shown in the following figure.



From the above circuit, we can write

$$I(s) = \frac{V_i(s) - V_o(s)}{R + sL}$$

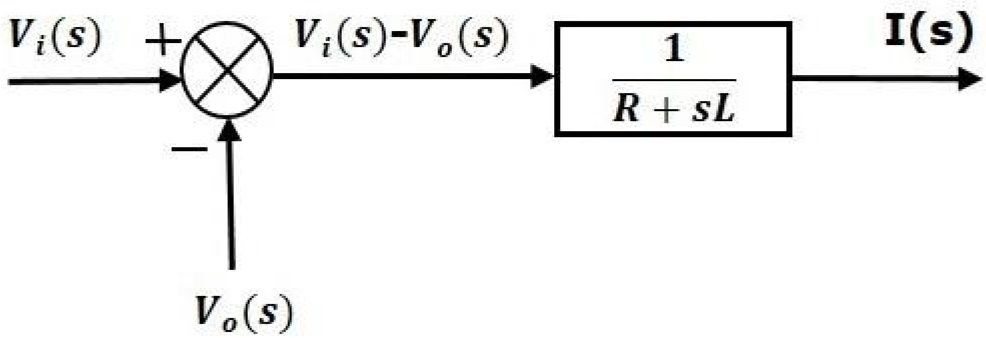
$$\Rightarrow I(s) = \left\{\frac{1}{R+sL}\right\}\{V_i(s) - V_o(s)\} \qquad \textbf{(Equation 1)}$$

$$V_o(s) = \left(\frac{1}{sC}\right) I(s) \qquad \textbf{(Equation 2)}$$

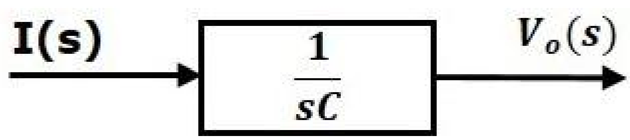
Let us now draw the block diagrams for these two equations individually. And then combine those block diagrams properly in order to get the overall block diagram of series of RLC Circuit (s-domain).

Equation 1 can be implemented with a block having the transfer function, $\frac{1}{R+sL}$. The input and output of this block are $\{V_i(s) - V_o(s)\}$ and $I(s)$. We require a summing point to get

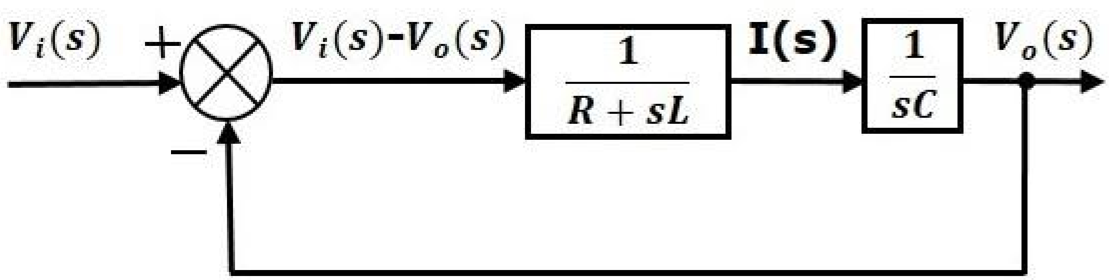
$\{V_i(s) - V_o(s)\}$. The block diagram of Equation 1 is shown in the following figure.



Equation 2 can be implemented with a block having transfer function, $\frac{1}{sC}$. The input and output of this block are $I(s)$ and $V_o(s)$. The block diagram of Equation 2 is shown in the following figure.



The overall block diagram of the series of RLC Circuit (s-domain) is shown in the following figure.



Similarly, you can draw the **block diagram** of any electrical circuit or system just by following this simple procedure.

- Convert the time domain electrical circuit into an s-domain electrical circuit by applying Laplace transform.

- Write down the equations for the current passing through all series branch elements and voltage across all shunt branches.
- Draw the block diagrams for all the above equations individually.
- Combine all these block diagrams properly in order to get the overall block diagram of the electrical circuit (s-domain).

Control Systems - Block Diagram Algebra

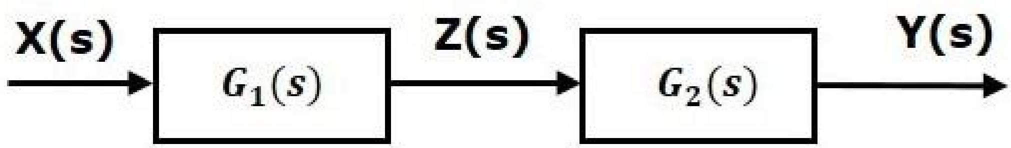
Block diagram algebra is nothing but the algebra involved with the basic elements of the block diagram. This algebra deals with the pictorial representation of algebraic equations.

Basic Connections for Blocks

There are three basic types of connections between two blocks.

Series Connection

Series connection is also called **cascade connection**. In the following figure, two blocks having transfer functions $G_1(s)$ and $G_2(s)$ are connected in series.



For this combination, we will get the output $Y(s)$ as

$$Y(s) = G_2(s)Z(s)$$

Where, $Z(s) = G_1(s)X(s)$

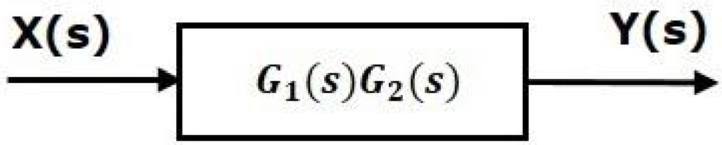
$$\Rightarrow Y(s) = G_2(s)[G_1(s)X(s)] = G_1(s)G_2(s)X(s)$$

$$\Rightarrow Y(s) = \{G_1(s)G_2(s)\}X(s)$$

Compare this equation with the standard form of the output equation, $Y(s) = G(s)X(s)$.

Where, $G(s) = G_1(s)G_2(s)$.

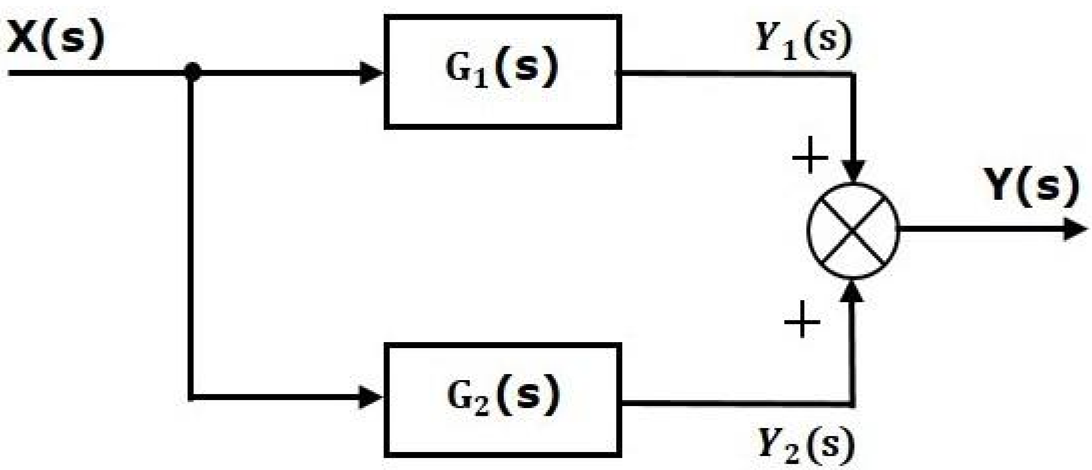
That means we can represent the **series connection** of two blocks with a single block. The transfer function of this single block is the **product of the transfer functions** of those two blocks. The equivalent block diagram is shown below.



Similarly, you can represent series connection of 'n' blocks with a single block. The transfer function of this single block is the product of the transfer functions of all those 'n' blocks.

Parallel Connection

The blocks which are connected in **parallel** will have the **same input**. In the following figure, two blocks having transfer functions $G_1(s)$ and $G_2(s)$ are connected in parallel. The outputs of these two blocks are connected to the summing point.



For this combination, we will get the output $Y(s)$ as

$$Y(s) = Y_1(s) + Y_2(s)$$

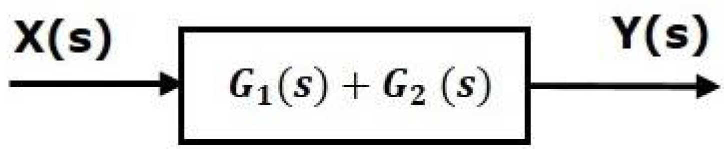
Where, $Y_1(s) = G_1(s)X(s)$ and $Y_2(s) = G_2(s)X(s)$

$$\Rightarrow Y(s) = G_1(s)X(s) + G_2(s)X(s) = \{G_1(s) + G_2(s)\}X(s)$$

Compare this equation with the standard form of the output equation, $Y(s) = G(s)X(s)$.

Where, $G(s) = G_1(s) + G_2(s)$.

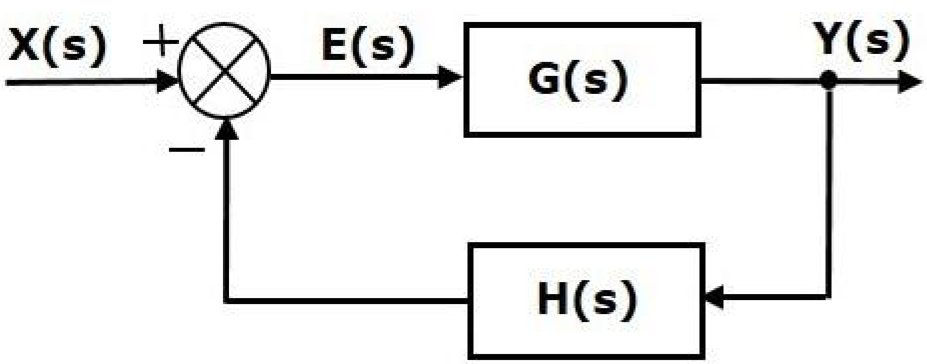
That means we can represent the **parallel connection** of two blocks with a single block. The transfer function of this single block is the **sum of the transfer functions** of those two blocks. The equivalent block diagram is shown below.



Similarly, you can represent parallel connection of 'n' blocks with a single block. The transfer function of this single block is the algebraic sum of the transfer functions of all those 'n' blocks.

Feedback Connection

As we discussed in previous chapters, there are two types of **feedback** — positive feedback and negative feedback. The following figure shows negative feedback control system. Here, two blocks having transfer functions $G(s)$ and $H(s)$ form a closed loop.



The output of the summing point is -

$$E(s) = X(s) - H(s)Y(s)$$

The output $Y(s)$ is -

$$Y(s) = E(s)G(s)$$

Substitute $E(s)$ value in the above equation.

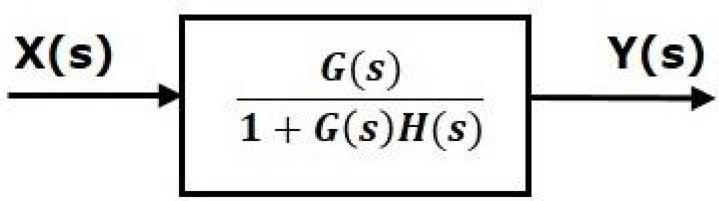
$$Y(s) = \{X(s) - H(s)Y(s)\}G(s)\}$$

$$Y(s)\left\{1 + G(s)H(s)\right\} = X(s)G(s)\}$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Therefore, the negative feedback closed loop transfer function is $\frac{G(s)}{1+G(s)H(s)}$

This means we can represent the negative feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the negative feedback. The equivalent block diagram is shown below.



Similarly, you can represent the positive feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the positive feedback, i.e., $\frac{G(s)}{1-G(s)H(s)}$

Block Diagram Algebra for Summing Points

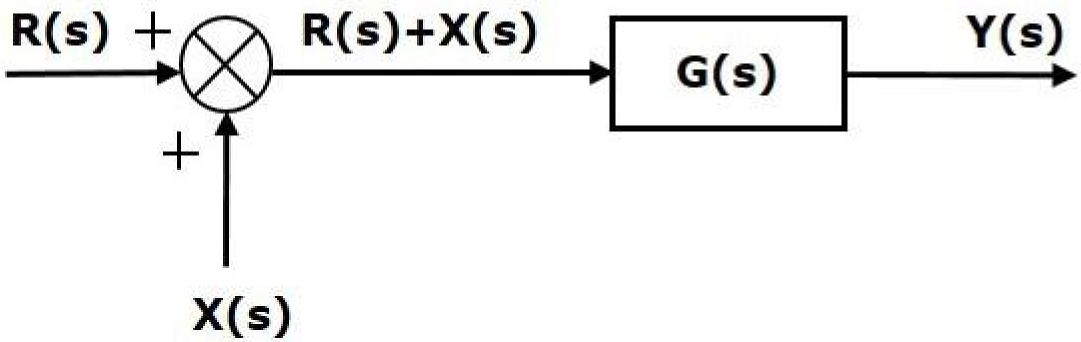
There are two possibilities of shifting summing points with respect to blocks −

- Shifting summing point after the block
- Shifting summing point before the block

Let us now see what kind of arrangements need to be done in the above two cases one by one.

Shifting Summing Point After the Block

Consider the block diagram shown in the following figure. Here, the summing point is present before the block.



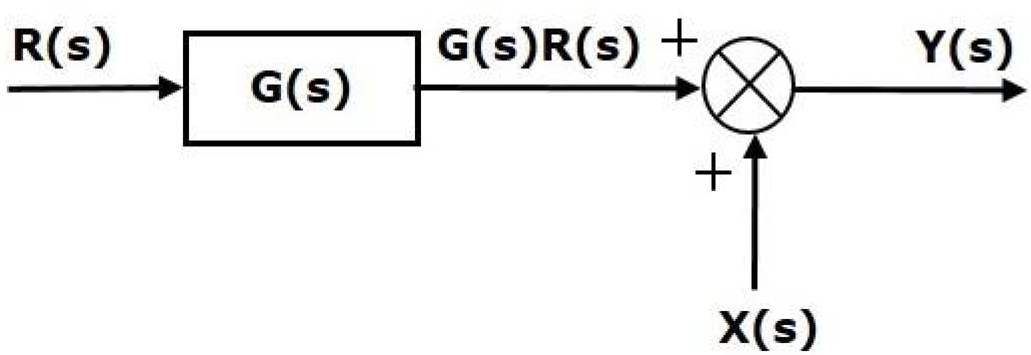
Summing point has two inputs $R(s)$ and $X(s)$. The output of it is $\left\{R(s) + X(s)\right\}$.

So, the input to the block $G(s)$ is $\left\{R(s) + X(s)\right\}$ and the output of it is –

$$Y(s) = G(s)\left\{R(s) + X(s)\right\}$$

$$\Rightarrow Y(s) = G(s)R(s) + G(s)X(s)$$ **(Equation 1)**

Now, shift the summing point after the block. This block diagram is shown in the following figure.



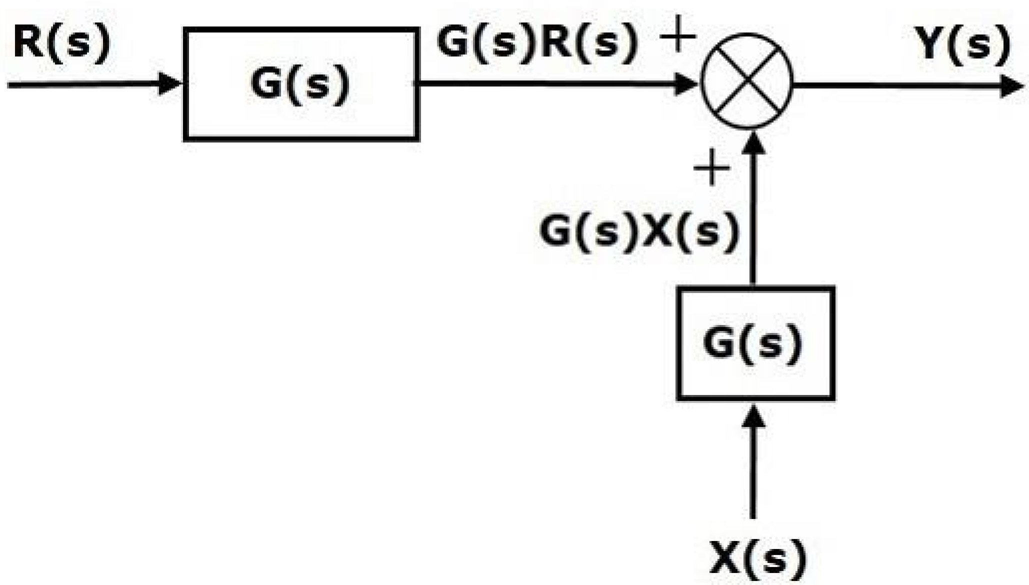
Output of the block $G(s)$ is $G(s)R(s)$.

The output of the summing point is

$$Y(s) = G(s)R(s) + X(s)$$ **(Equation 2)**

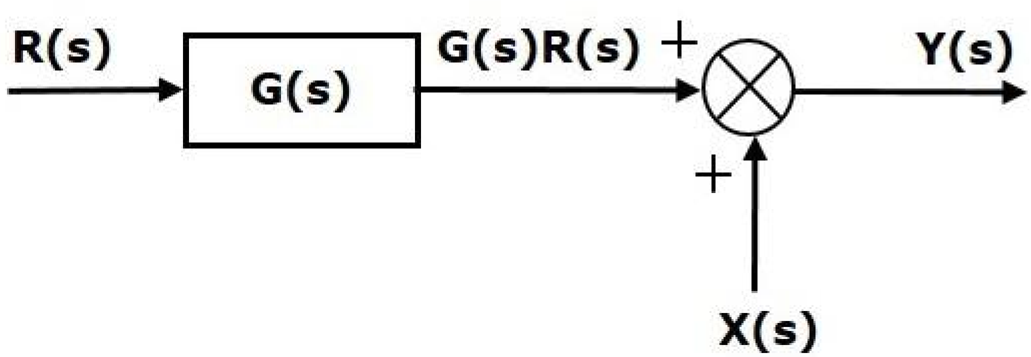
Compare Equation 1 and Equation 2.

The first term $`G(s)R(s)'$ is same in both the equations. But, there is difference in the second term. In order to get the second term also same, we require one more block $G(s)$. It is having the input $X(s)$ and the output of this block is given as input to summing point instead of $X(s)$. This block diagram is shown in the following figure.



Shifting Summing Point Before the Block

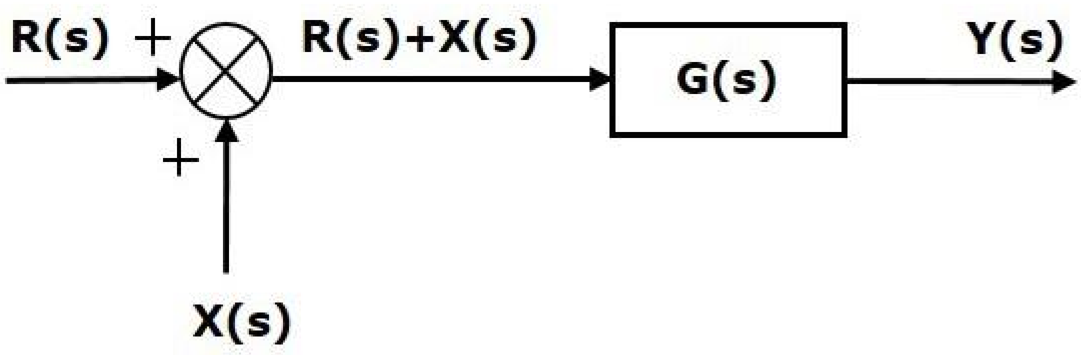
Consider the block diagram shown in the following figure. Here, the summing point is present after the block.



Output of this block diagram is -

$$Y(s) = G(s)R(s) + X(s) \qquad \textbf{(Equation 3)}$$

Now, shift the summing point before the block. This block diagram is shown in the following figure.



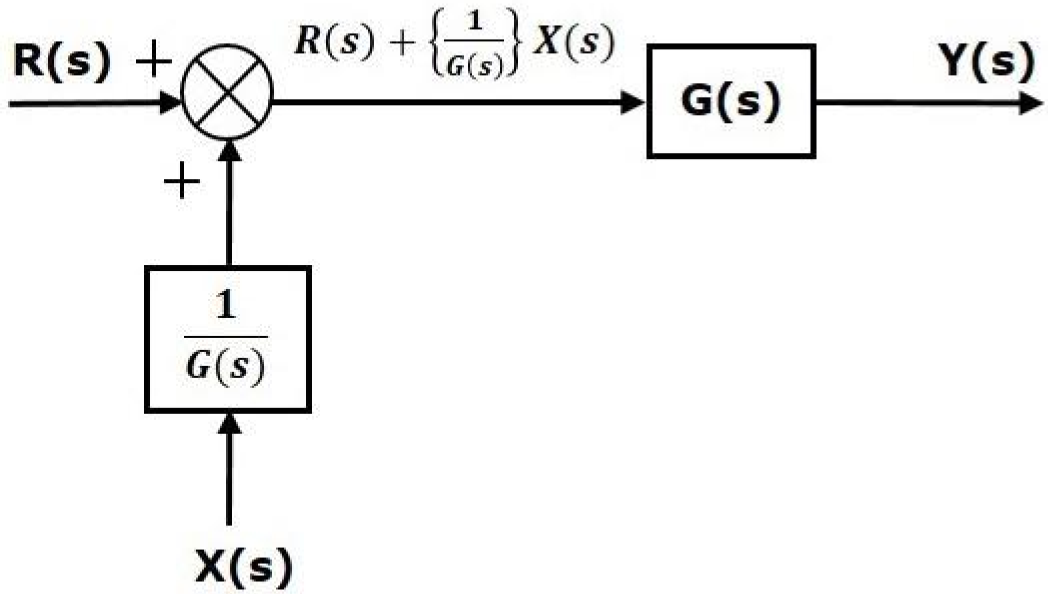
Output of this block diagram is -

$$Y(S) = G(s)R(s) + G(s)X(s) \qquad \textbf{(Equation 4)}$$

Compare Equation 3 and Equation 4,

The first term $`G(s)R(s)'$ is same in both equations. But, there is difference in the second term.

In order to get the second term also same, we require one more block $\frac{1}{G(s)}$. It is having the input $X(s)$ and the output of this block is given as input to summing point instead of $X(s)$. This block diagram is shown in the following figure.



Block Diagram Algebra for Take-off Points

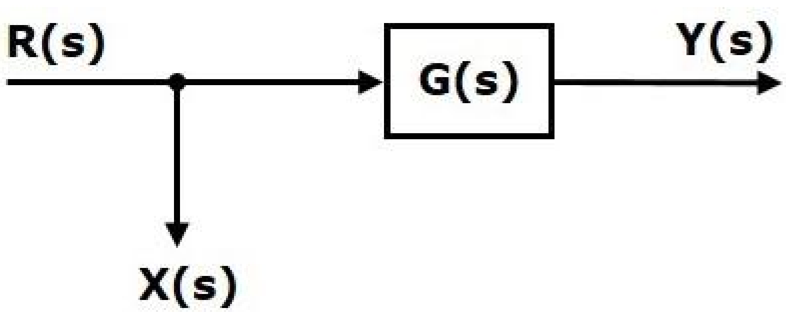
There are two possibilities of shifting the take-off points with respect to blocks −

- Shifting take-off point after the block
- Shifting take-off point before the block

Let us now see what kind of arrangements are to be done in the above two cases, one by one.

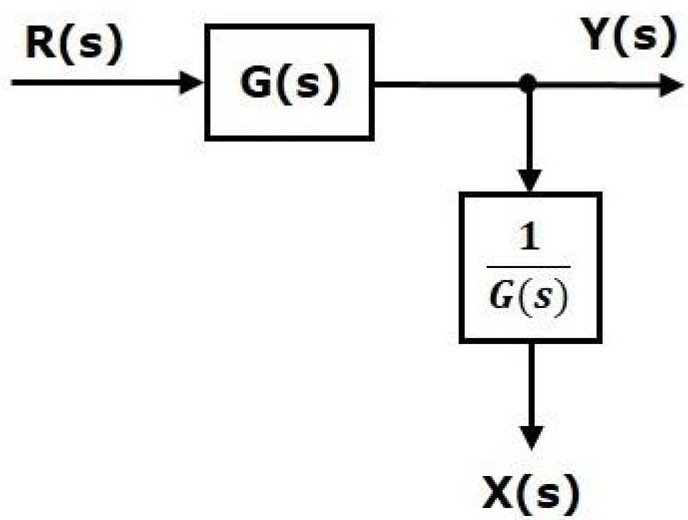
Shifting Take-off Point After the Block

Consider the block diagram shown in the following figure. In this case, the take-off point is present before the block.



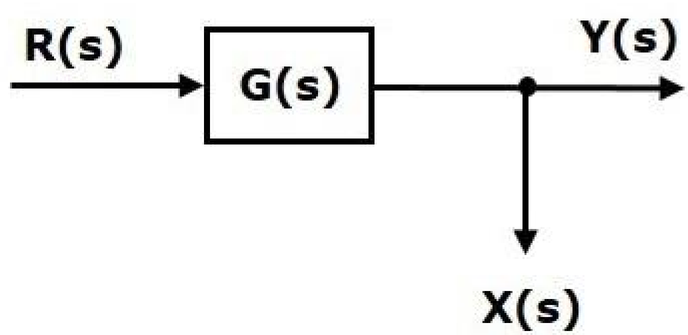
Here, $X(s) = R(s)$ and $Y(s) = G(s)R(s)$

When you shift the take-off point after the block, the output $Y(s)$ will be same. But, there is difference in $X(s)$ value. So, in order to get the same $X(s)$ value, we require one more block $\frac{1}{G(s)}$. It is having the input $Y(s)$ and the output is $X(s)$. This block diagram is shown in the following figure.



Shifting Take-off Point Before the Block

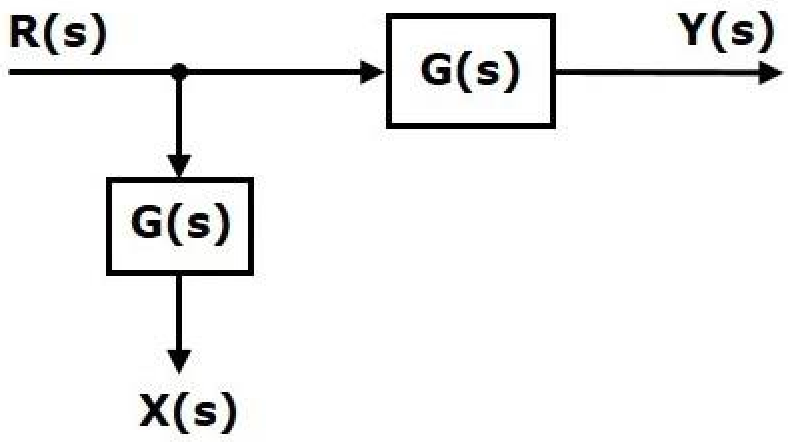
Consider the block diagram shown in the following figure. Here, the take-off point is present after the block.



Here, $X(s) = Y(s) = G(s)R(s)$

When you shift the take-off point before the block, the output $Y(s)$ will be same. But, there is difference in $X(s)$ value. So, in order to get same $X(s)$ value, we require one more block

$G(s)$. It is having the input $R(s)$ and the output is $X(s)$. This block diagram is shown in the following figure.



Control Systems - Block Diagram Reduction

The concepts discussed in the previous chapter are helpful for reducing (simplifying) the block diagrams.

Block Diagram Reduction Rules

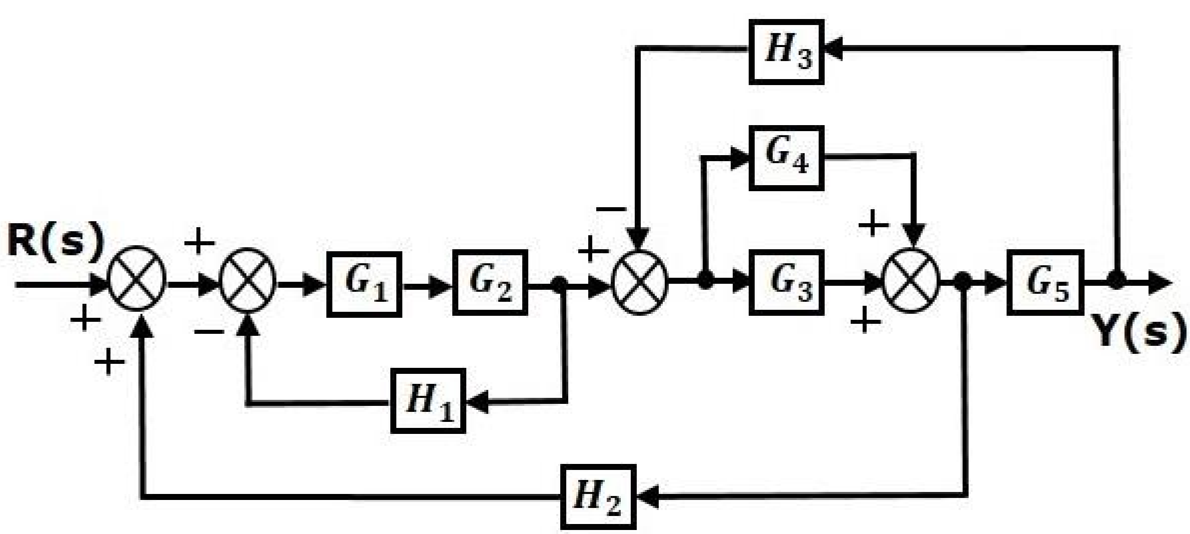
Follow these rules for simplifying (reducing) the block diagram, which is having many blocks, summing points and take-off points.

- **Rule 1** − Check for the blocks connected in series and simplify.
- **Rule 2** − Check for the blocks connected in parallel and simplify.
- **Rule 3** − Check for the blocks connected in feedback loop and simplify.
- **Rule 4** − If there is difficulty with take-off point while simplifying, shift it towards right.
- **Rule 5** − If there is difficulty with summing point while simplifying, shift it towards left.
- **Rule 6** − Repeat the above steps till you get the simplified form, i.e., single block.

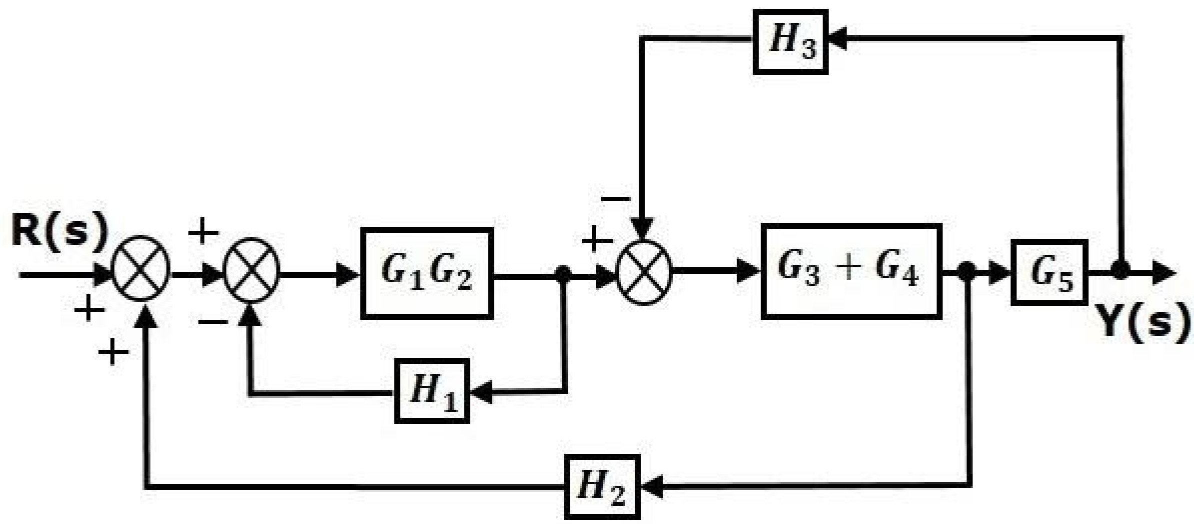
Note − The transfer function present in this single block is the transfer function of the overall block diagram.

Example

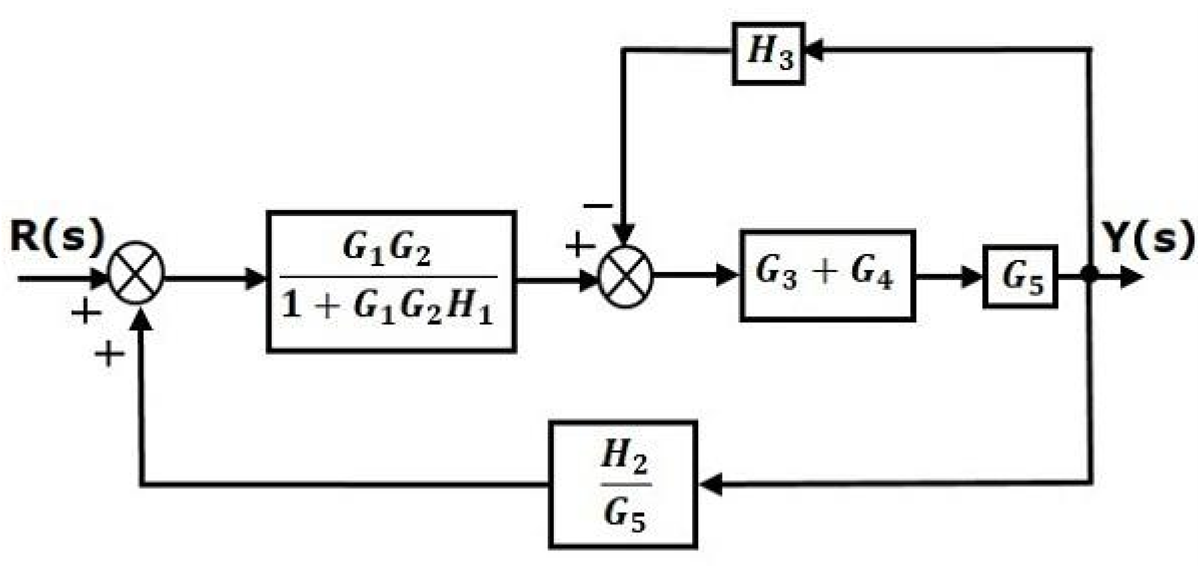
Consider the block diagram shown in the following figure. Let us simplify (reduce) this block diagram using the block diagram reduction rules.



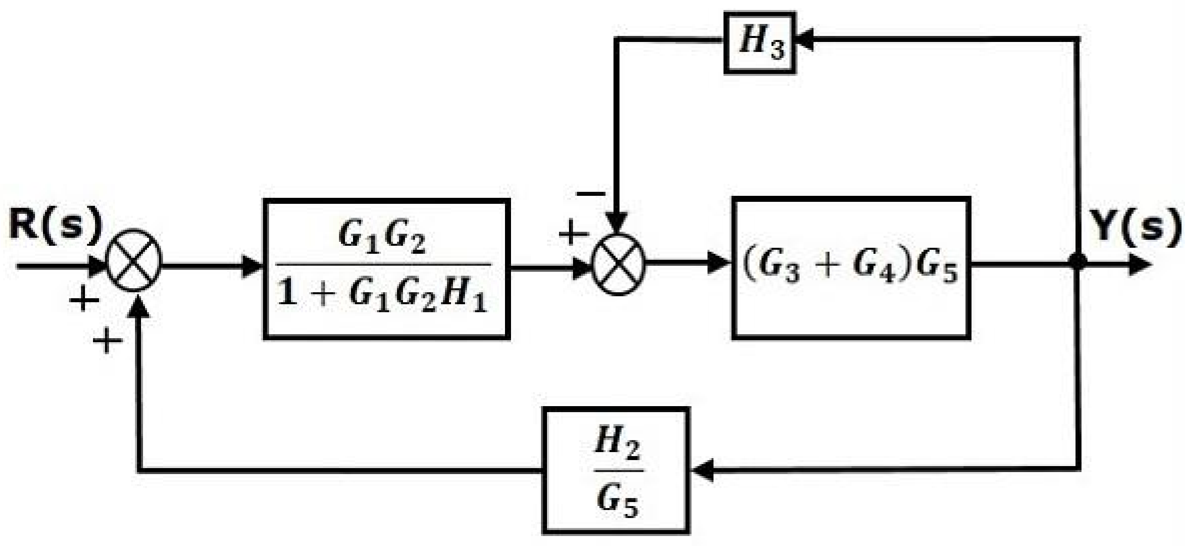
Step 1 − Use Rule 1 for blocks G_1 and G_2 . Use Rule 2 for blocks G_3 and G_4 . The modified block diagram is shown in the following figure.



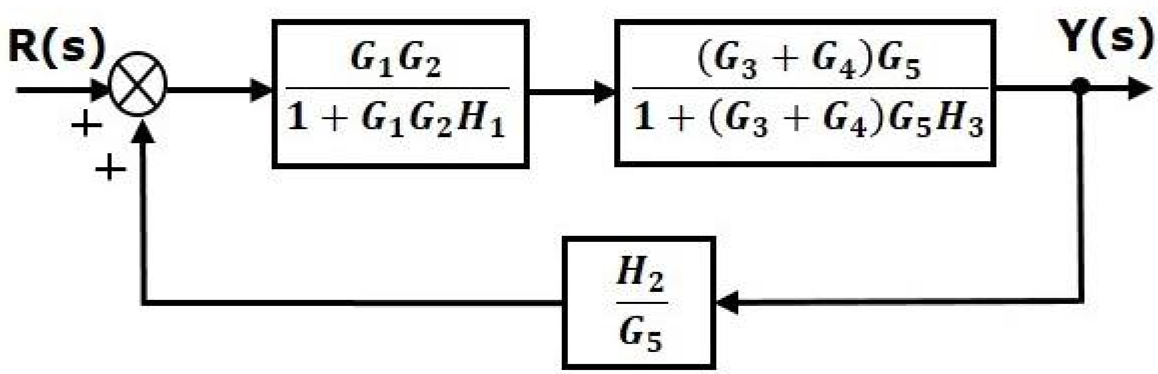
Step 2 − Use Rule 3 for blocks G_1G_2 and H_1 . Use Rule 4 for shifting take-off point after the block G_5 . The modified block diagram is shown in the following figure.



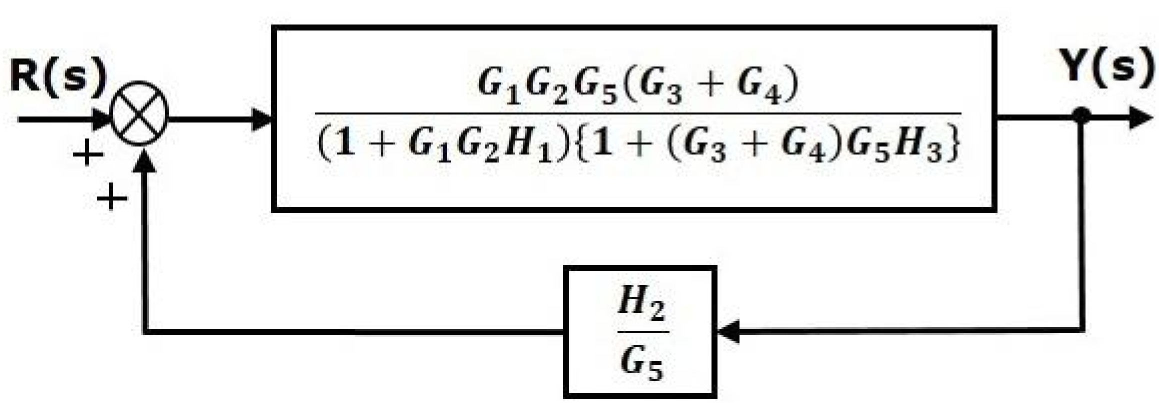
Step 3 − Use Rule 1 for blocks $(G_3 + G_4)$ and G_5 . The modified block diagram is shown in the following figure.



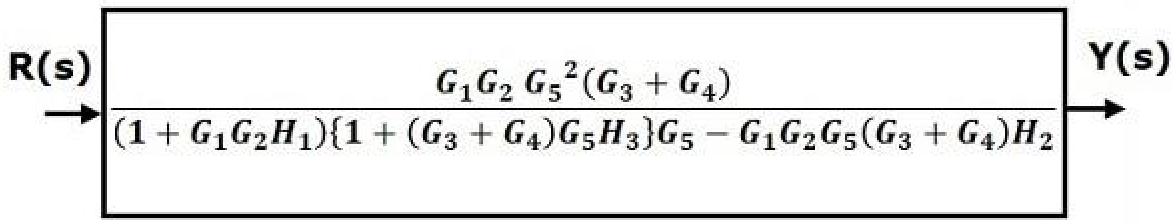
Step 4 − Use Rule 3 for blocks $(G_3 + G_4)G_5$ and H_3 . The modified block diagram is shown in the following figure.



Step 5 − Use Rule 1 for blocks connected in series. The modified block diagram is shown in the following figure.



Step 6 − Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.



Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_5^2(G_3+G_4)}{(1+G_1G_2H_1)\{1+(G_3+G_4)G_5H_3\}G_5-G_1G_2G_5(G_3+G_4)H_2}$$

Note − Follow these steps in order to calculate the transfer function of the block diagram having multiple inputs.

- **Step 1** − Find the transfer function of block diagram by considering one input at a time and make the remaining inputs as zero.
- **Step 2** − Repeat step 1 for remaining inputs.
- **Step 3** − Get the overall transfer function by adding all those transfer functions.

The block diagram reduction process takes more time for complicated systems. Because, we have to draw the (partially simplified) block diagram after each step. So, to overcome this drawback, use signal flow graphs (representation).

In the next two chapters, we will discuss about the concepts related to signal flow graphs, i.e., how to represent signal flow graph from a given block diagram and calculation of transfer function just by using a gain formula without doing any reduction process.

Control Systems - Signal Flow Graphs

Signal flow graph is a graphical representation of algebraic equations. In this chapter, let us discuss the basic concepts related signal flow graph and also learn how to draw signal flow graphs.

Basic Elements of Signal Flow Graph

Nodes and branches are the basic elements of signal flow graph.

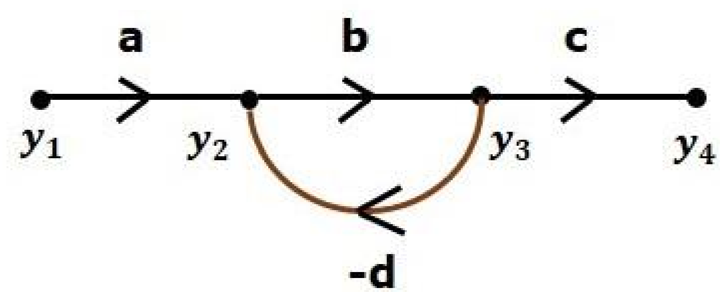
Node

Node is a point which represents either a variable or a signal. There are three types of nodes — input node, output node and mixed node.

- **Input Node** − It is a node, which has only outgoing branches.
- **Output Node** − It is a node, which has only incoming branches.
- **Mixed Node** − It is a node, which has both incoming and outgoing branches.

Example

Let us consider the following signal flow graph to identify these nodes.



- The **nodes** present in this signal flow graph are $\mathbf{y_1}$, $\mathbf{y_2}$, $\mathbf{y_3}$ and $\mathbf{y_4}$.
- $\mathbf{y_1}$ and $\mathbf{y_4}$ are the **input node** and **output node** respectively.
- $\mathbf{y_2}$ and $\mathbf{y_3}$ are **mixed nodes**.

Branch

Branch is a line segment which joins two nodes. It has both **gain** and **direction**. For example, there are four branches in the above signal flow graph. These branches have **gains** of **a, b, c** and **-d**.

Construction of Signal Flow Graph

Let us construct a signal flow graph by considering the following algebraic equations −

$$y_2 = a_{12}y_1 + a_{42}y_4$$

$$y_3 = a_{23}y_2 + a_{53}y_5$$

$$y_4 = a_{34}y_3$$

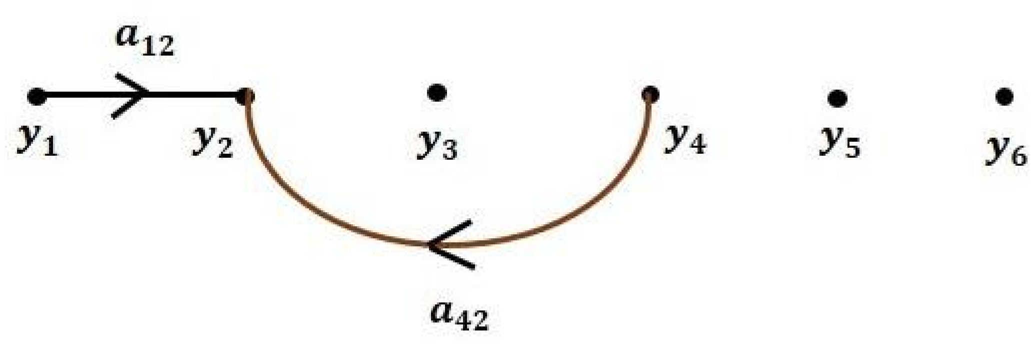
$$y_5 = a_{45}y_4 + a_{35}y_3$$

$$y_6 = a_{56}y_5$$

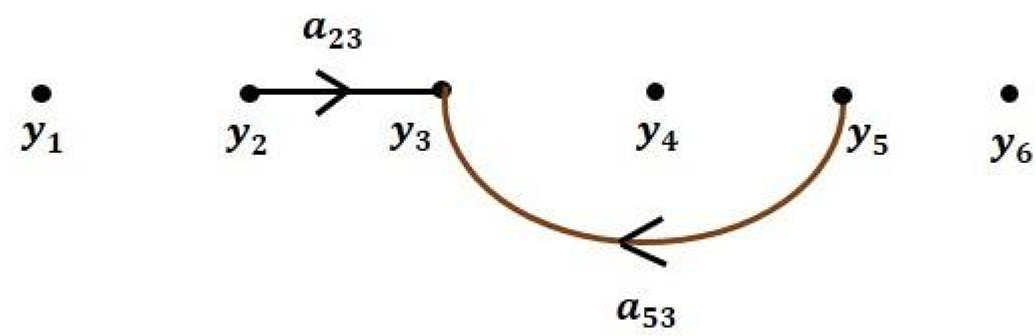
There will be six **nodes** (y_1, y_2, y_3, y_4, y_5 and y_6) and eight **branches** in this signal flow graph. The gains of the branches are a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{42}, a_{53} and a_{35}.

To get the overall signal flow graph, draw the signal flow graph for each equation, then combine all these signal flow graphs and then follow the steps given below −

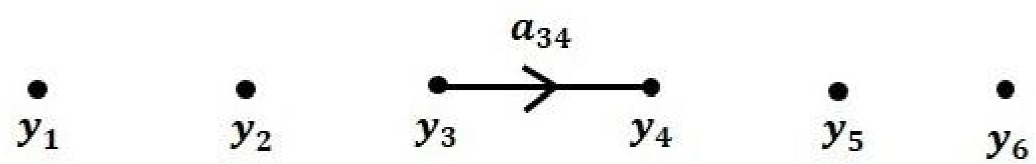
Step 1 − Signal flow graph for $y_2 = a_{13}y_1 + a_{42}y_4$ is shown in the following figure.



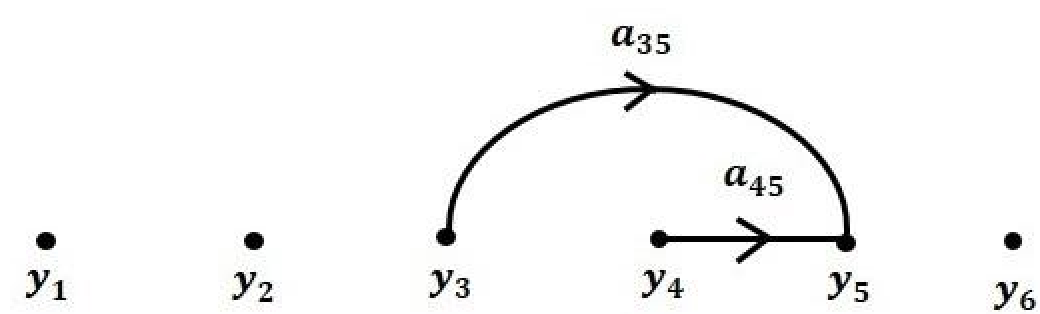
Step 2 − Signal flow graph for $y_3 = a_{23}y_2 + a_{53}y_5$ is shown in the following figure.



Step 3 − Signal flow graph for $y_4 = a_{34}y_3$ is shown in the following figure.

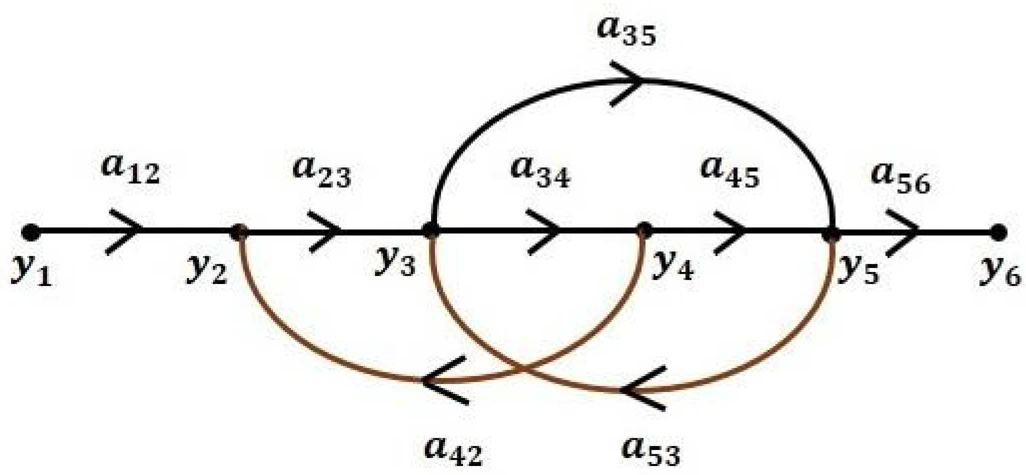


Step 4 − Signal flow graph for $y_5 = a_{45}y_4 + a_{35}y_3$ is shown in the following figure.



Step 5 − Signal flow graph for $y_6 = a_{56}y_5$ is shown in the following figure.

Step 6 − Signal flow graph of overall system is shown in the following figure.



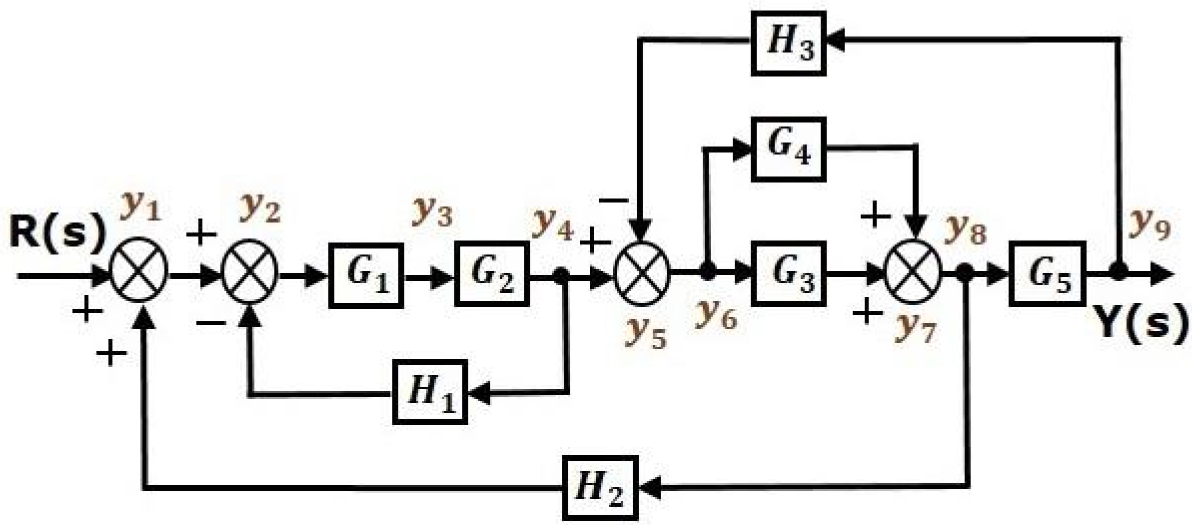
Conversion of Block Diagrams into Signal Flow Graphs

Follow these steps for converting a block diagram into its equivalent signal flow graph.

- Represent all the signals, variables, summing points and take-off points of block diagram as **nodes** in signal flow graph.
- Represent the blocks of block diagram as **branches** in signal flow graph.
- Represent the transfer functions inside the blocks of block diagram as **gains** of the branches in signal flow graph.
- Connect the nodes as per the block diagram. If there is connection between two nodes (but there is no block in between), then represent the gain of the branch as one. **For example**, between summing points, between summing point and takeoff point, between input and summing point, between take-off point and output.

Example

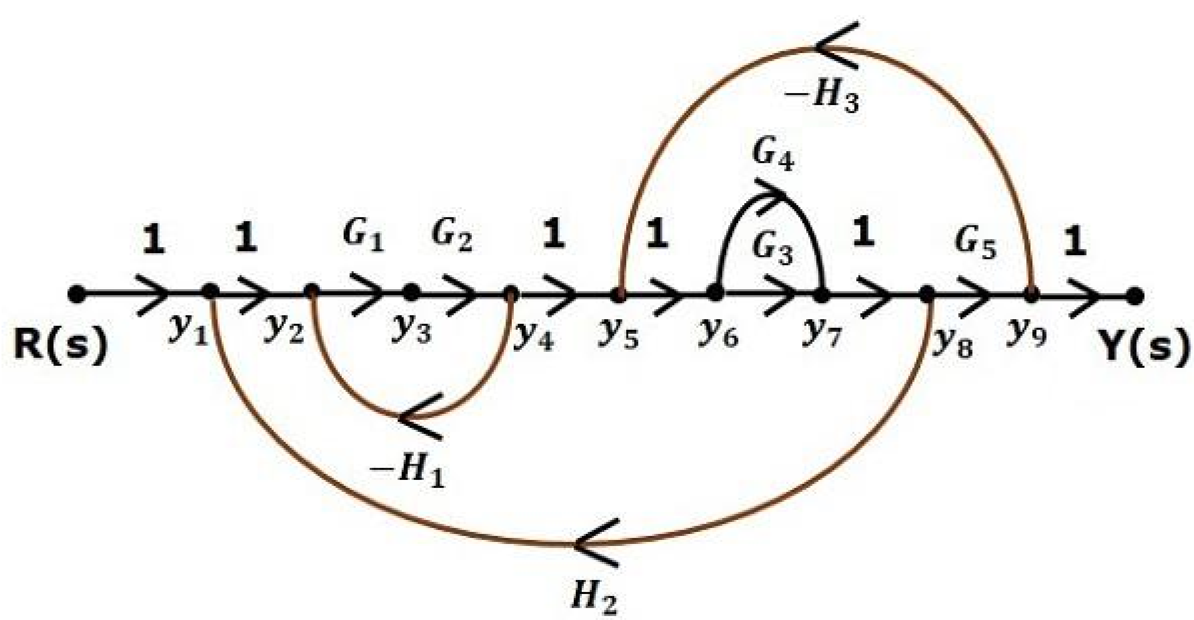
Let us convert the following block diagram into its equivalent signal flow graph.



Represent the input signal $R(s)$ and output signal $C(s)$ of block diagram as input node $R(s)$ and output node $C(s)$ of signal flow graph.

Just for reference, the remaining nodes (y_1 to y_9) are labelled in the block diagram. There are nine nodes other than input and output nodes. That is four nodes for four summing points, four nodes for four take-off points and one node for the variable between blocks G_1 and G_2 .

The following figure shows the equivalent signal flow graph.



With the help of Mason's gain formula (discussed in the next chapter), you can calculate the transfer function of this signal flow graph. This is the advantage of signal flow graphs. Here, we no need to simplify (reduce) the signal flow graphs for calculating the transfer function.